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Comportement microscopique de particules en interaction : gaz de Coulomb, Riesz et log-gases.

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devant le jury composé de :

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« Avec la mathématique, avec la géométrie, on explique presque tout. » Dans le « presque », il y a le battement même de notre vie, l'incertitude et l'élan du cœur (...). C'est par le « presque » que passe le souffle qui nous préserve de finir prisonniers du nombre. On peut dire cela, sans doute ; à condition de ne pas oublier que le nombre aussi est essentiel.

Philippe Jaccottet, *Le bol du pèlerin.*

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Pour le reste,

ce dont on ne peut parler, il faut le taire

. . . ou le chanter !

Table des matières

Chapitre 1

Introduction générale

Cette thèse est consacrée à l'étude mathématique du comportement microscopique de certains systèmes de particules en interaction provenant de la physique statistique et de la théorie des matrices aléatoires.

On considère des systèmes à l'équilibre de N particules classiques dans l'espace euclidien $\mathbb{R}^d$, qui interagissent par paires via un potentiel d'interaction g singulier et à longue portée. On utilisera souvent l'image suivante : les particules sont des charges ponctuelles et g un potentiel d'interaction "électrostatique". Ces particules sont piégées par un potentiel NV (par exemple un champ électrique externe d'intensité proportionnelle à N) qui évite qu'elles ne se repoussent mutuellement à l'infini. On s'intéresse au comportement aléatoire de ces systèmes au sens de la physique statistique, en tant que modèles présentant un intérêt physique intrinsèque mais aussi parce que pour certains choix de g, V et de la température, la loi des particules coïncide avec la loi des valeurs propres de modèles de matrices aléatoires Hermitiennes (en dimension 1) et non-Hermitiennes (en dimension 2).

Si le comportement global des particules (à l'échelle macroscopique $O(1)$) est bien compris, le comportement local (à l'échelle microscopique $O(N^{-1/d})$) est encore peu étudié en dehors d'un cas particulier (le gaz à interactions logarithmiques en dimension 1). La définition d'une "énergie renormalisée" par Sandier-Serfaty dans [SS12] (étendue ensuite dans [SS15b,SS15a,RS15,PS15]) permet de ré-écrire l'énergie d'interaction du système en termes d'une quantité (la norme du champ électrique associé, calculée de manière renormalisée pour compenser la divergence du champ près des charges ponctuelles) qui dépend naturellement de la configuration de points à l'échelle microscopique. Suivant les principes de la physique statistique, on peut alors chercher à discriminer entre les configurations rares, de grande énergie, et les configurations typiques, de basse énergie, tout en prenant en compte l'élément de volume dans l'espace des phases. On fait alors apparaître une fonctionnelle d'énergie libre $\mathcal{F}_\beta$ qui gouverne le comportement des particules à l'échelle microscopique. Plus précisément on montre que la loi du champ empirique (c'est-à-dire de la configuration de points observée lorsque l'on zoome d'un facteur $N^{1/d}$ autour d'un point choisi aléatoirement dans une petite zone) obéit à un Principe de Grandes Déviations (PGD) avec pour fonction de taux $\mathcal{F}_\beta$.

Pour cela, on combine des idées probabilistes appartenant à la théorie des grandes déviations avec des techniques qui relèvent du calcul des variations. Un avantage est que les méthodes utilisées fonctionnent indifféremment pour les *log-gases* en dimension 1 et 2 (qui correspondent, dans certains cas, à des modèles de matrices aléatoires) et pour des interactions plus générales comme le cas, particulièrement pertinent d'un point de vue physique, de l'interaction Coulombienne en dimension 3. Cette approche fournit également une interprétation physique de certains résultats connus en théorie des matrices aléatoires, tout en permettant parfois d'affaiblir les hypothèses de régularité sur le potentiel V.

Après avoir décrit les motivations et le contexte physico-mathématique de notre travail, on présente les principales méthodes mises en œuvre et les résultats obtenus au cours de cette thèse. En dernier lieu, on mentionnera quelques perspectives de recherches en relation avec les objets étudiés.

1.1 Contexte et motivations

1.1.1 Physique statistique

a. Contexte physique

Soit $d \geq 1$. On considère un système de N particules dans $\mathbb{R}^d$ interagissant par paires via un potentiel g et soumises à un champ extérieur NV. L'ensemble des états possibles du système, *l'espace des phases* est donc $(\mathbb{R}^d)^N$. À chaque état $\vec{X}_N = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$ on peut associer une énergie $\mathcal{H}_N(\vec{X}_N)$ donnée par

$$\mathcal{H}_N(\vec{X}_N) := \sum_{1 \leq i \neq j \leq N} g(x_i - x_j) + \sum_{i=1}^{N} NV(x_i).$$

Le premier terme du membre de droite correspond à l'énergie d'interaction des particules entre elles (la somme de l'énergie potentielle d'interaction entre x_i et x_j pour $i \neq j$) et le second terme correspond à l'effet du potentiel confinant NV sur chacune des particules.

À la température T, si l'on note $\beta = \frac{1}{T}$, les lois de la physique statistique stipulent que le comportement du système est caractérisé par une mesure de probabilité sur l'espace des phases, qui donne un poids proportionnel au facteur de Boltzmann $\exp\left(-\beta \mathcal{H}_N(\vec{X}_N)\right)$ à chaque état $\vec{X}_N$. On introduit donc la mesure de Gibbs *canonique* à N points et à température inverse β :

$$d\mathbb{P}_{N,\beta}(\vec{X}_N) := \frac{1}{Z_{N,\beta}} \exp\left(-\beta \left(\sum_{i \neq j} g(x_i - x_j) + \sum_{i=1}^{N} NV(x_i)\right)\right) d\vec{X}_N, \tag{1.1.1}$$

où $d\vec{X}_N := dx_1 \dots dx_N$ est la mesure de Lebesgue sur $(\mathbb{R}^d)^N$. La constante $Z_{N,\beta}$ dans (1.1.1) sert à normaliser $\mathbb{P}_{N,\beta}$, elle vaut donc

$$Z_{N,\beta} := \int_{(\mathbb{R}^d)^N} \exp\left(-\beta \left(\sum_{i \neq j} g(x_i - x_j) + \sum_{i=1}^{N} NV(x_i)\right)\right) d\vec{X}_N, \tag{1.1.2}$$

et est nommée "fonction de partition" dans le langage de la physique statistique. Pour que (1.1.1) fasse sens, il faut en particulier que l'intégrale de (1.1.2) converge.

b. Questions typiques

Soit $\mathcal{O}_N$ une quantité observable, c'est-à-dire une fonction mesurable de l'état $\vec{X}_N$ du système. Par abus de terminologie, on parlera encore de l'observable $\mathcal{O}_N$ pour désigner la variable aléatoire $\mathcal{O}_N(\vec{X}_N)$, lorsque $\vec{X}_N$ est distribuée selon la mesure de Gibbs $\mathbb{P}_{N,\beta}$. Citons quelques exemples :

— La mesure empirique des particules $\vec{X}_N \mapsto \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$.
— La plus petite distance entre deux particules $\vec{X}_N \mapsto \min_{i \neq j} |x_i - x_j|$.
— Le plus grand module des particules $\vec{X}_N \mapsto \max_{i=1}^{N} |x_i|$.
— Le nombre de particules se trouvant dans un certain domaine Ω.

Plusieurs questions naturelles peuvent être posées :

Comportement typique. Y a-t-il une (ou plusieurs) limite(s), en un sens à préciser, à l'observable quand $N \to \infty$? Cette limite est-elle déterministe ou bien aléatoire ? Peut-on la décrire comme la solution d'un problème variationnel ? À quelle vitesse y a-t-il convergence ?

Universalité. Comment la limite dépend-elle de V (et de g), c'est-à-dire de la nature exacte du système physique ?

Rôle de la température. Comment la limite dépend-elle de β ? Y a-t-il des *transitions de phases* ? Que peut-on dire des cas particuliers $\beta \to 0$ et $\beta \to \infty$?

C'est à ce type de questions que l'on va s'intéresser par la suite. Le choix de l'observable $\mathcal{O}_N$ peut se révéler délicat : on veut que $\mathcal{O}_N(\vec{X}_N)$ encode assez d'informations sur le système pour être "intéressante", mais plus une observable est précise, plus il peut être difficile de caractériser sa limite.

c. Modèles étudiés

Les interactions entre particules étudiées dans cette thèse présentent deux particularités techniques :

1. Le potentiel est singulier en 0, et le système est défini dans tout $\mathbb{R}^d$ (et pas seulement sur un réseau).
2. Les interactions sont à longue portée (en anglais *long-range*) : elles tendent vers 0 quand $|x-y| \to \infty$ comme $|x-y|^{-s}$ avec $s < d$, voire pas du tout. En particulier, l'effet d'une particule placée en $y = 0$ est *a priori* ressenti dans tout l'espace, puisque $x \mapsto g(x)$ n'est pas intégrable sur $\mathbb{R}^d$.

On distinguera, dans notre étude, trois types d'interactions.

Le gas logarithmique uni-dimensionnel.

$$\boxed{d = 1,\ g(x-y) = -\log|x-y|}$$

Ce modèle est appelé un *one-dimensional log-gas* ou β-ensemble, et il a été largement étudié en raison de ses liens avec la théorie des matrices aléatoires, que l'on développera en Section 1.1.2. La pertinence physique de ce modèle n'est pas primordiale, mais il peut être vu comme un "modèle jouet" de physique statistique, voir par exemple [For10] pour un présentation exhaustive du *log-gas* et de son lien avec les matrices aléatoires. Il est aussi relié à des questions de physique concernant le *Calogero-Sutherland model*, pour lequel la fonction d'onde de l'état fondamental coïncide avec la loi d'un *log-gas* avec potentiel Gaussien (voir [For98]).

Le gaz de Coulomb bi-dimensionnel.

$$\boxed{d = 2,\ g(x-y) = -\log|x-y|}$$

Ce modèle est appelé un *two-dimensional log-gas*. Dans la littérature physique on trouve aussi l'appellation de *two-dimensional one-component plasma*, dont on utilisera par la suite l'abrévation en 2DOCP (voir par exemple [AJ81, SM76]), ou de *2D Dyson Gas* [ZW06]. En dehors de son rôle comme modèle jouet pour la physique statistique dans $\mathbb{R}^2$, l'intérêt physique du 2DOCP est plus immédiat, puisque l'interaction logarithmique est (à un facteur multiplicatif près) l'interaction Coulombienne pour la physique de dimension deux (on peut penser par exemple à la vue en coupe de fils électriques "infiniment"

longs), ce type d'interaction apparaît notamment entre les vortex dans l'étude de la supraconductivité (cf. [Ser15]) ou de la mécanique des fluides bi-dimensionnels [MP82]. Le gaz de Coulomb bi-dimensionnel est également étudié en raison de ses liens avec la théorie des matrices aléatoires non-Hermitiennes, que l'on explicitera en Section 1.1.2.

Les gaz de Riesz.

$$\boxed{d \geq 1,\ g(x-y) = |x-y|^{-s}, \quad \max(d-2,0) \leq s < d}$$

Le cas $s = d-2$ (pour $d \geq 3$) correspond (à un facteur multiplicatif près) à l'interaction Coulombienne en dimension d, en particulier le cas $d = 3, s = 1$ présente un intérêt physique évident. Les cas Riesz généralisent cette interaction et permettent notamment d'observer l'influence de la nature exacte des interactions sur les propriétés du système.

On fera parfois référence au cas des interactions logarithmiques (en une ou deux dimensions) comme "les cas logarithmiques", on parlera aussi des "cas Coulomb" pour le 2DOCP ou pour $d \geq 3, s = d-2$ et, par opposition, des cas "non-Coulomb".

On impose peu de conditions directes sur le potentiel V. La principale exigence est qu'il soit fortement confinant, c'est-à-dire qu'il vérifie une certaine condition de croissance à l'infini qui a pour effet d'éviter qu'une fraction non nulle des particules ne s'échappent.

1.1.2 Matrices aléatoires

Une motivation importante pour l'étude des gaz logarithmiques est leur lien avec la théorie des matrices aléatoires, dans laquelle une idée centrale est que "les valeurs propres se repoussent logarithmiquement". Une étude historique récente [DF15] fait remonter l'apparition des matrices aléatoires en mathématiques aux travaux d'Hurwitz en 1897, on cite généralement l'article [Wis28] de 1928 dû à Wishart pour les applications en statistiques, et Wigner [Wig58] pour une motivation liée à la physique quantique, qui est celle que nous allons présenter succintement. L'étude des matrices aléatoires présente de nombreuses autres applications, que ce soit en physique, en télécommunications ou encore en finance.

Pour définir une matrice aléatoire de taille $N \times N$, on peut par exemple :

— Fixer la loi jointe de ses N^2 coefficients.
— Considérer une mesure de probabilité "abstraite", par exemple la mesure de Haar sur un groupe compact de matrices.
— Chercher une variable aléatoire dont la loi soit invariante sous certaines opérations matricielles (par exemple, sous l'action du groupe unitaire ou orthogonale par conjugaison).

Les modèles présentés dans cette section possèdent la propriété importante que la loi jointe de leurs valeurs propres peut être calculée exactement et correspondent à la loi de particules avec interaction logarithmique en dimension 1 ou 2.

a. Une motivation physique des matrices aléatoires

Dans le formalisme de la mécanique quantique, un système physique est représenté par un vecteur dans un espace de Hilbert $\mathcal{H}$ et son évolution est encodée par *l'opérateur Hamiltonien*, qui est un opérateur Hermitien A sur $\mathcal{H}$ correspondant aux interactions du système. Le spectre de A joue alors un rôle crucial dans l'analyse, les valeurs propres correspondant aux "niveaux d'énergie" du système. Si le système est très complexe (par exemple un noyau lourd), déterminer théoriquement les valeurs propres de A devient une tâche difficile dès que l'on dépasse les niveaux d'énergie les plus bas.

> Recent theoretical analyses have had impressive success in interpreting the detailed structure of the low-lying excited states of complex nuclei. Still, there must come a point beyond which such analyses of individual levels cannot usefully go. [Dys62, p.1]

Le changement de paradigme suivant est alors proposé : au lieu de déterminer exactement A et ses niveaux d'énergie, on cherche plutôt à connaître les propriétés statistiques des niveaux d'énergie d'un opérateur A "général", vérifiant certaines conditions de symétrie correspondant aux invariances du système physique.

> The result of such an inquiry will be a statistical theory of energy levels. The statistical theory will not predict the detailed sequence of levels in any one nucleus, but it will describe the general appearance and the degree of irregularity of the level structure that is expected to occur in any nucleus which is too complicated to be understood in detail. [Dys62, p.1]

Cela évoque l'idée fondamentale de la physique statistique à la différence suivante près : tandis que la physique statistique renonce à connaître *l'état exact* du système - tous les états de même énergie étant équiprobables, la théorie statistique des niveaux d'énergie renonce, elle, à connaître *la nature exacte* du système et de ses interactions. Pour reprendre les mots de [Dys62, p.1] cela constitue

> a mathematical idealization of the notion of "all physical systems with equal probability". (...) We picture a complex nucleus as a "black box" in which a large number of particles are interacting according to unknown laws. The problem is then to define in a mathematically precise way an ensemble of systems in which all possible laws of interactions are equally probable.

D'une part, puisque les interactions sont représentées par un opérateur, on fait l'hypothèse que des interactions générales peuvent être représentées par un opérateur tiré au hasard dans un certain ensemble ; d'autre part on peut chercher à approcher le spectre d'un opérateur aléatoire en considérant la limite du spectre d'une *matrice aléatoire* de taille N quand $N \to \infty$. On ne tentera pas dans la suite de justifier physiquement les modèles de matrices aléatoires considérés, mais une idée importante de la théorie est l'espoir que les propriétés statistiques intéressantes ne dépendent pas trop fortement de la manière précise dont la modélisation est effectuée et sont, en un certain sens, universelles.

b. Cas Hermitien

Le cas des matrices aléatoires à valeurs propres réelles est de loin le plus étudié, on se limitera ici à présenter quelques modèles et résultats en rapport avec notre étude.

À N fixé, on définit trois modèles de matrices aléatoires en fixant la loi des coefficients $(M_{i,j})_{1\leq i,j\leq N}$ de la manière suivante (voir par exemple [AGZ10, Section 2.5] pour une définition précise) :

G.O.E. Les coefficients $M_{i,j}$ sont des variables aléatoires Gaussiennes centrées *réelles* de variance proportionnelle à $1/N$, indépendantes à cela près que M doit être symétrique.

G.U.E. Les coefficients $M_{i,j}$ sont des variables aléatoires Gaussiennes centrées *complexes* de variance proportionnelle à $1/N$, indépendantes à cela près que M doit être Hermitienne (en particulier, la diagonale est réelle).

G.S.E. Les coefficients $M_{i,j}$ sont des variables aléatoires Gaussiennes centrées *quaternioniques* de variance proportionnelle à $1/N$, indépendantes à cela près que M doit être auto-duale.

Les conditions de symétrie imposées équivalent chacune à une certaine invariance, au sens de la mécanique quantique, du système physique dont la matrice est supposée représenter l'interaction. Par ailleurs, cela garantit que les N valeurs propres de M sont réelles. La loi jointe des N valeurs propres peut être exprimée exactement et s'écrit

$$dP^{\mathrm{mat}}_{N,\beta}(\vec{X}_N) = C_\beta \exp\left(-\beta \sum_{1\leq i\neq j\leq N} -\log|x_i - x_j| - \sum_{i=1}^{N} \frac{N}{2}|x_i|^2\right) d\vec{X}_N \tag{1.1.3}$$

où C_β est une constante de normalisation dont l'expression est connue. La valeur de β est à choisir en fonction du modèle considéré, les valeurs classiques sont $\beta = 1$ (GOE), $\beta = 2$ (GUE) et $\beta = 4$ (GSE) mais avec notre convention d'écriture pour l'énergie d'interaction (la somme portant sur $i \neq j$ et non sur $i < j$) ces valeurs doivent être divisées par 2. On voit que (1.1.3) correspond à la loi des particules d'un *one-dimensional log-gas* avec un potentiel quadratique et un bon choix de β (notons que la convention usuelle en matrices aléatoires est de ne faire porter le facteur β que sur l'interaction par paires et non sur le potentiel quadratique, contrairement à la convention choisie pour la mesure de Gibbs $\mathbb{P}_{N,\beta}$ (1.1.1)).

Il est alors naturel de se demander si l'on peut dépasser le cadre de ces trois valeurs classiques et trouver, pour tout $\beta > 0$, un modèle de matrices aléatoires à coefficients indépendants dont la loi jointe des valeurs propres serait donnée par $P_{N,\beta}$. Les auteurs de [DE02] construisent un modèle de matrices tridiagonales répondant par l'affirmative à cette question, ce qui prolonge à des valeurs arbitraires de β le lien entre matrices aléatoires et gaz de particules avec interaction logarithmique (même si les constructions de [DE02] correspondent toujours à un choix de V quadratique).

c. Cas non-Hermitien

> "An ensemble of matrices whose elements are complex, quaternion, or real numbers, but with no other restrictions as to their Hermitian or unitary character, is of no immediate physical interest, for their eigenvalues may lie anywhere on the complex plane. However, efforts have been made (..) to investigate them and the results are interesting in their own right." [Meh04, Chapitre 15, p.266]
>
> "Applications and studies of matrix ensembles with complex eigenvalues are numerous." [WZ06]

Sans chercher à trancher la question de l'*intérêt* de leur étude, mentionnons une application possible des matrices aléatoires non-Hermitiennes : comme expliqué ci-dessous, la loi des valeurs propres complexes de certains modèles de matrices aléatoires (l'ensemble de Ginibre) coïncide avec la loi des particules d'un gaz de Coulomb bi-dimensionnel pour une certaine température (ce fait est déjà observé par Dyson dans [Dys62]). Or une matrice est un objet potentiellement plus simple à étudier qu'un système de particules, ne serait-ce que d'un point de vue numérique. On renvoie aux références citées par [ZW06] pour de plus amples connections avec les modèles de croissance (*Laplacian growth*), l'analyse complexe ou encore l'effet Hall quantique.

Le principal modèle de matrice aléatoire non-Hermitienne est obtenu en distribuant les coefficients de $M = (M_{i,j})_{1\leq i,j\leq N}$ comme N^2 variables aléatoires indépendantes Gaussiennes complexes centrées et en imposant une normalisation sur la variance, par exemple $\mathbf{E}[M_{1,1}^2] = \frac{1}{N}$. Cela correspond à une loi de densité $\exp(-N\mathrm{tr}(A^*A))$ (à normalisation près) par rapport à la mesure de Lebesgue $\prod_{1\leq i,j,\leq N} dA_{i,j}$ sur $(\mathbb{R}^2)^{N^2}$ (où A représente la matrice $(A_{i,j})_{i,j}$ et A^* son adjoint), et on peut alors déterminer explicitement la loi jointe des valeurs propres (en suivant

par exemple le calcul de [Meh04, Chapitre 15])

$$dP_N^{\mathrm{Gin}}(z_1,\dots,z_N) = C\exp\left(-\sum_{1\le i\le j\le N} -\log|z_i-z_j| - N\sum_{i=1}^{N}|z_i|^2\right) dz_1\dots dz_N, \qquad (1.1.4)$$

où C est une certaine constante de normalisation dont l'expression est connue. On voit que (1.1.4) coïncide avec la loi des particules d'un gaz de Coulomb bi-dimensionnel en choisissant V quadratique et β convenable (avec nos conventions d'écriture on trouve $\beta = 1$ mais les conventions usuelles en matrices aléatoires donnent $\beta = 2$). Ce modèle, introduit dans [Gin65], est appelé "ensemble de Ginibre". On peut le généraliser en choisissant un potentiel différent de $V(A) = A^*A$ dans l'expression de la loi comme $\exp(-N\mathrm{tr}(A^*A))\prod_{1\le i,j,\le N} dA_{i,j}$ (à un facteur de normalisation près). On obtient alors le "Random normal matrix model" tel qu'étudié dans [AHM11, AHM15] qui correspond à une densité proportionnelle à $\exp(-N\mathrm{tr}(V(A)))\prod_{1\le i,j,\le N} dA_{i,j}$, ce qui revient à changer le potentiel de confinement du gaz.

Une particularité de ces modèles (plus précisement de la valeur de β correspondante) est que le processus ponctuel associé aux valeurs propres, c'est à dire la loi sur $(\mathbb{R}^2)^N$ de densité $dP_N^{\mathrm{Gin}}(z_1,\dots,z_N)$ est *déterminantal* (voir par exemple [HKPV09]) . Sans rentrer dans les détails, mentionnons simplement que cela permet un accès explicite à de nombreuses quantités comme les fonctions de corrélations et fournit des techniques analytico-algébriques d'étude qui font défaut pour une valeur générale de β.

d. Quelques autres modèles

Les ensembles de Wishart définissent des "matrices aléatoires de covariance empirique". La loi jointe de leurs valeurs propres correspond à un gaz de particules restreint à la demi-droite $\mathbb{R}_+$, avec interaction logarithmique et sous l'effet d'un certain potentiel confinant, pour certaines valeurs particulières de β. Ce modèle admet une généralisation à $\beta > 0$, voir par exemple [DEKV13] ou [DE02].

Comme mentionné plus haut, il est naturel de considérer une matrice aléatoire selon la mesure de Haar sur le groupe unitaire. On appelle Circular Unitary Ensemble (C.U.E.) la loi d'une telle matrice unitaire, ses valeurs propres appartiennent au cercle unité et leur loi jointe peut être explicitement calculée, elle correspond physiquement à un gaz de particules sur le cercle avec interactions logarithmiques, pour une certaine valeur de la température β. En autorisant $\beta > 0$ quelconque on obtient la loi dite du β-C.U.E. et le modèle de matrices associé est construit dans [KN04].

On a vu que dans le cas non-Hermitien, la connection entre matrices aléatoires et gaz de Coulomb n'était établie que pour une valeur particulière de β. L'existence d'un modèle matriciel associé au gaz de Coulomb bi-dimensionnel pour $\beta > 0$ quelconque (avec, par exemple, un potentiel quadratique) reste ouverte, l'extension des heuristiques utilisées dans la construction de [DE02] et dévelopées par exemple dans [Ede10] n'étant pas suffisante. Un tel modèle donnerait certainement lieu à de nouvelles voies dans l'étude numérique et théorique du 2DOCP.

e. Pourquoi ce lien ?

On peut se demander quelle est l'origine de cette "répulsion logarithmique" entre les valeurs propres, qui donne lieu à l'identification entre valeurs propres aléatoires et particules en interaction. Sans chercher une raison ontologique, mentionnons toujours une raison pratique : partons d'une matrice aléatoire dont on connaît la loi des coefficients (par exemple, réels) sous la forme d'une densité $P(A_{i,j})\prod_{1\le i,j,\le N} dA_{i,j}$ sur $\mathbb{R}^{N^2}$. Pour déterminer la loi induite des valeurs propres, on cherche à diagonaliser A, ce qui revient à faire le changement de variables

$A \mapsto$ (valeurs propres, vecteurs propres). La partie du Jacobien correspondant au passage des *coefficients* $\{A_{i,j}\}_{1\leq i,j\leq N}$ aux *racines* $(\lambda_1,\dots,\lambda_N)$ du polynôme caractéristique (i.e. aux valeurs propres) fait apparaître un déterminant de Vandermonde

$$\mathtt{VDM}(\lambda_1,\dots,\lambda_N) := \det\left(\{\lambda_i^{j-1}\}_{1\leq i,j\leq N}\right),$$

dont l'expression est bien connue et s'écrit effectivement comme l'exponentielle des interactions logarithmiques entre les λ_i. De manière très simplifiée, on peut dire que le volume infinitésimal sur $\mathbb{R}^{N^2}$ en coordonnées "valeurs propres, vecteurs propres" fait apparaître $\mathtt{VDM}(\lambda_1,\dots,\lambda_N)$ (en particulier il est rare d'avoir des valeurs propres très proches les unes des autres).

1.1.3 Une autre motivation : la théorie de l'approximation

La limite $\beta\to\infty$ dans (1.1.1) revient formellement à ne considérer que les minimiseurs de l'énergie $\mathcal{H}_N$. Dans le cas d'un gas de Coulomb bi-dimensionnel, ces configurations d'énergie minimales, appelées *points de Fekete*, jouent un rôle important en théorie de l'approximation. Pour simplifier, supposons que le potentiel V vaut $+\infty$ en-dehors d'un compact K de $\mathbb{C}$ sur lequel il vaut 0. Déterminer les points de Fekete revient alors à minimiser l'interaction logarithmique entre N points de K. Cette question est reliée à la théorie de l'approximation de la manière suivante : cherchons à déterminer N points d'interpolation $\{z_1,\dots,z_N\}$ dans K tels que la relation

$$\int_K f(z)dz = \sum_{j=1}^N w_j f(z_j)$$

soit exacte pour les polynômes de degré au plus $N-1$. On voit qu'il faut (et il suffit) de calculer les coefficients w_j tels que $\int_K z^k = \sum_{j=1}^N w_j z_j^k$ pour $0\leq k\leq N-1$, et ce calcul est facile si l'on sait inverser la matrice de Vandermonde des $\{z_j\}_{j=1\dots N}$. La stabilité numérique de cette opération est d'autant plus forte que le conditionnement (*condition number*) de la matrice de Vandermonde est petit, c'est à dire que $\det\mathtt{VDM}(z_1,\dots,z_N)$ est grand. Vu l'expression de $\mathtt{VDM}(z_1,\dots,z_N)$, les meilleurs points (au sens du conditionnement) pour l'interpolation sont donc les points de Fekete. On renvoie à [ST97] pour une présentation détaillée (avec un choix de V plus général) et des références. Bien sûr, ces questions d'approximation se posent aussi sur des variétés, on renvoie par exemple à [HS04] pour une introduction au rôle joué par les interactions de type Riesz.

1.1.4 Le cas à deux composantes

Le dernier résultat de cette thèse, présenté au chapitre 6 , traite d'un cas différent où les charges ponctuelles peuvent prendre deux valeurs opposées ± 1. Si on dispose de N charges positives $x_1,\dots,x_N$ et N charges négatives $y_1,\dots,y_N$ dans un compact Λ de $\mathbb{R}^d$, leur énergie d'interaction vaut

$$\mathcal{H}_N^{2CP}(\vec{X}_N,\vec{Y}_N) := \sum_{1\leq i\neq j\leq N} g(x_i-x_j) + \sum_{1\leq i\neq j\leq N} g(y_i-y_j) - 2\sum_{1\leq i,j\leq N} g(x_i-y_j),$$

et il est clair que cette quantité peut prendre des valeurs arbitrairement négatives, et vaut même $-\infty$ dès que se forme un *dipôle* (c'est à dire qu'une particule chargée positivement et une particule chargée négativement occupent la même position dans l'espace). En particulier, la convergence de la fonction de partition associée

$$Z_{N,\beta} := \int_{\Lambda^{2N}} \exp(-\beta\mathcal{H}_N^{2CP}(\vec{X}_N,\vec{Y}_N))d\vec{X}_N d\vec{Y}_N$$

n'est pas immédiate. Il est d'ailleurs facile de voir qu'aucun des choix précédents de g ne peut donner lieu à une valeur finie de $Z_{N,\beta}$ (et ce pour aucun β), hormis les interactions logarithmiques. On se limitera au cas $d = 2$ et $g(x-y) = -\log|x-y|$ et on prend pour Λ le carré $[0,1]^2$. Ce modèle est appelé *two-dimensional two-component plasma* ou 2D2CP, et bien défini pour des températures assez hautes ($\beta < 2$).

Le 2D2CP, en plus d'être un modèle jouet de la physique statistique à deux dimensions (relié entre autres à des systèmes de vortex provenant de la supraconductivité, ou de la mécanique des fluides $2D$ cf. [BG99]), est intimement lié à d'autres modèles importants : le modèle de Sine-Gordon et le modèle XY. Une particularité remarquable de ces modèles est l'existence d'une transition de phase "d'ordre infini", qui n'est pas reliée à un manque de régularité de la fonction de partition mais à un changement critique du comportement à longue distance des corrélations, voir [KT73] pour l'article fondateur et [FS81] pour une discussion dans le cadre du 2D2CP, voir aussi [Spe97] pour une synthèse. Une autre motivation vient du lien entre le 2D2CP et la fonction de partition du "chaos Gaussian multiplicatif" (cf. [LRV15]), que l'on peut écrire formellement comme $\int e^{i\beta h(x)}\,dx$, où $h(x)$ est un champ libre Gaussien (*Gaussian Free Field*). Ces liens sont discutés en détail dans l'annexe de [LSZW15] (voir chapitre 6).

1.1.5 État de l'art

a. Échelle macroscopique

On présente dans cette section les principaux résultats concernant le comportement macroscopique des systèmes de particules avec interaction Coulomb, Riesz, ou logarithmiques (en fait la plupart des énoncés qui suivent sont valables dans un cas plus général). On renvoie à [Ser15, Chapitre 2] pour une exposition plus détaillée, des références historiques et une preuve de ces résultats.

Le comportement global (à l'échelle macroscopique $O(1)$) des systèmes de particules décrits plus hauts est bien compris. Faisons les hypothèses suivantes sur V :

- **(A1)** V est semi-continu inférieurement et minoré.
- **(A2)** On a $\lim_{|x|\to\infty} V(x) + 2g(x) = +\infty$.
- **(A3)** L'ensemble $\{x \in \mathbb{R}^d | V(x) < +\infty\}$ est de capacité strictement positive.
- **(A4)** Il existe $\alpha > 0$ tel que $\int_{\mathbb{R}^d} e^{-\alpha V(x)} dx < +\infty$.

Sans chercher à définir la notion de "capacité", notons que **(A3)** est en particulier vraie si V ne prend jamais la valeur $+\infty$, ou bien si l'ensemble en question est de mesure de Lebesgue strictement positive, mais **(A3)** autorise aussi, par exemple, le cas où $V = +\infty$ en dehors du cercle unité (ou d'une demi-droite) sur lequel il est constant.

On note $\mathcal{P}(\mathbb{R}^d)$ l'ensemble des mesures de probabilité sur $\mathbb{R}^d$ et on définit la fonctionnelle d'énergie suivante

$$I(\mu) := \iint g(x-y)d\mu(x)d\mu(y) + \int V d\mu(x). \tag{1.1.5}$$

Le théorème suivant relève de la théorie classique du potentiel.

Théorème 1. *Sous les hypothèses **(A1)–(A3)**, le minimum de I sur $\mathcal{P}(\mathbb{R}^d)$ existe et est fini, il est atteint par une unique mesure μ_{eq} dont le support Σ est compact et de capacité positive.*

On appelle μ_{eq} la mesure d'équilibre associée au potentiel V. Mentionnons quelques cas particuliers :

- Dans le cas d'un gas logarithmique $1D$ avec potentiel quadratique la mesure d'équilibre obtenue est (à une mise à l'échelle près) la loi du demi-cercle de Wigner (*Wigner's semi-circular law*), omniprésente en théorie des matrices aléatoires Hermitiennes et de densité $\sqrt{4-x^2}dx\mathbf{1}_{|x|\leq 2}$.

— Dans le cas d'un gaz logarithmique $2D$ avec potentiel quadratique, ce qui correspond à l'ensemble de Ginibre (défini en Section 1.1.2) pour une certaine température, la mesure d'équilibre est (à une éventuelle mise à l'échelle près) la loi circulaire (*circular law*), uniforme dans le disque unité, qui joue en théorie des matrices aléatoires non-Hermitiennes un rôle analogue à la loi du demi-cercle.
— Si $d \geq 3$ et V quadratique, dans le cas Coulombien la mesure d'équilibre est uniforme dans une certaine boule de dimension d.
— Plus généralement, si V est radial et assez régulier, la mesure d'équilibre est, dans le cas Coulombien, supportée sur une boule et de densité proportionnelle à ΔV.

On établit maintenant le lien entre ce résultat de théorie du potentiel "abstraite" et les systèmes de physique statistique décrits plus haut. Pour cela, on considère la mesure empirique des particules, définie par

$$\mu_N(\vec{X}_N) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}.$$

On a bien sûr $\mu_N \in \mathcal{P}(\mathbb{R}^d)$. Rappelons que l'on voit $\vec{X}_N$ comme une variable aléatoire dans $(\mathbb{R}^d)^N$ de loi $\mathbb{P}_{N,\beta}$ et μ_N comme une variable aléatoire à valeurs mesures.

Théorème 2. *Sous les hypothèses **(A1)–(A4)**, les variables aléatoires $\{\mu_N\}_N$ convergent presque sûrement vers μ_{eq} quand $N \to \infty$. De plus, la loi de μ_N obéit à un principe de grandes déviations à vitesse N^2 avec pour fonction de taux $\beta(I - \min I)$.*

On définira un "principe de grandes déviations" en section 1.2.3, pour l'instant on peut penser à la seconde partie du Théorème 2 comme un résultat exprimant formellement l'approximation suivante

$$\mathbb{P}_{N,\beta}(\mu_N \approx \mu) \sim_{N\to\infty} e^{-N^2(I(\mu)-\min I)}.$$

En particulier, si $\mu \neq \mu_{\text{eq}}$ on a $I(\mu) > I(\mu_{\text{eq}})$ et la probabilité que μ_N "ressemble" à μ décroît comme $\exp(-N^2\alpha)$ avec $\alpha = I(\mu) - I(\mu_{\text{eq}})$.

Le résultat du Théorème 2 est établi dans [BAG97] dans le cas particulier des matrices aléatoires Hermitiennes (et dans [BAZ98] pour des matrices de Ginibre réelles, qui diffèrent légèrement du modèle non-Hermitien présenté dans 1.1.2), une preuve plus générale est donnée dans [HP00], voir aussi [CGZ14] et [Ser15, Chapitre 2] pour des interactions g assez générales. Cela conclut très largement l'étude comportement macroscopique encodé par l'observable μ_N. En effet :

— $\{\mu_N\}_N$ converge vers une limite déterministe μ_{eq} qui est l'unique minimiseur de la fonctionnelle I, et l'on connaît un principe de grandes déviations autour de μ_{eq} à vitesse N^2.
— Il n'y a pas universalité à l'échelle macroscopique, car μ_{eq} et Σ dépendent fortement du potentiel V (et de l'interaction g). La dépendance de Σ par rapport à V, par exemple, est "compliquée" et non régulière en général.
— La température (si on ne la fait pas dépendre de N) ne joue aucun rôle à l'échelle macroscopique, puisque les minimiseurs de I et βI sont évidemment les mêmes pour tout $\beta > 0$.

b. Le *one-dimensional log-gas*

Le cas $d = 1$, $g(x - y) = -\log|x - y|$ a été particulièrement étudié, notamment en raison de son lien avec les matrices aléatoires. On résume ici quelques résultats en relation avec notre propos.

Processus ponctuel limite. L'observable microscopique est ici le processus ponctuel à l'échelle N^{-1}. Plus précisément, prenons le cas d'un potentiel quadratique tel que la mesure d'équilibre soit la loi du demi-cercle de Wigner. Fixons $x \in]-2, 2[$ (l'intérieur du support de la mesure d'équilibre) et considérons l'observable

$$\mathcal{C}_N^x : \vec{X}_N \mapsto \sum_{i=1}^{N} \delta_{N(x_i - x)} \ , \tag{1.1.6}$$

à valeurs dans $\mathcal{X}$, l'espace des configurations de points. L'existence d'un comportement limite (en loi) pour ces configurations de points était connue pour les trois valeurs "classiques" de β correspondant aux modèles classiques de matrices aléatoires (GOE, GUE, GSE), grâce à la structure particulière de la loi des valeurs propres dans ces cas. Dans l'article fondamental [VV09], Valko et Virag ont montré l'existence d'une limite pour la loi de $\mathcal{C}_N^x$ pour tout $\beta > 0$, dans le cas d'un potentiel quadratique. Cette limite, le "*Sine-beta process*", est une mesure de probabilité sur $\mathcal{X}$ (un processus ponctuel aléatoire) notée Sine_β et ne dépend de x que par un changement d'échelle d'un facteur $\sqrt{4 - x^2}$, qui est la densité de la loi du demi-cercle au point x.

Universalité. Dans les travaux [BEY14, BEY12] (voir aussi le résumé [Bou13]), Bourgade, Erdös et Yau ont étudié l'universalité du comportement microscopique des β-ensemble par rapport au potentiel V. Le résultat peut s'énoncer informellement ainsi : si V est analytique, de dérivée seconde minorée, et vérifie certaines conditions génériques, alors le comportement microscopique du *log-gas* $1D$ avec potentiel V et celles du *log-gas* $1D$ avec potentiel quadratique sont asymptotiquement les mêmes (à une mise à l'échelle près). En particulier, en ce sens, le processus Sine_β est *universel*.

Fluctuations. Les fluctuations des valeurs propres autour de la mesure d'équilibre μ_{eq} sont bien comprises dans le cas uni-dimensionnel.

1. Le cas de la plus grande (ou des plus grandes) valeurs propres a fait l'objet de beaucoup d'attention. Quand μ_{eq} est la loi du demi-cercle, on s'attend à ce que la plus grande valeur propre soit ≈ 2. Il y a effectivement convergence presque sûre vers 2, et les grandes déviations autour de cette valeur sont connues (voir par exemple [BADG01]), ainsi que les fluctuations à l'échelle $N^{-2/3}$ qui sont données par la loi de Tracy-Widom définie dans [TW94, TW96] et étendue à tout β dans [RRV11]. Plus généralement, le processus ponctuel des plus grandes valeurs propres, c'est à dire la loi de $\mathcal{C}_N^x$ pour $x = 2$ dans (1.1.6) admet une limite qui est le processus d'Airy (généralisé à tout β dans [RRV11, KRV15]).
2. Les statistiques linéaires, c'est à dire la loi de $\sum_{i=1}^N f(x_i) - N \int f(x) d\mu_{\mathrm{eq}}(x)$ pour une fonction test assez régulière f, obéissent à un théorème central limite d'un genre particulier, puisque contrairement au cas de variables aléatoires indépendantes, il n'y a pas de normalisation en $1/\sqrt{N}$ (la variance des fluctuations reste bornée quand N tend vers l'infini, du fait des interactions entre valeurs propres). Un tel théorème n'est valide que dans le cas où le support de la mesure d'équilibre est connexe (cas *one-cut*), sinon (cas *multi-cut*) il peut y avoir un comportement quasi-périodique des fluctuations. Un premier résultat de ce type a été établi dans [Joh98], voir aussi [Shc13, Shc14, BG13b, BG13a] pour des généralisations, qui vont de pair avec un développement asymptotique de $\log Z_{N,\beta}$.

Mentionnons aussi une autre direction possible dans l'étude des systèmes à une dimension, qui consiste à remplacer l'interaction logarithmique par des interactions plus générales qui possèdent le même type de singularité en 0, et donc à étudier l'universalité du comportement microscopique en fonction de g, voir [GV14, Ven13]

Les principales techniques utilisées dans les travaux mentionnés ci-dessus sont assez particulières au cas de la dimension 1 : utilisation de la représentation matricielle des β-ensembles (qui n'existe pas en général pour le cas complexe, par exemple), polynômes orthogonaux, mouvement Brownien de Dyson (*Dyson Brownian motion*, voir par exemple [Tao12, Section 3.1]), et équations de boucles (qui peuvent s'écrire en toute dimension mais sont particulièrement maniables en dimension 1).

c. Le gaz de Coulomb bidimensionnel

Le gaz de Coulomb à deux dimensions a surtout été étudié mathématiquement pour la valeur particulière de la température à laquelle il coïncide avec la loi des valeurs propres de matrices aléatoires non Hermitiennes.

L'ensemble de Ginibre. Considérons encore l'observable "processus ponctuel microscopique" définie par

$$\mathcal{C}_N^x : \vec{X}_N \mapsto \sum_{i=1}^{N} \delta_{N^{1/2}(x_i - x)} \ ,$$

pour $x \in \mathbb{R}^2$ et rappelons que la mesure d'équilibre associée à l'ensemble de Ginibre est la mesure uniforme sur le disque unité. Dans ce contexte, les processus ponctuels $\mathcal{C}_N^x$ ont une limite en loi :

1. Si $|x| > 1$, cette limite est triviale (il n'y a aucune particule, on voit presque sûrement la configuration vide).
2. Si $|x| = 1$, cette limite est un processus déterminantal de noyau explicite que l'on peut trouver par exemple dans l'annexe de [BS09]. Elle ne dépend pas de la position de x sur le cercle unité.
3. Si $|x| < 1$, cette limite est un processus déterminantal (le processus ponctuel de Ginibre) de noyau explicite qui est déjà mentionné dans l'article fondateur [Gin65]. Elle ne dépend pas de x dans l'intérieur du disque.

La structure déterminantale de ces processus ponctuels permet de calculer explicitement de nombreuses quantités. On renvoie à [HKPV09] et notamment [HKPV09, Section 4.3.7].

En particulier, le comportement des fluctuations est bien compris : l'analogue des résultats $1d$ est le suivant : le plus grand module des valeurs propres suit une loi de Gumbel autour de la valeur 1 [Kos92, Rid03] (pour un résultat d'universalité dans cette direction, voir [CP14]) et les statistiques linéaires obéissent à un théorème central limite [RV07].

Le "Random Normal Matrix model". Comme mentionné dans la Section 1.1.2, il s'agit d'une généralisation naturelle de l'ensemble de Ginibre en remplaçant le potentiel quadratique par un potentiel confinant V supposé analytique. Le processus ponctuel des valeurs propres est encore déterminantal, mais son noyau est plus compliqué que dans le cas V quadratique. Néanmoins, dans [AHM11, AHM15], Ameur, Hedenmalm et Makarov établissent un résultat d'universalité : si x est dans l'intérieur de Σ alors $\mathcal{C}_N^x$ converge en loi (après un scaling dépendant de la mesure d'équilibre) vers la loi du processus ponctuel de Ginibre. Ils montrent de plus un théorème central limite analogue à [RV07]. Les méthodes utilisées sont très dépendantes de la valeur particulière de β et du caractère analytique de V.

Quelques résultats physiques. On mentionne ici quelques pistes de réfléxion (mathématique) inspirées par la littérature (physique) autour du 2DOCP.

Fluctuations de charges. La question de savoir à quelle vitesse et à quelle précision les charges ponctuelles se répartissent dans le support Σ de manière à ressembler à μ_{eq} est bien sûr fondamentale. Dans [JLM93], Jancovici, Lebowitz et Manificat étudient la probabilité d'avoir une non-neutralité locale, c'est à dire que le nombre de points N_Λ dans un domaine $\Lambda \subset \Sigma$ diffère de $\mu_{\text{eq}}(\Lambda)$, et distinguent trois régimes selon la valeur

de $\frac{N_\Lambda - N\mu_{\text{eq}}(\Lambda)}{|\Lambda|}$. On verra par la suite qu'on peut obtenir rigoureusement des estimées dans le même esprit, mais moins précises, et la question de justifier mathématiquement les résultats de [JLM93] reste, à notre connaissance, ouverte. Il est notable que cette question ait été résolue dans [NSV08] pour le cadre légérèment différent des zéros de fonctions analytiques aléatoires, dont l'étude présente des similarités avec les valeurs propres de matrices aléatoires : les auteurs trouvent, dans ce contexte, exactement les trois régimes de déviation prédits par [JLM93] dans le contexte du 2DOCP.

Cristallisation. Une question fondamentale dans l'étude du gaz de Coulomb à deux dimensions est celle de son comportement à basse température (formellement, quand $\beta = +\infty$). La mesure $\mathbb{P}_{N,\beta}$ doit alors se concentrer sur les minimiseurs de $\mathcal{H}_N$, et on s'attend à ce que ces minimiseurs aient une structure ordonnée. La conjecture généralement admise est la suivante (formulée ici informellement, voir par exemple l'article de survol [BL15] pour des précisions et des références)

Conjecture 1. *Quand $N \to \infty$ les minimiseurs de $\mathcal{H}_N$ prennent la forme d'un réseau triangulaire, appelé aussi "réseau d'Abrikosov".*

La littérature physique considère ce problème comme résolu (voir par exemple [AJ81]) et pousse la question plus loin : quel est le comportement du système quand β est très grand, mais fini ? Certains résultats affirment que le système "cristallise" au-delà d'un certain β critique dont la valeur est estimée autour de 140 [BST66,PH73,GCC79,dLP82], mais le sens à donner à cette "cristallisation" est ambigu (il ne peut, pour des raisons évidentes d'entropie, s'agir d'un cristal parfait). Voir aussi [Sti98] pour un avis critique sur ces résultats. Le point crucial pour mieux comprendre ces questions serait d'avoir accès aux fonctions de corrélations à deux points du système pour $\beta \to \infty$, ou du moins à leur vitesse de décroissance quand $|x-y| \to \infty$. Il se pourrait que pour β assez grand, ces fonctions possèdent (au moins sur des distances assez grandes) un comportement périodique assimilable à une "cristallisation" du système, sans pour autant que les particules ne forment un cristal parfait.

1.1.6 Le comportement microscopique

D'après la section précédente, on sait que les N particules vont (typiquement) se concentrer dans un compact Σ de $\mathbb{R}^d$ et se répartir selon une mesure μ_{eq} *qui ne dépend pas de la température.* Mais pour ce faire, les particules peuvent s'arranger de multiples façons à l'échelle microscopique $N^{-1/d}$. Les simulations numériques (ou l'étude théorique dans les cas où l'on dispose de formules explicites, par exemple pour les modèles classiques de matrices aléatoires) montrent que cette répartition microscopique dépend, elle, de la température. Grossièrement, plus la température est haute (β petit), plus le système est désordonné à l'échelle $N^{-1/d}$, voir par exemple les figures 1.1.6 et 1.1.6 à la page suivante. Le principal but de cette thèse est de comprendre quels phénomènes régissent le comportement microscopique en fonction de la température. On mettra en évidence le rôle d'une fonctionnelle d'énergie libre $\mathcal{F}_\beta := \beta\mathbb{W}^{\text{elec}} + \mathsf{ent}$ qui gouverne la loi des configurations de points et fait apparaître leur dépendance en β. Ici $\mathbb{W}^{\text{elec}}$ est un terme d'énergie qui correspond à la limite de l'énergie d'interaction $\mathcal{H}_N$ et ent est un terme d'entropie qui correspond à la limite (en un certain sens) des éléments de volume $d\vec{X}_N$.

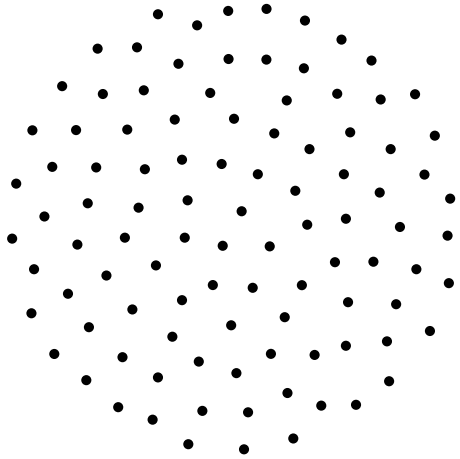

FIGURE 1.1 – Un 2DOCP avec $N = 100$ particules, sous potentiel quadratique, à basse température $\beta = 400$

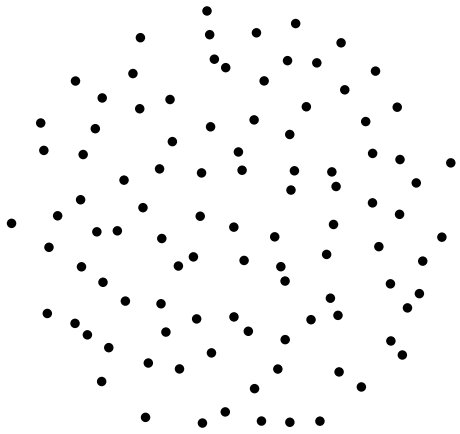

FIGURE 1.2 – Un 2DOCP avec $N = 100$ particules, sous potentiel quadratique, à haute température $\beta = 5$

1.2 Méthodes utilisées

1.2.1 Quelques notations et définitions

a. Généralités

Si X est un espace métrique, on le munit de sa tribu Borélienne et on note $\mathcal{P}(X)$ l'espace des mesures de probabilité sur X. On note $\mathrm{Lip}_1(X)$ l'ensemble des fonctions F qui sont 1-Lipschitz sur X et telles que $||F||_\infty \leq 1$. On définit une distance sur $\mathcal{P}(X)$ par

$$d_{\mathcal{P}(X)}(P_1, P_2) := \sup\left\{\int_X F(dP_1 - dP_2) \mid F \in \mathrm{Lip}_1(X)\right\}.$$

Il est bien connu que $d_{\mathcal{P}(X)}$ métrise la convergence faible sur $\mathcal{P}(X)$. Si $P \in \mathcal{P}(X)$ et $f : X \to \mathbb{R}^m$ est mesurable sur X on note $\mathbf{E}_P[f]$ l'espérance de f sous P. Si $A \subset X$, on note $\bar{A}$ l'adhérence de A et $\mathring{A}$ son intérieur.

On note **Leb** la mesure de Lebesgue dans $\mathbb{R}^d$. Si $A \subset \mathbb{R}^d$ est mesurable on note $|A|$ sa masse sous **Leb**. On dénote la diagonale d'un produit cartésien $A \times A$ par $\triangle$.

Pour $R > 0$, on note C_R l'hypercube $[-R/2, R/2]^d$. Pour tout $m > 0$ on note $\mathbf{\Pi}^m$ la loi d'un processus de Poisson ponctuel d'intensité m dans $\mathbb{R}^d$.

On note a_+ la partie positive d'un nombre réel i.e. $a_+ = \max(0, a)$.

b. Processus ponctuels (aléatoires)

On note $\mathcal{X}$ l'espace des configurations de points dans $\mathbb{R}^d$. Formellement, c'est l'espace des mesures de Radon purement atomiques qui donnent une masse entière aux singletons (cf. [DVJ88]). Muni de la topologie de la convergence vague des mesures de Radon, c'est un espace Polonais (et on note $d_{\mathcal{X}}$ une distance associée). On notera généralement $\mathcal{C}$ une configuration de points et, avec un léger abus de notation, on confondra souvent l'ensemble $\mathcal{C}$ des points avec la mesure de Radon $\sum_{p\in\mathcal{C}} \delta_p$. On définit un processus ponctuel aléatoire (*random point process*) comme la loi d'une configuration de points, donc un élément de $\mathcal{P}(\mathcal{X})$. On dit que $P \in \mathcal{P}(\mathcal{X})$ est *stationnaire* (et on note $P \in \mathcal{P}_s(\mathcal{X})$) s'il est invariant par l'action de $\mathbb{R}^d$ sur $\mathcal{X}$ par translations. Si P est stationnaire, le rapport du nombre moyen de points dans un domaine Λ non vide et du volume de Λ est constant et on appelle "intensité" de P cette quantité.

Pour $N \geq 1$, soit $\sim_N$ la relation d'équivalence sur $(\mathbb{R}^d)^N$ définie par $(x_1, \ldots, x_N) \sim_N (y_1, \ldots, y_N)$ si et seulement si il existe une permutation $\sigma \in \mathfrak{S}_N$ (le groupe symétrique à N éléments) telle que $x_i = y_{\sigma(i)}$ pour $i = 1, \ldots, N$. On dénote par $(\mathbb{R}^d)^N/\mathfrak{S}_N$ l'ensemble quotient et par π_N la projection canonique $(\mathbb{R}^d)^N \to (\mathbb{R}^d)^N/\mathfrak{S}_N$. L'ensemble des configurations finies dans $\mathcal{X}$ peut être identifié à $\{\emptyset\} \cup \bigcup_{N=1}^{+\infty} (\mathbb{R}^d)^N/\mathfrak{S}_N$.

Si $\mathcal{A} \subset B^N/\mathfrak{S}_N$ on définit $\hat{\mathcal{A}} \subset B^N$ comme $\hat{\mathcal{A}} := \bigcup_{\mathcal{C}\in\mathcal{A}} \mathcal{C}$. Il est facile de vérifier que $\hat{\mathcal{A}}$ est le plus grand sous-ensemble de $(\mathbb{R}^d)^N$ telle que l'image directe de $\hat{\mathcal{A}}$ par π_N est $\mathcal{A}$. On appelle "volume de $\mathcal{A}$" et on écrit (avec un léger abus de notation) $\mathbf{Leb}^{\otimes N}(\mathcal{A})$ la quantité $\mathbf{Leb}^{\otimes N}(\hat{\mathcal{A}})$.

1.2.2 Observables considérées

Il s'agit ici de discuter la notion de "comportement microscopique". Pour caractériser le comportement des particules (ou des valeurs propres) à l'échelle $N^{-1/d}$, plusieurs quantités peuvent être considérées.

1. Le processus ponctuel (non-moyenné) obtenu en zoomant autour d'un point fixé. Cela revient à fixer $x \in \mathring{\Sigma}$ et à considérer l'observable

$$\mathcal{C}_N^x : \vec{X}_N \mapsto \sum_{i=1}^N \delta_{N^{1/d}(x_i - x)} \ , \tag{1.2.1}$$

à valeurs dans $\mathcal{X}$. Cette observable encode la configuration de points vue à l'échelle $N^{-1/d}$ autour du point x.

2. Le *champ empirique* obtenu en moyennant $\mathcal{C}_N^x$ sur le voisinage d'un certain point. De manière équivalente, cela revient à choisir le point de centrage aléatoirement sur un petit voisinage de x. Formellement, on fixe $x \in \mathring{\Sigma}$ et un voisinage Ω de x, et on définit

$$\mathcal{C}_{N,\Omega}^x := \frac{1}{|\Omega|} \int_\Omega \delta_{\mathcal{C}_N^z} dz \tag{1.2.2}$$

à valeurs dans $\mathcal{P}(\mathcal{X})$.
— Si Ω est de taille ε indépendante de N, on dit que la moyenne est *macroscopique*.
— Si Ω est de taille $\varepsilon = N^{-\frac{1}{d}+\delta}$ pour $\delta \in (0, \frac{1}{d})$ on dit que la moyenne est *mésoscopique*.

Nos méthodes ne permettent pas de caractériser la limite des processus non-moyennés si son existence n'est pas connue *a priori*, mais on peut en revanche essayer d'établir des principes de grandes déviations pour les champs empiriques. Commençons par donner la définition d'un principe de grandes déviations.

1.2.3 Grandes déviations

a. Principe de grandes déviations

Définition 1.2.1. *Soit $\mathcal{S}$ un espace métrique, et $I : \mathcal{S} \to [0, +\infty]$ une fonctionnelle semi-continue inférieurement. Soit $\{r_N\}_N$ une suite de réels positifs et $\{\mu_N\}_N$ une suite de mesures de probabilité sur $\mathcal{S}$. On dit que $\{\mu_N\}_N$ obéit à un principe de grandes déviations (PGD) à vitesse r_N avec fonction de taux I si pour tout $A \subset S$ on a*

$$-\inf_{\mathring{A}} I \leq \liminf_{N\to\infty} \frac{1}{r_N} \log \mu_N(A) \leq \limsup_{N\to\infty} \frac{1}{r_N} \log \mu_N(A) \leq -\inf_{\bar{A}} I, \tag{1.2.3}$$

où $\mathring{A}$ (resp. $\bar{A}$) désigne l'intérieur (resp. l'adhérence) de A dans $\mathcal{S}$. On dit que la fonction de taux I est "bonne" si ses ensembles de sous-niveaux sont compacts.

On peut penser à la chaine d'inégalités (1.2.3) comme

$$\mu_N(A) \approx \exp(-r_N \inf_A I).$$

Une référence pour les questions de grandes déviations est [DZ10], on peut aussi citer [RAS09] pour une introduction tournée vers les applications à la physique statistique.

Si la loi d'une suite de variables aléatoires $\{X_N\}_N$ obéit à un principe de grandes déviations avec fonction de taux I, on voit que X_N va se concentrer, avec grande probabilité, autour des minimiseurs de la fonction de taux I. En particulier, en utilisant le lemme de Borel-Cantelli, il n'est pas difficile de voir que si I admet un unique minimiseur alors c'est la limite presque sûre des X_N. Dans le cas général, si I est une bonne fonction de taux, l'ensemble des ses minimiseurs est compact, et la distance de X_N à ce compact tend vers 0 presque sûrement. C'est en celà qu'un principe de grandes déviations "gouverne" le comportement asymptotique d'une suite de variables aléatoires : le comportement typique est obtenu en cherchant les minimiseurs de la fonctionnelle de taux, et la probabilité d'être à distance fixée des minimiseurs décroît exponentiellement avec une vitesse quantifiable.

b. Entropie relative et théorème de Sanov

Soit $\mathcal{S}$ un espace métrique (par exemple Polonais), et μ, ν deux mesures de probabilités (Boréliennes) sur $\mathcal{S}$. L'entropie relative (ou divergence de Kullback-Leibler) de μ par rapport à ν est définie par

$$\mathrm{Ent}[\mu|\nu] := \int_X \log \frac{d\mu}{d\nu} d\mu,$$

si μ est absolument continue par rapport à ν (on note alors $\frac{d\mu}{d\nu}$ la dérivée de Radon-Nikodym de μ par rapport à ν) et $+\infty$ sinon. Il n'est pas difficile de vérifier que l'entropie relative est toujours positive, et semi-continue inférieurement en la variable μ, de plus on a $\mathrm{Ent}[\mu|\nu] = 0$ si et seulement si $\mu = \nu$.

Donnons maintenant une interprétation probabiliste (informelle) de $\mathrm{Ent}[\mu|\nu]$. On suppose qu'on simule N fois la loi ν et qu'on obtient les résultats $x_1, \ldots, x_N$. Quand N est grand, l'histogramme des valeurs obtenues doit ressembler à ν elle-même : c'est la loi des grands nombres. Quelle est la probabilité que cet échantillon ressemble plutôt à la distribution de μ ? La réponse décroît exponentiellement vite avec N, et en fait est (au premier ordre) comme $\exp(-N\ \mathrm{Ent}[\mu|\nu])$. Une formulation rigoureuse est donnée par le théorème de Sanov (voir [San61], on cite ici [RAS09, Section 5.2]).

Théorème 3 (Sanov)**.** *Soit $\mathcal{S}$ un espace polonais et ν une mesure de probabilité sur X. Soit $\{X_k\}_{k\geq 1}$ des variables aléatoires i.i.d sur $\mathcal{S}$ de loi ν. On note L_N la mesure empirique des N premières variables*

$$L_N := \frac{1}{N} \sum_{k=1}^{N} \delta_{x_k},$$

qui est une mesure de probabilité aléatoire sur $\mathcal{S}$. La loi de $\{L_N\}_N$ obéit un principe de grandes déviations à la vitesse N avec pour bonne fonction de taux $\mu \mapsto \mathrm{Ent}[\mu|\nu]$.

D'après la loi des grands nombres, si f est continue bornée sur $\mathcal{S}$, on a $\int f dL_N \to \int f d\mu$ presque sûrement quand $N \to \infty$. De plus, $\{L_N\}_N$ converge presque sûrement vers μ au sens de la convergence faible des mesures de probabilités. Le théorème de Sanov précise les grandes déviations autour de ce comportement typique, qui sont gouvernées par l'entropie relative.

c. Comment établir un principe de grandes déviations ?

Ce paragraphe n'est pas rigoureux mais sert à éclairer et illustrer la démarche qu'on mettra en œuvre à plusieurs reprises dans cette thèse afin d'établir un principe de grandes déviations.

De manière générale, on voit que pour montrer (1.2.3) il faut établir deux inégalités :

- Une borne supérieure, qui consiste à montrer que les événements "déviants" n'arrivent pas avec trop grande probabilité.
- Une borne inférieure, qui consiste à montrer qu'un événement, même non typique, arrive quand même - parfois - avec une probabilité quantifiable.

Plaçons-nous dans le cas d'un système de particules en interaction, supposons qu'on dispose d'une fonctionnelle d'énergie $\mathcal{H}_N$ définie sur $(\mathbb{R}^d)^N$ et qu'on considère la mesure de probabilité de type Gibbs donnée par $dP_N(\vec{X}_N) := \frac{1}{Z_N} \exp(-\mathcal{H}_N) d\vec{X}_N$ (on oublie ici le rôle de la température), où Z_N est la constante de normalisation (la fonction de partition). Soit $\mathcal{O}_N$ une observable à valeurs dans $\mathcal{S}$, et r_N une vitesse, on se demande si la loi de $\mathcal{O}_N$ obéit à un principe de grandes déviations à vitesse r_N.

Soit $A \subset \mathcal{S}$ mesurable. On peut écrire, par définition, que

$$P_N(\mathcal{O}_N \in A) = \frac{1}{Z_N} \int_{\mathcal{O}_N \in A} \exp(-\mathcal{H}_N) d\vec{X}_N.$$

Cherchons d'abord à ré-exprimer l'énergie en fonction de l'observable considérée, c'est-à-dire d'avoir $\mathcal{H}_N(\vec{X}_N) = \mathcal{H}'_N(\mathcal{O}_N(\vec{X}_N))$ pour une certaine fonctionnelle $\mathcal{H}'_N$ définie sur $\mathcal{S}$. On peut au moins espérer montrer une minoration du type $\mathcal{H}_N(\vec{X}_N) \geq \mathcal{H}'_N(\mathcal{O}_N(\vec{X}_N))$. Cela donne

$$P_N(\mathcal{O}_N \in A) \leq \frac{1}{Z_N} \int_{\mathcal{O}_N(\vec{X}_N)\in A} \exp(-\mathcal{H}'_N(\mathcal{O}_N(\vec{X}_N)))d\vec{X}_N.$$

Il faut ensuite déterminer une fonctionnelle $\mathcal{H}'$ définie sur $\mathcal{S}$ qui soit la limite de $\frac{1}{r_N}\mathcal{H}'_N$. Une bonne notion de limite serait ici celle de Γ-convergence (au sens de DeGiorgi, voir par exemple [Bra02, Mas93]). Demandons simplement que la fonctionnelle d'énergie limite $\mathcal{H}'$ vérifie

$$\liminf_{N\to\infty} \frac{1}{r_N}\mathcal{H}'_N(\mathcal{O}_N(\vec{X}_N)) \geq \mathcal{H}'(\mathcal{O}) \text{ si } \lim_{N\to\infty} \mathcal{O}_N(\vec{X}_N) = \mathcal{O} \text{ dans } \mathcal{S}, \tag{1.2.4}$$

ce qui correspond à une Γ-lim inf dans le langage de la Γ-convergence. On en déduit que

$$\liminf_{N\to\infty} \inf_{\mathcal{O}_N(\vec{X}_N)\in A} \left(\frac{1}{r_N}\mathcal{H}'_N(\mathcal{O}_N(\vec{X}_N))\right) \geq \inf_{\bar{A}} \mathcal{H}'$$

si bien que l'on peut majorer $P_N(\mathcal{O}_N \in A)$ par

$$\frac{1}{Z_N} \int_{\mathcal{O}_N(\vec{X}_N)\in A} \exp\left(-r_N(\inf_{\bar{A}} \mathcal{H}' + o(1))\right) d\vec{X}_N.$$

En passant au logarithme et en divisant par la vitesse r_N, il vient

$$\frac{1}{r_N}\log P_N(\mathcal{O}_N \in A) \leq -\frac{1}{r_N}\log Z_N - \inf_{\bar{A}} \mathcal{H}' + \frac{1}{r_N}\log \int_{\mathcal{O}_N(\vec{X}_N)\in A} d\vec{X}_N + o(1).$$

Intéressons-nous maintenant au terme de "volume" $\frac{1}{r_N}\log \int_{\mathcal{O}_N(\vec{X}_N)\in A} d\vec{X}_N$ dans le membre de droite, qui ne dépend plus du tout de la nature des interactions. L'intégrale dont on prend le logarithme représente la masse de l'événement $\{\mathcal{O}_N(\vec{X}_N) \in A\}$ pour la mesure de Lebesgue (bien sûr cette masse pourrait être infinie, et en général il faut se restreindre *a priori* par d'autres arguments à un domaine borné de l'espace). On espère pouvoir montrer un principe de grandes déviations pour la loi de l'observable $\mathcal{O}_N$ quand les particules $\vec{X}_N$ sont distribuées par rapport à cette mesure de référence "sans interaction" (par opposition à P_N). C'est le cas par exemple si $\mathcal{O}_N(\vec{X}_N)$ est la mesure empirique des particules comme dans le Théorème 3. Admettons qu'il existe une telle fonctionnelle de "volume" (ou d'entropie) Vol définie sur $\mathcal{S}$, vérifiant

$$\frac{1}{r_N}\log \int_{\mathcal{O}_N(\vec{X}_N)\in A} d\vec{X}_N \approx -\inf_{A} \text{Vol}.$$

Notre borne supérieure de PGD s'écrit maintenant

$$\frac{1}{r_N}\log P_N(\mathcal{O}_N \in A) \leq -\frac{1}{r_N}\log Z_N - \inf_{\bar{A}}(\mathcal{H}' + \text{Vol}) + o(1).$$

Pour montrer la borne inférieure correspondante, il faudrait générer une famille assez grande de configurations $\vec{X}_N$ dans A telles que

$$\frac{1}{r_N}\mathcal{H}_N(\vec{X}_N) \leq \inf_{\mathring{A}} \mathcal{H}' + o(1). \tag{1.2.5}$$

Par "famille assez grande", on entend un sous-ensemble $B \subset A$ tel que le "volume" de B au sens précédent est du même ordre que celui de A à l'échelle logarithmique considérée, i.e. que

$$\frac{1}{r_N} \log \int_{\mathcal{O}_N(\vec{X}_N)\in B} d\vec{X}_N = \frac{1}{r_N} \log \int_{\mathcal{O}_N(\vec{X}_N)\in A} d\vec{X}_N + o(1).$$

Notons que cela fonctionne même si B est "beaucoup" plus petit que A, par exemple si seulement une configuration de A sur un million vérifie (1.2.5). On peut alors minorer $P_N(\mathcal{O}_N \in A)$ et en déduire une borne inférieure de PGD. Cette démarche évoque la construction d'une *recovery sequence* (ou Γ-lim sup) dans le contexte de la Γ-convergence, à la différence près qu'ici on veut construire toute une famille (de volume prescrit) de configurations dont on contrôle l'énergie en termes de $\mathcal{H}'$.

Le terme associé à la fonction de partition est ensuite facile à traiter, on montre qu'il vaut asymptotiquement $\min(\mathcal{H}' + \mathrm{Vol})$. La fonction de taux du PGD est donc $(\mathcal{H}' + \mathrm{Vol}) - \min(\mathcal{H}' + \mathrm{Vol})$.

En résumé, la tâche est triple :

1. Établir un principe de grandes déviations pour la loi de $\mathcal{O}_N$ sous la mesure de référence sans interactions (obtenir un terme de volume).
2. Dériver un Hamiltonien "effectif" $\mathcal{H}'_N$ qui s'exprime simplement en fonction de l'observable $\mathcal{O}_N$ puis trouver une limite $\mathcal{H}'$ vérifiant (1.2.4) (minoration de l'énergie).
3. Trouver des configurations en nombre suffisant dans A telles que $\frac{1}{r_N}\mathcal{H}_N(\vec{X}_N) \leq \inf_A \mathcal{H}' + o(1)$ (majoration de l'énergie).

Dans cette thèse, la situation sera souvent la suivante :

— Le principe de grandes déviations sous la mesure de référence "sans interaction" existe et fait apparaître un analogue de l'entropie relative.
— La borne supérieure du PGD, c'est à dire la minoration de l'énergie à N fixé par une énergie limite, se fait par des arguments d'analyse fonctionnelle.
— La borne inférieure du PGD, c'est à dire la construction de configurations convenables en nombre suffisant, se fait en combinant un argument abstrait de grandes déviations (pour disposer d'un matériau de base, c'est à dire un volume suffisant de configurations presque convenables) et des techniques *ad hoc* pour modifier un peu ces configurations afin de contrôler précisément leur énergie sans trop perdre de volume.

d. Grandes déviations pour la mesure de référence

Dans ce paragraphe, on s'intéresse à la première des tâches mentionnées plus haut : comment obtenir un terme de volume pour les observables qui nous intéressent, sous une mesure de référence "uniforme" ? Pour cela, on va utiliser la notion *d'entropie relative spécifique*.

Définition 1.2.2 (Entropie relative spécifique)**.** *Soit P un processus ponctuel stationnaire et $m > 0$. On appelle "entropie relative spécifique de P par rapport au processus de Poisson d'intensité m" et on note $\mathsf{ent}[P|\mathbf{\Pi}^m]$ la quantité*

$$\mathsf{ent}[P|\mathbf{\Pi}^m] := \lim_{R\to\infty} \frac{1}{R^d} \mathrm{Ent}[P_{|R}|\mathbf{\Pi}^m_{|R}], \tag{1.2.6}$$

où $P_{|R}$ désigne la loi du processus ponctuel induit dans l'hypercube $C_R = [-R/2, R/2]^d$.

Non seulement la limite (1.2.6) existe mais on a aussi, par sur-additivité,

$$\mathsf{ent}[P|\mathbf{\Pi}^m] = \sup_{R>0} \frac{1}{R^d} \mathrm{Ent}[P_{|R}|\mathbf{\Pi}^m_{|R}].$$

La fonctionnelle $\mathsf{ent}[\cdot|\mathbf{\Pi}^m]$ est positive, affine, semi-continue inférieurement sur $\mathcal{P}_s(\mathcal{X})$ et on a $\mathsf{ent}[P|\mathbf{\Pi}^m] = 0$ si et seulement si $P = \mathbf{\Pi}^m$. Pour une preuve de ces faits, on renvoie à [RAS09].

L'entropie relative spécifique peut se comprendre comme l'analogue *en volume infini* de l'entropie relative "classique" définie plus haut. Elle apparaît comme fonction de taux pour les grandes déviations des champs empiriques. Le résultat suivant est démontré dans [Föl88, GZ93]

Proposition 1.2.3 (Grandes déviations pour les champs empiriques)**.** *Soit $\{\Lambda_N\}_N$ une suite exhaustive croissante d'hypercubes dans $\mathbb{R}^d$. Pour tout x, on note $\theta_x \cdot \mathcal{C} := \sum_{p\in\mathcal{C}} \delta_{p-x}$ l'application de translation (ou de recentrage) d'une configuration de points par $\mathcal{C}$ par x. Le poussé-en-avant de $\mathbf{\Pi}^m$ par l'application "champ empirique"*

$$\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x\cdot\mathcal{C}} dx$$

(à comparer avec (1.2.2)) obéit à un principe de grandes déviations à vitesse $|\Lambda_N|$ avec pour bonne fonction de taux $\mathsf{ent}[\cdot|\mathbf{\Pi}^m]$.

1.2.4 Énergie renormalisée

Dans cette section, on répond à la deuxième tâche mentionnée plus haut : ré-écrire la fonctionnelle d'énergie en termes de la configuration de points microscopique, et lui trouver une limite quand $N \to \infty$. On suit pour cela les travaux [SS12, SS15b] (pour l'interaction logarithmique en dimension 2) étendus dans [SS15a] (interaction logarithmique en dimension 1), [RS15] (pour l'interaction Coulombienne en dimension $d \geq 3$) et [PS15] (qui traite le cas Riesz général).

a. Décomposition de l'énergie

Soit $\vec{X}_N$ dans $(\mathbb{R}^d)^N$, et ν_N la configuration de points associée vue à l'échelle macroscopique $\nu_N := \sum_{i=1}^N \delta_{x_i}$. On peut ré-écrire la fonctionnelle d'énergie $\mathcal{H}_N$ comme

$$\mathcal{H}_N(\vec{X}_N) = \iint_{\triangle^c} g(x-y)d\nu_N(x)d\nu_N(y) + N\int V(x)d\nu_N.$$

D'après les résultats à l'ordre macroscopique on sait que, avec grande probabilité, ν_N est proche de $N\mu_{\mathrm{eq}}$ (à $o(N)$ près), décomposons donc ν_N comme $\nu_N = N\mu_{\mathrm{eq}} + (\nu_N - N\mu_{\mathrm{eq}})$. On obtient (on rappelle que $\triangle$ désigne la diagonale)

$$\begin{aligned}\mathcal{H}_N(\vec{X}_N) = N^2 \iint_{\triangle^c} g(x-y)d\mu_{\mathrm{eq}}d\mu_{\mathrm{eq}} + N^2\int V d\mu_{\mathrm{eq}} \\ + \iint_{\triangle^c} g(x-y)(d\nu_N - Nd\mu_{\mathrm{eq}})(x)(d\nu_N - Nd\mu_{\mathrm{eq}})(y) \\ + 2N\iint g(x-y)(d\nu_N - Nd\mu_{\mathrm{eq}})(x)d\mu_{\mathrm{eq}}(y) + N\int V(d\nu_N - Nd\mu_{\mathrm{eq}}).\end{aligned}$$

On peut ré-écrire la première ligne du membre de droite comme $N^2 I(\mu_{\mathrm{eq}})$ (avec I définie par (1.1.5)). La troisième ligne du membre de droite admet une ré-écriture maniable, en effet les équations d'Euler-Lagrange associées à la minimisation de I garantissent que la fonction $x \mapsto h_{\mu_{\mathrm{eq}}}(x) + \frac{1}{2}V(x)$, avec $h_{\mu_{\mathrm{eq}}}(x) := \int g(x-y)d\mu_{\mathrm{eq}}(y)$, est égale à une certaine constante c presque partout (en fait, *quasiment partout* au sens de la théorie du potentiel) sur le support Σ de la mesure d'équilibre. De plus, en posant

$$\zeta(x) = h_{\mu_{\mathrm{eq}}}(x) + \frac{1}{2}V(x) - c$$

on a $\zeta(x) \geq 0$ presque partout sur $\mathbb{R}^d$. On en déduit que

$$2N \iint g(x-y)(d\nu_N - Nd\mu_{\rm eq})(x)d\mu_{\rm eq}(y) + N\int V(d\nu_N - Nd\mu_{\rm eq}) = 2N\int \zeta(x)d\nu_N.$$

Cette quantité est positive et joue le rôle d'un potentiel confinant au second ordre, puisqu'elle a pour effet de contenir les particules dans l'ensemble $\{\zeta = 0\}$ (qui contient Σ mais peut être strictement plus grand). Finalement, on peut écrire

$$\mathcal{H}_N(\vec{X}_N) = N^2 I(\mu_{\rm eq}) + \iint_{\triangle^c} g(x-y)(d\nu_N - Nd\mu_{\rm eq})(x)(d\nu_N - Nd\mu_{\rm eq})(y) + 2N\int \zeta(x)d\nu_N, \quad (1.2.7)$$

et le terme $N^2 I(\mu_{\rm eq})$ ne dépend évidemment pas de la configuration. Dans (1.2.7), $\mathcal{H}_N$ est toujours exprimé en fonction des quantités macroscopiques ν_N et $\mu_{\rm eq}$. Pour tout $\vec{X}_N$, on pose $\nu'_N = \sum_{i=1}^N \delta_{N^{1/d}x_i}$ (la configuration de points microscopiques) et $\mu'_{\rm eq}(x) = N\mu_{\rm eq}(N^{-1/d}x)$ (qui est une mesure de masse N correspondant à $\mu_{\rm eq}$ vue à l'échelle microscopique). On utilise le fait que

$$g(\alpha x) = \begin{cases} -\log(\alpha) + g(x) & \text{dans les cas logarithmiques,} \\ \alpha^{-s} g(x) & \text{dans les cas où } g(x) = |x|^{-s}, \end{cases}$$

et on obtient, par un changement de variables

$$\mathcal{H}_N(\vec{X}_N) = N^2 I(\mu_{\rm eq}) - \frac{N\log N}{d} + \iint_{\triangle^c} g(x-y)(d\nu'_N - d\mu'_{\rm eq})(x)(d\nu'_N - d\mu'_{\rm eq})(y) + 2N\int \zeta(x)d\nu_N,$$

dans les cas logarithmiques (avec $d = 1$ ou 2) et

$$\mathcal{H}_N(\vec{X}_N) = N^2 I(\mu_{\rm eq}) + N^{s/d} \iint_{\triangle^c} g(x-y)(d\nu'_N - d\mu'_{\rm eq})(x)(d\nu'_N - d\mu'_{\rm eq})(y) + 2N\int \zeta(x)d\nu_N,$$

dans les autres cas.

Notons w_N la fonctionnelle d'énergie définie sur $\mathcal{X}$ par

$$w_N(\mathcal{C}) := \iint_{\triangle^c} g(x-y)(d\mathcal{C} - d\mu'_{\rm eq})(x)(d\mathcal{C} - d\mu'_{\rm eq})(y),$$

si la configuration est finie (et $+\infty$ sinon). Posons aussi $\tilde{\zeta}(\mathcal{C}) := \int \zeta d\mathcal{C}$. La formule de décomposition (*splitting*) de $\mathcal{H}_N$ s'écrit, en résumé

$$\boxed{\mathcal{H}_N(\vec{X}_N) = \begin{cases} N^2 I(\mu_{\rm eq}) - \frac{N\log N}{d} + w_N(\nu'_N) + 2N\tilde{\zeta}(\nu_N) & \text{dans les cas logarithmiques,} \\ N^2 I(\mu_{\rm eq}) + N^{s/d} w_N(\nu'_N) + 2N\tilde{\zeta}(\nu_N) & \text{dans les autres cas.} \end{cases}}$$

Le terme $2N\tilde{\zeta}(\nu_N)$ jouera le rôle d'un potentiel confinant et entrera dans les considérations sur le volume, mais il n'est pas déterminant puisque ζ vaut zéro pour des points "typiques" i.e. dans le support de la mesure d'équilibre. La fonctionnelle d'énergie ("au second ordre", par opposition au premier ordre de l'énergie qui correspond à $N^2 I(\mu_{\rm eq})$) qui nous intéresse est w_N. Les travaux de Serfaty *et al.* cités plus haut montrent qu'elle est typiquement (en un sens à préciser) d'ordre N et déterminent une fonctionnelle limite pour $\frac{1}{N} w_N$.

b. Le champ électrique et une intégration par parties

Un deuxième point clef de l'analyse est la ré-écriture de l'énergie d'interaction au second ordre $w_N(\mathcal{C})$ comme la norme d'un certain champ de vecteurs calculée de façon renormalisée. Pensons aux points $\vec{X}_N$ comme à des particules portant une charge $+1$ et à l'interaction g comme à une énergie d'interaction électrostatique (l'analogie n'est physiquement valide que dans le cas Coulombien, mais on utilisera le même langage dans tous les cas). La mesure $-\mu'_{\rm eq}$ peut alors être vue comme une densité de charge négative de masse totale $-N$, et le système "charges ponctuelles et arrière-plan (*background*) chargé négativement" forme un système électrique dont w_N donne l'énergie d'interaction électrostatique. On introduit alors le potentiel $H^{\rm loc}$

$$H^{\rm loc}(x) := \int g(x-y)(d\nu'_N - d\mu'_{\rm eq})(y)$$

qui est le potentiel électrostatique généré par le système de charges. Son gradient donne le champ électrique associé $E^{\rm loc}$ (que l'on appelera "champ local") [1], on a

$$E^{\rm loc}(x) := \int \nabla g(x-y)(d\nu'_N - d\mu'_{\rm eq})(y).$$

Extension de l'espace. Dans le cas Coulombien en dimension $d \geq 2$, il est facile d'observer que $E^{\rm loc}$ vérifie l'équation suivante (équation de Poisson)

$$-{\rm div}(E^{\rm loc}) = c_d \left(\nu'_N - \mu'_{\rm eq}\right),$$

pour une certaine constante c_d dépendant de la dimension. Cela est lié au fait que le potentiel g est alors (à une constante multiplicative près) la solution fondamentale du Laplacien en dimension d. Pour disposer d'une équation analogue dans les autres cas, il faut ajouter une dimension d'espace, suivant [CS07].

On va noter $k \in \{0,1\}$ un paramètre indiquant si l'on travaille dans l'espace $\mathbb{R}^d$ ambiant ou bien dans l'espace augmenté d'une dimension $\mathbb{R}^{d+1}$. On aura $k=0$ dans les cas Coulomb et $k=1$ sinon (i.e. pour l'interaction logarithmique en dimension 1 et pour les cas Riesz avec $s > d-2$). Quand $k=1$ on notera souvent X un élement de $\mathbb{R}^{d+1}$, avec $X=(x,y) \in \mathbb{R}^d \times \mathbb{R}$, on notera également $\mu_{\rm eq}\delta_{\mathbb{R}^d}$ la mesure $\mu_{\rm eq}$ vue dans l'espace augmenté (on a $\int f(X)d(\mu_{\rm eq}\delta_{\mathbb{R}^d}) = \int f(x,0)d\mu_{\rm eq}$). Si l'on étend g, $H^{\rm loc}$ et $E^{\rm loc}$ à $\mathbb{R}^{d+1}$ de la manière naturelle, on dispose de la relation suivante :

$$-{\rm div}(|y|^\gamma \nabla g) = c_{d,s}\delta_0$$

où γ est tel que $d-2+k+\gamma = s$ et $c_{d,s}$ est une constante dépendant de d, s et connue explicitement (voir [PS15, Section 1.2]).

Troncature et intégration par parties renormalisée. Revenons un instant au cas Coulombien (pour lequel les précédentes considérations d'extension ne sont pas nécessaires), et à l'égalité $-\Delta H^{\rm loc} = -{\rm div}(E^{\rm loc}) = c_d\left(\nu'_N - \mu'_{\rm eq}\right)$. Heuristiquement, l'énergie $w_N(\nu'_N)$ s'écrit

$$w_N(\nu'_N) \simeq -\frac{1}{c_d}\int_{\mathbb{R}^d} H^{\rm loc}\Delta H^{\rm loc}.$$

En intégrant par parties, il vient

$$w_N(\nu'_N) \simeq \frac{1}{c_d}\int_{\mathbb{R}^d} |\nabla H^{\rm loc}|^2 = \frac{1}{c_d}\int_{\mathbb{R}^d} |E^{\rm loc}|^2.$$

1. Physiquement, il faudrait bien sûr prendre *l'opposé* du gradient du potentiel pour obtenir le champ.

Ainsi l'énergie électrostatique (dans le cas Coulombien) s'écrit-elle formellement comme la norme L^2 du champ électrique local E^{loc} engendré par le système charges + *background*. Cependant il est facile de voir que les champs E^{loc} ne sont jamais de carré intégrable autour de chaque particule, et en effet le raisonnement précédent est incorrect car on a oublié que w_N est défini en *excluant* la diagonale dans le domaine d'intégration, or l'auto-interaction d'une charge avec elle-même est infinie. Une manière de rendre rigoureuse l'heuristique précédente est de calculer la norme L^2 du champ de manière renormalisée. Pour cela on va tronquer le champ près des charges à une distance η et soustraire à la norme L^2 de ce champ tronqué une quantité divergente (quand $\eta \to 0$) correspondant à l'auto-interaction de chaque charge.

Soit $\eta \in (0,1)$, on définit le potentiel tronqué par $f_\eta(X) = (g(X) - g(\eta))_+$ pour $X \in \mathbb{R}^{d+k}$. La quantité

$$\delta_0^{(\eta)} := \frac{1}{c_{d,s}} \mathrm{div}(|y|^\gamma \nabla f_\eta) + \delta_0$$

est une mesure positive de masse 1 supportée sur $\partial B(0,\eta)$ (la boule étant prise dans $\mathbb{R}^{d+k}$).

Si une configuration de points ν'_N est donnée, on définit H^{loc}_η et E^{loc}_η à l'aide de cette troncature.

$$H^{\mathrm{loc}}_\eta(X) := H^{\mathrm{loc}}(X) - \sum_{i=1}^N f_\eta(X - x_i), \quad E^{\mathrm{loc}}_\eta(X) := E^{\mathrm{loc}}(X) - \sum_{i=1}^N \nabla f_\eta(X - x_i). \tag{1.2.8}$$

On peut maintenant ré-écrire la fonctionnelle w_N dans l'espace $\mathbb{R}^{d+k}$ de la manière suivante

Proposition 1.2.4. *Soit $\vec{X}_N$ dans $(\mathbb{R}^d)^N$ et E^{loc}_η comme défini en* (1.2.8)*. On a*

$$w_N(\nu'_N) = \lim_{\eta \to 0} \frac{1}{c_{d,s}} \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |E^{\mathrm{loc}}_\eta|^2 - N g(\eta) \right)$$

Cela permet de ré-écrire la fonctionnelle d'énergie en fonction d'un objet microscopique (le champ électrique local) qui encode, en particulier, la configuration de points puisqu'on peut retrouver ν'_N à partir de E^{loc} en calculant $-\mathrm{div}(|y|^\gamma E^{\mathrm{loc}}) + c_{d,s}\mu'_{\mathrm{eq}}\delta_{\mathbb{R}^d}$.

c. Énergie renormalisée limite

On tente maintenant de définir une fonctionnelle d'énergie similaire à w_N mais qui puisse calculer l'énergie d'interaction d'un système électrique *infini*, avec une infinité de charges dans $\mathbb{R}^d$ et un *background* négatif dans tout l'espace.

Soit $\mathcal{C}$ une configuration de points et $m > 0$. Soit E un champ de vecteurs dans $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$. On dit que E est un champ électrique compatible avec $\mathcal{C}, m$ lorsque

$$-\mathrm{div}(|y|^\gamma E) = c_{d,s} \left(\mathcal{C} - m\delta_{\mathbb{R}^d} \right).$$

Pour tout $\eta \in (0,1)$ on définit alors sa troncature E_η de manière semblable à (1.2.8)

$$E_\eta(x) = E(x) - \sum_{p \in \mathcal{C}} \nabla f_\eta(x - p),$$

et son énergie renormalisée $\mathcal{W}(E)$ par

$$\mathcal{W}(E) = \lim_{\eta \to 0} \mathcal{W}_\eta(E) \quad \text{avec} \quad \mathcal{W}_\eta(E) := \limsup_{R \to \infty} \frac{1}{R^d} \left(\frac{1}{c_{d,s}} \int_{R^d \times \mathbb{R}^k} |y|^\gamma |E_\eta|^2 - m g(\eta) \right).$$

À $\eta \in (0,1)$ fixé, $\mathcal{W}_\eta(E)$ est bien défini comme lim sup. Il n'est pas évident, en revanche, que la limite de ces quantités existe quand $\eta \to 0$, mais un calcul (voir par exemple [PS15, Proposition 2.4]) montre que $\mathcal{W}_\eta(E)$ est presque monotone en η (à $O(\eta)$ près), donc $\mathcal{W}(E)$ est bien défini. Cela procure une notion d'énergie (calculée là encore de façon renormalisée) pour les champs électriques associés à un système infini de charges ponctuelles + *background* négatif.

On peut ensuite définir l'énergie d'une configuration de points $\mathcal{C}$ (et d'un background de densité m) comme l'infimum de l'énergie renormalisée (calculée au sens précédent) des *champs* compatibles avec $\mathcal{C}, m$

$$\mathbb{W}_m(\mathcal{C}) := \inf_{E \text{ compatible avec } \mathcal{C},m} \mathcal{W}(E).$$

Enfin, si P est un processus ponctuel aléatoire d'intensité m, on définit naturellement son énergie renormalisée comme l'espérance sous P de $\mathbb{W}_m$

$$\mathbb{W}_m^{\text{elec}}(P) := \mathbf{E}_P[\mathbb{W}_m].$$

d. Connection entre w_N et $\mathbb{W}^{\text{elec}}$

Pour énoncer la connexion entre w_N et $\mathbb{W}^{\text{elec}}$, il est utile d'introduire l'espace des processus ponctuels *étiquetés* $\mathcal{P}(\Sigma \times \mathcal{X})$. Ces processus correspondent à des observables de type "champ empirique" pour lesquelles on garde une information supplémentaire (l'étiquette) à savoir le point autour duquel on a réalisé le zoom. On supposera toujours que la première marginale de $\bar{P} \in \mathcal{P}(\Sigma \times \mathcal{X})$ est la mesure de Lebesgue sur Σ. On peut alors considérer la *désintégration* de $\bar{P}$ par rapport à la mesure de Lebesgue sur Σ, ce qui donne une famille $\{\bar{P}^x\}_{x\in\Sigma}$ d'éléments de $\mathcal{P}(\mathcal{X})$ tels que

$$\mathbf{E}_{\bar{P}}[f(x,\mathcal{C})] = \int_\Sigma \mathbf{E}_{\bar{P}^x}\left[f(x,\mathcal{C})\right] \frac{dx}{|\Sigma|},$$

pour toute fonction f continue bornée sur $\Sigma \times \mathcal{X}$. On dit que $\bar{P}$ est stationnaire si les $\bar{P}^x$ sont stationnaires pour Lebesgue-presque tout $x \in \Sigma$.

Si $\vec{X}_N \in (\mathbb{R}^d)^N$ on définit $\bar{P}_N(\vec{X}_N)$ comme

$$\bar{P}_N := \frac{1}{|\Sigma|}\int_\Sigma \delta_{(x,\mathcal{C}_N^x)}dx \ \in \mathcal{P}(\Sigma \times \mathcal{X}).$$

On peut maintenant énoncer le lien entre w_N et $\mathbb{W}^{\text{elec}}$.

Proposition 1.2.5. *On a les deux propriétés suivantes :*

A) *Soit $\{\vec{X}_N\}_N$ une suite de N-uplets vérifiant $\sup \frac{w_N(\vec{X}_N)}{N} < +\infty$. Alors la suite des $\bar{P}_N$ converge, à extraction près, vers un processus ponctuel étiqueté aléatoire $\bar{P}$ tel que*

1. *$\bar{P}$ est stationnaire.*
2. *$\bar{P}^x$ est d'intensité $m_{\text{eq}}(x)$ pour Lebesgue-presque tout $x \in \Sigma$.*
3. *On a*

$$\liminf_{N\to\infty} \frac{w_N(\vec{X}_N)}{N} \geq \int \mathbb{W}^{\text{elec}}_{m_{\text{eq}}(x)}(\bar{P}^x)\frac{dx}{|\Sigma|}. \tag{1.2.9}$$

B) *Réciproquement, si $\bar{P}$ vérifie 1. et 2. et si $\int \mathbb{W}^{\text{elec}}_{m_{\text{eq}}(x)}(\bar{P}^x)\frac{dx}{|\Sigma|}$ est fini, il existe une suite $\{\vec{X}_N\}_N$ telle que $\bar{P}_N$ converge vers $\bar{P}$ et*

$$\limsup_{N\to\infty} \frac{1}{N} w_N(\vec{X}_N) \leq \int \mathbb{W}^{\text{elec}}_{m_{\text{eq}}(x)}(\bar{P}^x)\frac{dx}{|\Sigma|}.$$

L'énoncé précédent est très similaire à un résultat de Γ-convergence. Il a été formulé initialement à l'aide des champs électriques (voir [Ser15, Chapitres 5,6]), et la présentation ci-dessus s'en déduit en passant des champs électriques aux configurations de points sous-jacentes.

La partie A) de la Proposition 1.2.5 est une propriété de semi-continuité inférieure de l'énergie, qui est démontrée par des arguments abstraits d'analyse fonctionnelle. L'affirmation B), c'est-à-dire l'existence d'une suite *recovery sequence* qui montre que (1.2.9) est parfois optimale, nécessite elle des constructions explicites. Dans la section suivante, on présente le principal outil technique pour réaliser ces constructions : les méthodes d'écrantage.

1.2.5 Le rôle de l'écrantage

Du fait de son caractère *à longue portée*, l'énergie d'interaction est difficile à localiser. Prenons pour exemple le cas du 2DOCP : soit $\vec{X}_N$ un N-uplet de $\mathbb{R}^2$ et E^{loc} le champ local engendré dans $\mathbb{R}^2$. Supposons qu'on a partitionné Σ selon une famille finie I de domaines réguliers, connexes Ω_i. On a bien sûr

$$\int_\Sigma |E_\eta^{\mathrm{loc}}|^2 = \sum_{i\in I} \int_{\Omega_i} |E_\eta^{\mathrm{loc}}|^2,$$

mais cette décomposition ne permet pas de localiser l'énergie car la valeur de E^{loc} sur Ω_i dépend *a priori* de toute la configuration. On peut relaxer le problème et écrire

$$\int_{\Omega_i} |E_\eta^{\mathrm{loc}}|^2 \geq F_i(\nu'_N \cap \Omega_i),$$

où F_i est l'infimum des $\int_{\Omega_i} |E_\eta^{(i)}|^2$ pris parmi l'ensemble des champs électriques $E^{(i)}$ compatibles **dans** Ω_i avec ν'_N et μ'_{eq}, c'est-à-dire tels que

$$-\mathrm{div}(E^{(i)}) = c_2 \left(\nu'_N - \mu'_{\mathrm{eq}}\right) \text{ dans } \Omega_i.$$

Une telle relaxation permet de minorer l'énergie d'interaction (donc de majorer les probabilités) à l'aide d'énergies "localisées". Pour obtenir des bornes inférieures de PGD on doit majorer les énergies, et montrer que l'inégalité

$$\int_\Sigma |E_\eta^{\mathrm{loc}}|^2 \geq \sum_{i\in I} F_i(\nu'_N \cap \Omega_i)$$

peut être renversée (à une petite erreur près) assez souvent. Cela nécessite de construire "à la main" des configurations et d'estimer leur énergie. La question est alors la suivante : partant de la donnée de champs $\{E^{(i)}\}_{i\in I}$ et de configurations de points sous-jacentes $\{\mathcal{C}^{(i)}\}_{i\in I}$, peut-on construire un champ total E^{tot} et une configuration sous-jacente $\mathcal{C}^{\mathrm{tot}}$ dont l'énergie soit bornée par la somme de celles des $E^{(i)}$? Il n'est pas possible, en général, de recoller deux champs électriques si leur composante normale le long de la frontière ne coïncide pas : le champ de vecteurs obtenu n'est plus compatible avec la configuration totale (il apparaît de la divergence le long de la frontière). Pour résoudre ce problème, on utilise des techniques d'*écrantage* qui consistent à modifier les configurations de points et les champs électriques dans un petit voisinage tubulaire des frontières des domaines $\{\Omega_i\}$, de manière à rendre la composante normale de *tous* les champs électriques nulle au bord des domaines, et de pouvoir ainsi les recoller. On dit que les champs sont "écrantés", et leur énergie devient alors additive. Cette construction demande un bon contrôle de l'énergie de $E^{(i)}$ dans Ω_i et, dans le cas des interactions non-Coulombiennes, un bon contrôle de sa décroissance le long de l'axe additionnel dans $\mathbb{R}^{d+k}$.

Ces techniques d'écrantage ont été introduites dans [SS12, SS15b] pour le cas du 2DOCP et généralisée dans [RS15] pour le cas Coulombien ($d \geq 3$), dans [SS15a] pour le cas du *log-gas*

quand $d = 1$ (où le besoin d'ajouter une dimension d'espace apparaît). Le cas général (incluant les interactions Riesz) est traité, avec de meilleures estimées, dans [PS15].

1.3 Résultats obtenus, perspectives et problèmes ouverts

1.3.1 Présentation des résultats

a. Grandes déviations pour les champs empiriques avec moyennes macroscopiques

On résume ici le contenu du chapitre 2, qui correspond à l'article [LS15], écrit avec S. Serfaty. Par simplicité, on présentera ici les résultats dans le cas d'une interaction logarithmique en dimension 1 ou 2. On suppose toujours que V satisfait les hypothèses du Théorème 2, ainsi que quelques hypothèses de régularité, notamment on demande que $\mu_{\rm eq}$ ait une densité $m_{\rm eq}$ assez régulière par rapport à la mesure de Lebesgue et que $\partial\Sigma$ soit C^1. On introduit une fonctionnelle d'énergie libre $\mathcal{F}_\beta$, définie sur $\mathcal{P}_s(\mathcal{X})$ par

$$\mathcal{F}_\beta^m(P) := \beta \mathbb{W}_m^{\rm elec}(P) + \mathsf{ent}[P|\mathbf{\Pi}^m] \quad \text{pour tout } m > 0.$$

Soit $x \in \mathring{\Sigma}$ et soit $\varepsilon > 0$ tel que $B(x,\varepsilon) \subset \Sigma$. Pour la simplicité de l'exposition, supposons temporairement que $m_{\rm eq}$ soit constante sur $B(x,\varepsilon)$.

Théorème 4. *La loi de $\mathcal{C}^x_{N,B(x,\varepsilon)}$ satisfait un PGD à vitesse $|B(x,\varepsilon)|N$ avec bonne fonction de taux $\mathcal{F}_\beta^{m_{\rm eq}(x)} - \min \mathcal{F}_\beta^{m_{\rm eq}(x)}$.*

En général $m_{\rm eq}$ n'est pas constante, et il faut énoncer le résultat d'une façon différente. Pour cela, on enrichit l'observable "champ empirique" en gardant trace du point autour duquel on zoome. Plus précisément on introduit encore le "champ empirique étiqueté" (*tagged empirical field*) comme

$$\bar{P}_N : \vec{X}_N \mapsto \frac{1}{|\Sigma|}\int_\Sigma \delta_{(x,\mathcal{C}^x_N)}dx.$$

Pour tout $\vec{X}_N$, $\bar{P}_N(\vec{X}_N)$ est une mesure de probabilité sur $\Sigma\times\mathcal{X}$. En testant contre des fonctions qui approximent $\mathbf{1}_\Omega$ (selon la première variable), on peut localiser cette observable et obtenir des informations sur les champs empiriques étiquetés *locaux* du type

$$\bar{P}_{N,\Omega} : \vec{X}_N \mapsto \frac{1}{|\Omega|}\int_\Omega \delta_{(x,\mathcal{C}^x_N)}dx.$$

Le principe de grandes déviations énoncé dans [LS15] porte sur la loi du poussé-en-avant de $\mathbb{P}_{N,\beta}$ par $\bar{\mathcal{C}}_N$ et est gouverné par une fonction de taux $\bar{\mathcal{F}}_\beta$ qui est la version "étiquetée" de $\mathcal{F}_\beta$, définie tranche par tranche par rapport à la désintégration d'une mesure $\bar{P} \in \mathcal{P}(\Sigma\times\mathcal{X})$ contre la mesure de Lebesgue sur Σ.

Le Théorème 4 est valide pour des interactions de Riesz à condition d'effectuer un changement d'échelle sur la température dans la définition de $\mathbb{P}_{N,\beta}$. À notre connaissance, c'est la première justification rigoureuse d'un principe de grandes déviations au niveau *microscopique* pour un système continu de particules avec des interactions singulières et à longue portée. On obtient une forme d'universalité, puisque la fonction de taux ne dépend du potentiel V qu'à travers la densité $m_{\rm eq}(x)$.

Naturellement, la connaissance du comportement typique moyenné (comme en (1.2.2)) est moins précise que la loi du processus ponctuel non-moyenné (comme en (1.2.1)). Dans les cas où l'existence d'une limite en loi au processus non-moyenné est connue, il n'est pas difficile de montrer que ces processus limites doivent minimiser la fonctionnelle d'énergie libre. On obtient

en particulier une nouvelle propriété, variationnelle, des processus Sine_β, dont on donnera une application dans la section suivante.

Corollaire 1.3.1. *La loi des processus Sine_β (en dimension 1) et la loi du processus de Ginibre (en dimension 2) minimisent (après mise à l'échelle) l'énergie libre $\mathcal{F}^1_\beta$ pour un bon choix de β.*

Une seconde conséquence du Théorème 4 est qu'on obtient une forme explicite pour le terme d'ordre N dans le développement asymptotique de $\log Z_{N,\beta}$, en termes de $\mathcal{F}^1_\beta$ et de la densité d'équilibre.

Corollaire 1.3.2. *On a, dans les cas logarithmiques pour $d = 1, 2$*

$$\log Z_{N,\beta} = -N^2 I_\beta(\mu_{\mathrm{eq}}) + \frac{\beta N}{d}\log N - N\min \mathcal{F}^1_\beta \\ - N\left(1 - \frac{\beta}{d}\right)\int m_{\mathrm{eq}}(x)\log m_{\mathrm{eq}}(x)\,dx + o(N). \quad (1.3.1)$$

L'existence d'un développement asymptotique à tout ordre a été démontrée (pour le cas du *log-gas* à une dimension) dans [BG13b, BG13a], mais avec (1.3.1) on peut interpréter le terme d'ordre N à l'aide de l'énergie libre. On retrouve le fait, déjà prédit par Dyson dans [Dys62], que pour une valeur particulière de β ($\beta = d$ avec nos conventions, $\beta = \frac{d}{2}$ pour les conventions usuelles en matrices aléatoires) la dépendance en m_{eq} de ce terme disparaît de $\log Z_{N,\beta}$. Dans les cas Riesz, on obtient aussi, pour la première fois, un développement de $\log Z_{N,\beta}$ au second ordre.

b. Comportement microscopique local pour le 2DOCP

Dans le chapitre 3 , qui correspond à l'article [Leb15a], on donne un résultat qui améliore le principe de grandes déviations avec moyennes macroscopiques décrit plus haut, dans le cas du 2DOCP. On montre qu'un principe de grandes déviations similaire est vérifié par la loi des champs empiriques avec moyennes mésoscopiques jusqu'à l'échelle la plus fine $N^{-1/2+\delta}$ pour $\delta > 0$. Dans l'énoncé suivant on utilise la notation (1.2.2).

Théorème 5. *Soit $x \in \mathring{\Sigma}$ et $\delta \in (0, 1/2)$ fixés. La loi de $\mathcal{C}^x_{N,B(x,N^{-1/2+\delta})}$ obéit à un PGD à vitesse $N^{2\delta}$ avec bonne fonction de taux $\mathcal{F}^{m_{\mathrm{eq}}(x)}_\beta - \min \mathcal{F}^{m_{\mathrm{eq}}(x)}_\beta$.*

Diminuer ainsi la taille sur laquelle on moyenne en espace représente d'une part un pas vers l'étude hypothétique de l'observable "non-moyennée" (qui est réellement "la plus fine" possible), et permet d'un autre côté d'obtenir des lois locales, au sens suivant.

Corollaire 1.3.3. *Soit $x \in \mathring{\Sigma}$ et $\delta \in (0, 1/2)$ fixés. Soit $f(\cdot) = \tilde{f}(N^{1/2-\delta}(\cdot - x_0))$ pour une fonction $\tilde{f}$ supposée C^1 à support compact dans $\mathbb{R}^2$. On a*

$$\left|\int_{\mathbb{R}^2} f(d\nu_n - N d\mu_{\mathrm{eq}})\right| = o\left(N^{2\delta}\right).$$

L'estimée de [Leb15a] est en fait quantitative et le membre de droite dépend explicitement de f et de son gradient. Ce type de lois locales a été très étudié dans le cadre des matrices aléatoires (voir par exemple [BYY14a,BYY14b]). Pour le 2DOCP, une loi locale (sans le principe de grandes déviations associé) de même nature a été démontrée simultanément (et indépendamment) par Bauerschmidt, Bourgade, Nikula et Yau dans [BBNY15].

L'idée de base de la preuve du Théorème 5 est de réaliser un bootstrap le long des échelles considérées. Un argument semblable avait été utilisé dans [RNS15] pour l'étude des minimiseurs,

et il est intéressant de noter que la preuve de la loi locale pour les matrices aléatoires repose aussi sur un bootstrap (voir [BGK16] pour une introduction à ces méthodes). On introduit une notion de "bon contrôle à l'échelle δ", qui exprime que l'énergie du système et le nombre de particules sont proportionnelles au volume quand on considère des domaines d'échelle δ (c'est à dire de longueur caractéristique N^δ après mise à l'échelle d'un facteur $N^{1/2}$), avec bonne probabilité. On montre qu'il existe $t < 1$ tel que

$$\text{Bon contrôle à l'échelle } \delta \longrightarrow \begin{cases} \text{PGD à l'échelle } \delta_1 \\ \text{Bon contrôle à l'échelle } \delta_1 \end{cases} \quad \text{pour tout } t\delta \leq \delta_1 < \delta.$$

L'initialisation repose sur des considérations énergétiques simples, mais l'argument de descente en lui-même nécessite une légère adaptation des techniques d'écrantages présentées plus haut. On doit se limiter à traiter le cas de l'interaction logarithmique pour des raisons techniques : d'une part la gestion de la troncature est plus facile quand la singularité en 0 est modérée et d'autre part l'ajout d'une dimension d'espace dans les cas non-Coulombiens rend les arguments d'écrantages plus délicats.

c. Application aux limites de haute et basse température

On présente ici les chapitres 4 et 5 qui correspondent aux articles [Leb15b, Leb15c].

Un inconvénient de l'énergie libre $\mathcal{F}_\beta$ est qu'elle est difficile à calculer explicitement. En particulier, il est difficile d'évaluer le terme d'énergie en dehors des cas où la configuration de points est périodique (cas pour lequel il existe des formules exactes qui permettent une évaluation, fût-elle numérique). Dans [Leb15b], on introduit une définition alternative, plus intuitive, de l'énergie d'un processus ponctuel, sans passer par les champs électriques. Cette énergie $\mathbb{W}^{\text{int}}$ correspond réellement à la "limite en volume infini" de l'interaction électrostatique d'une configuration de points et d'un arrière-plan uniformément négativement chargé, dans l'esprit de [BS13]. Si P est la loi d'un processus ponctuel stationnaire d'intensité 1, $\mathbb{W}^{\text{int}}(P)$ s'exprime uniquement en fonction de $\rho_{2,P}$, la fonction de corrélation à deux points, par la formule suivante

$$\mathbb{W}^{\text{int}}(P) := \liminf_{R\to\infty} \frac{1}{R^d} \int_{[-R,R]^d\setminus\{0\}} g(v)(\rho_{2,P}(v) - 1) \prod_{i=1}^{d} (R - |v_i|)dv,$$

où $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ et où on écrit, avec un léger abus de notation $\rho_{2,P}(v) := \rho_{2,P}(0, v)$ (comme P est stationnaire, $\rho_{2,P}(x, y)$ ne dépend que de $x - y$). Dans les cas logarithmiques, on doit définir aussi

$$\mathcal{D}^{\log}(P) := C \limsup_{R\to\infty} \left(\frac{1}{R^d} \iint_{[-R/2,R/2]^d} (\rho_{2,P}(x, y) - 1)dxdy + 1 \right) \log R,$$

avec C une certaine constante dont la valeur n'est pas importante. Cette quantité supplémentaire est une complication technique mais nécessaire, sur laquelle il n'est pas important de s'attarder dans un premier temps.

Une fonctionnelle d'énergie similaire a été introduite (dans les cas logarithmiques) par Borodin et Serfaty dans [BS13], mais sans donner de connection rigoureuse avec l'énergie renormalisée $\mathbb{W}^{\text{elec}}$. Ici, on prouve

Théorème 6. *Les énergies $\mathbb{W}_1^{\text{elec}}$ et $\mathbb{W}^{\text{int}}$ sont reliées comme suit.*

- *Dans le cas logarithmique $1D$, $\mathbb{W}_1^{\text{elec}}$ est la régularisée semi-continue inférieurement de $\mathbb{W}^{\text{int}} + \mathcal{D}^{\log}$.*
- *Dans le cas logarithmique $2D$ (2DOCP), on a $\mathbb{W}_1^{\text{elec}} \leq \mathbb{W}^{\text{int}} + \mathcal{D}^{\log}$.*

— *Dans les autre cas non-Coulombiens,* $\mathbb{W}_1^{\mathrm{elec}}$ *est la régularisée semi-continue inférieurement de* $\mathbb{W}^{\mathrm{int}}$.
— *Dans les cas Coulomb pour* $d \geq 3$*, on a* $\mathbb{W}_1^{\mathrm{elec}} \leq \mathbb{W}^{\mathrm{int}}$.

On utilise ensuite l'expression "explicite" de $\mathbb{W}^{\mathrm{int}}$ en termes de la fonction de corrélation à deux points, et son lien avec $\mathbb{W}^{\mathrm{elec}}$ tel qu'exprimé ci-dessus, pour effectuer des estimées d'énergies simples (mais impossibles à mener directement avec $\mathbb{W}^{\mathrm{elec}}$). Cela permet de prouver les deux résultats suivants concernant la minimisation de $\mathcal{F}_\beta$ dans la limite $\beta \to 0$ et $\beta \to \infty$.

Théorème 7. *Les minimiseurs de* $\mathcal{F}_\beta^1$ *convergent, quand* $\beta \to 0$*, vers la loi d'un processus ponctuel de Poisson d'intensité* 1*. De plus, cette convergence a lieu "en entropie", i.e.*

$$\lim_{\beta \to 0} \sup_{\mathcal{F}_\beta(P_\beta) = \min \mathcal{F}_\beta} \mathsf{ent}[P_\beta | \mathbf{\Pi}^1] = 0.$$

Dans le cas particulier du *log-gas* $1D$, comme expliqué plus haut (Corollaire 1.3.1), un minimiseur de $\mathcal{F}_\beta^1$ est la loi du processus Sine_β de Valko et Virag [VV09]. Notre méthode donne donc une nouvelle preuve, plus "physique", d'un résultat récent de Allez et Dumaz [AD14] :

Corollaire 1.3.4. *Quand* $\beta \to 0$*, le processus* Sine_β *converge en loi vers un processus ponctuel de Poisson d'intensité* 1 *sur* $\mathbb{R}$.

Si l'on se restreint maintenant à la dimension 1, on peut également caractériser la limite $\beta \to \infty$ (limite de basse température) des minimiseurs de $\mathcal{F}_\beta$. Définissons $P_{\mathbb{Z}}$ comme la loi du processus ponctuel stationnaire associé au réseau $\mathbb{Z}$ i.e. $P_{\mathbb{Z}} := \int_0^1 \delta_{x+\mathbb{Z}}\, dx$.

Théorème 8 (Cristallisation pour $d = 1$)**.** *Pour* $d = 1$ *(et* g *logarithmique ou Riesz),* $P_{\mathbb{Z}}$ *est l'unique minimiseur de* $\mathbb{W}_1^{\mathrm{elec}}$ *parmi les processus ponctuels aléatoires stationnaires d'intensité* 1*. De plus, si* $\{P_\beta\}_\beta$ *est une famille de minimiseurs de* $\mathcal{F}_\beta$*, on a*

$$\lim_{\beta \to \infty} P_\beta = P_{\mathbb{Z}}.$$

Le Théorème 8 est un résultat de *cristallisation*, puisqu'il établit la convergence vers l'unique cristal uni-dimensionnel quand $\beta \to \infty$. L'article antérieur [Leb15c] est dévolu à la preuve d'un résultat similaire, restreint au cas où g est logarithmique. La preuve de [Leb15c] s'appuie sur les expressions explicites valables dans le cas de configurations périodiques et procède ensuite par l'approximation d'un processus stationnaire par des (combinaisons de) processus périodiques. La méthode de [Leb15b] est analogue dans l'esprit, mais suit une approche plus simple et qui s'adapte au cas Riesz.

d. Grandes déviations pour le 2D2CP

L'article [LSZW15], écrit avec S. Serfaty, O. Zeitouni et W. Wu, constitue le chapitre 6 et traite du contexte légèrement différent d'un gaz de Coulomb bi-dimensionnel avec des particules de charge ± 1.

Du fait de la présente de charges ponctuelles de signes opposés, le système est instable à basse température car l'excitation thermique ne compense pas l'attrait des configurations d'énergie très négative (avec des dipôles de charges opposées très proches l'une de l'autre). Le domaine de stabilité (en température) a été déterminé dans [DL74] et l'existence d'une limite thermodynamique pour la fonction de partition a été établie dans [Frö76] par des méthodes de théorie des champs et par Gunson et Panta dans [GP77] par des méthodes "classiques". On s'appuie beaucoup sur les estimées de [GP77] pour contrôler les moments exponentiels de quantités *a priori* non bornées comme les interactions entre particules proches de charges opposées.

À part l'existence d'une limite thermodynamique, le comportement macroscopique du système n'était pas étudié, ni les lois microscopiques. Par exemple il n'était pas connu rigoureusement que les charges positives et négatives se répartiraient (à l'échelle macroscopique $O(1)$) de manière à remplir le domaine (ici, le carré unité) de manière uniforme (voir cependant [BG99] pour une question similaire avec un scaling différent sur la température). La question de déterminer leur comportement à l'échelle $N^{-1/2}$ n'avait pas non plus été abordée.

Dans [LSZW15], on établit un PGD à la vitesse N pour le comportement microscopique, après une moyenne à l'échelle macroscopique. La fonction de taux est analogue à l'énergie libre $\mathcal{F}_\beta$ étudiée plus haut, mais le terme d'énergie demande une attention supplémentaire car il n'est pas minoré (il existe même des configurations d'énergie $-\infty$). En utilisant les calculs de [GP77] on arrive à montrer que pour des températures suffisamment hautes (en fait, dans tout le domaine de stabilité du système), l'excitation thermique est suffisante pour rendre négligeables le volume (sous la mesure de Gibbs) dans l'espace des phases de ces configurations d'énergie très négatives. Une fois que l'on connaît le PGD qui gouverne le comportement microscopique, on étudie la dépendance de la fonction de taux en la densité *macroscopique* de charges, et on résout un problème variationnel simple qui montre que le comportement optimal pour les particules est de se répartir uniformément à l'échelle $O(1)$. On déduit donc le comportement macroscopique asymptotique à partir de la détermination d'une contrainte sur le comportement microscopique, et à notre connaissance c'est pour l'instant la seule façon de procéder.

1.3.2 Perspectives de recherche

Présentons maintenant quelques questions reliées aux objets étudiés dans cette thèse.

a. Zéros de polynômes aléatoires

L'étude des zéros de polynômes aléatoires ou de fonctions entières aléatoires possède certaines similarités avec cette des valeurs propres de matrices aléatoires (voir par exemple [HKPV09], ou encore comparer [TV11] et [TV14]). Une raison qui peut expliquer cette similitude est que le changement de variables coefficients-racines fait apparaître un déterminant de Vandermonde en les racines, d'où une structure algébrique particulière de la loi de probabilité et - du point de vue de la physique statistique - une interaction logarithmique entre racines considérées comme des particules.

De la même manière qu'il existe de nombreux modèles de matrices aléatoires, on peut considérer plusieurs familles de polynômes aléatoires. Un cas particulier est celui où P_N est un "polynôme de Kac" et s'écrit

$$P_N(X) := \sum_{k=0}^{N} a_k X^k$$

avec pour coefficients $\{a_k\}_{k=1,\dots,N}$ des variables aléatoires Gaussiennes complexes i.i.d centrées de variance $1/2$. Dans ce cas (et, en fait, le résultat vaut pour des familles plus générales) la mesure empirique des zéros

$$\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i}$$

converge presque sûrement vers la mesure uniforme sur le cercle unité quand $N \to \infty$ (voir par exemple [Kos93]).

Dans [ZZ10] (voir aussi [But15] pour une ré-écriture plus élementaire de la preuve), Zeitouni et Zelditch établissent un principe de grandes déviations pour la loi de μ_N, qui fournit l'équivalent du théorème 2 dans ce cadre. En particulier, ce résultat fait apparaître la similarité de la loi

jointe des racines de P_N avec un modèle de physique statistique. Introduisons la fonctionnelle d'énergie

$$\mathcal{H}_N^{\mathrm{Kac}}(z_1,\dots,z_N) := \sum_{1\le i\neq j\le N} -\log|z_i-z_j| + (N+1)\log\int_{\mathbb{S}^1}\left(\prod_{i=1}^N |z-z_i|^2\right) d\nu_{\mathbb{S}^1}(z),$$

avec $\nu_{\mathbb{S}^1}$ la mesure uniforme sur le cercle unité, et la mesure de Gibbs associée (à température inverse $\beta = 1$)

$$d\mathbb{P}_N^{\mathrm{Kac}}(\vec{Z}_N) := \frac{1}{Z_N}\exp\left(-\mathcal{H}_N^{\mathrm{Kac}}(z_1,\dots,z_N)\right) d\vec{Z}_N,$$

où $\vec{Z}_N = (z_1,\dots,z_N) \in \mathbb{C}^N$ et $d\vec{Z}_N$ est la mesure de Lebesgue sur $\mathbb{C}$ (avec toujours Z_N une constante de normalisation). Alors la loi jointe des zéros de la famille de polynômes aléatoires définie plus haut coïncide avec $\mathbb{P}_N^{\mathrm{Kac}}$. On peut donc voir les zéros de P_N comme des particules dans $\mathcal{C}$ interagissant via un potentiel logarithmique, et soumises à un confinement non linéaire

$$V(\vec{Z}_N) := (N+1)\log\int_{\mathbb{S}^1}\left(\prod_{i=1}^N |z-z_i|^2\right) d\nu_{\mathbb{S}^1}(z).$$

Dans un projet avec Raphaël Butez, nous aimerions donner un principe de grandes déviations au second ordre, concernant le comportement *microscopique* du système. Le caractère particulier du "potentiel" V est intéressant : outre sa nature non-linéaire, il est "faiblement" confinant et pourtant piège les particules dans un compact du plan. De plus, le problème est bi-dimensionnel mais la mesure limite, elle, est concentrée sur une sous-variété de dimension 1, l'observable microscopique doit donc tenir compte à la fois de la disposition $1d$ des particules (mesurée par leur angle) et de leur distance au cercle unité.

b. Unicité des minimiseurs et caractérisation des processus Sine$_\beta$

Dans [LS15] il est prouvé que les processus ponctuels Sine$_\beta$ définis par Valko et Virag dans [VV09] forment une famille de minimiseurs pour la fonctionnelle d'énergie libre $\mathcal{F}_\beta^1$ dans le cas du *log-gas* à une dimension. Si l'on prouvait que les minimiseurs de $\mathcal{F}_\beta$ sont uniques en dimension 1, cela donnerait une *caractérisation* variationnelle de ces processus. Comme $\mathcal{F}_\beta$ est affine, on ne peut pas utiliser un argument direct de (stricte) convexité pour obtenir l'unicité des minimiseurs. Cependant, suivant une suggestion d'Alice Guionnet, il serait possible de montrer que $\mathcal{F}_\beta$ est *convexe par déplacement*, une notion introduite par Mc Cann [McC97] dans le contexte du transport optimal. On dit qu'une fonctionnelle $\mathcal{F} : \mathcal{P}(X) \to \mathbb{R}$ est convexe par déplacement si $\mathcal{F}$ est convexe le long des géodésiques $\{\mu_t\}_{t\in[0,1]}$ avec $\mu_t := ((1-t)\mathrm{Id} + tT)\#\mu$ (# désigne l'opération de pousser-en-avant), quand $\mu \in \mathcal{P}(X)$ et T est l'application de transport optimal de μ à une autre mesure ν.

Il est très plausible que la fonctionnelle d'énergie libre associée à notre définition alternative $\mathbb{W}^{\mathrm{int}}$ de l'énergie (voir le chapitre 4) soit effectivement convexe par déplacement (pour une bonne notion de transport optimal sur les processus ponctuels aléatoires) et a donc un unique minimiseur. Il resterait à montrer que cette propriété s'étend à la "vraie" fonction de taux $\mathcal{F}_\beta$.

Plus généralement, la question de l'unicité des minimiseurs est d'importance physique puisqu'une réponse négative (plusieurs minimiseurs) correspond physiquement à une *transition de phase* : différents comportements limites possibles coexistent à une température donnée.

c. Théorème central limite dans les cas logarithmiques

Un champ d'étude récemment actif consiste à établir un Théorème Central Limite (TCL) pour les fluctuations de statistiques linéaires i.e. pour la loi de $\sum_{i=1}^N f(x_i) - N\int f d\mu_{\mathrm{eq}}$ (pour

une fonction test f assez régulière) given test function f, dans le contexte des *log-gases* $1D$ et $2D$, ou pour des modèles de matrices aléatoires. On peut citer par exemple [Joh98, For99, Shc13, Shc14, AHM15, BG13b, BG13a], qui ne couvrent pas toutes les valeurs de β (en dimension 2) et imposent généralement des conditions fortes de régularité sur le potentiel V.

La transformée de Laplace de la loi de ces fluctuations sous la mesure de Gibbs $\mathbb{P}_{N,\beta}$ coïncide avec la mesure de Gibbs d'un *log-gas* où le potentiel a été perturbé. Dans un travail en cours avec Sylvia Serfaty, on cherche à établir un TCL pour les *log-gases* en dimension 1 et 2 en utilisant notamment les méthodes présentées plus haut, qui permettraient entre autres d'assouplir les conditions sur V (par exemple sans supposer V analytique) et de s'appliquer indifféremment à toutes les valeurs de $\beta > 0$.

d. PGD local pour le *log-gas* à une dimension

Le comportement microscopique local (i.e. avec des moyennes réalisées à échelle mésoscopique arbitrairement petite comme dans) pour les *log-gases* $1D$ a été établi dans [BEY14, BEY12], où les auteurs prouvent l'universalité (en fonction du potentiel) et établissent des lois locales (dans le même sens que) précises jusqu'à l'échelle la plus fine $N^{-1+\varepsilon}$. Un objectif serait de retrouver ces résultats par des arguments vartationnels plus "physiques" similaires à ceux mis en œuvre dans [Leb15a] et d'obtenir de plus un principe de grandes déviations à toute échelle mésoscopique. Un avantage serait que nos méthodes ne nécessitent pas d'hypothèses fortes de régularité sur V et sont, pour l'essentiel, insensibles au nombre de *cuts* (i.e. au nombre de composantes connexes du support Σ de μ_{eq}).

La principale difficulté rencontrée est d'obtenir un bon contrôle de la décroissance du champ électrique (ou de la transformée de Stieltjes, ce qui est essentiellement le même objet) le long de l'axe "supplémentaire" (quand on étend $\mathbb{R}$ en $\mathbb{R}^{1+1}$), afin de permettre aux techniques d'écrantage de fonctionner. Pour l'instant, les résultats obtenus ne permettent que de descendre à certaines échelles mésoscopiques $N^{-\varepsilon}$, mais pas aux échelles les plus fines $N^{-1+\varepsilon}$.

e. Gaz de Riesz hypersinguliers

Une question naturelle est de chercher à généraliser le principe de grandes déviations de [LS15] à des potentiels d'interaction plus généraux. Dans un travail en cours avec Doug Hardin, Edward Saff et Sylvia Serfaty, on traite le cas où g est un potentiel de Riesz *hypersingulier* du type $g(x-y) = |x-y|^{-s}$ avec $s > d$. Ce potentiel est plus répulsif (puisque la singularité en 0 est de plus en plus forte quand s est grand) et décroît plus vite à l'infini (il n'est plus à longue portée, ce qui simplifie la localisation de l'énergie). Le comportement macroscopique est différent (il n'y a pas, en général, de mesure d'équilibre à support compact), mais à l'échelle microscopique on peut établir un résultat analogue à celui de [LS15] concernant les grandes déviations des champs empiriques.

1.3.3 Pour rêver

Pour conclure, on mentionne deux questions qui paraissent particulièrement intéressantes, mais aussi particulièrement difficiles à résoudre.

a. Processus ponctuels limites en dimension 2.

On se place ici dans le cas du 2DOCP. En dimension 2, on connaît l'existence d'un processus ponctuel (non-moyenné) limite, dans le cas de l'ensemble de Ginibre et plus généralement du "Random Normal Matrix model" tel qu'étudié dans [AHM11, AHM15]. Ces résultats correspondent à une valeur précise de β, et la question de l'existence de la limite en loi des processus

ponctuels limites non-moyennés (tels que définis dans (1.2.1)) est ouverte pour toutes les autres valeurs de β. Cette question est indirectement liée à l'unicité des minimiseurs de $\mathcal{F}_\beta$ en dimension 2, puisque l'unicité impliquerait (via le principe de grandes déviations) que les processus moyennés admettent une limite, qui serait un candidat évident pour être la limite des processus non-moyennés.

b. Le problème de cristallisation

Ce problème est bien connu : il consiste à démontrer rigoureusement que les configurations qui minimisent l'énergie (en dimension 2) forment (ou convergent vers, si l'on étudie la limite $N \to \infty$) un réseau triangulaire (appelé aussi réseau d'Abrikosov). On renvoie à l'article de synthèse [BL15] pour une présentation de ces questions et de la littérature concernée.

Dans [SS12], il est démontré, en utilisant des résultats poussés de théorie des nombres, que le réseau triangulaire est l'unique minimiseur de $\mathbb{W}$ parmi les **réseaux** de densité fixée, mais la question de minimiser $\mathbb{W}$ sur les configurations de points (de densité moyenne fixée, par exemple) reste ouverte. Il est d'ailleurs clair que le minimiseur ne saurait être unique, puisqu'une petite perturbation d'une configuration de points laisse $\mathbb{W}$ inchangé. Il est peut-être plus pratique de poser la question au niveau des processus et de chercher à minimiser $\mathbb{W}^{\mathrm{elec}}$ parmi les processus ponctuels aléatoires stationnaires d'intensité fixée. On peut commencer par tenter de répondre à la même question pour la fonctionnelle alternative $\mathbb{W}^{\mathrm{int}}$ (telle que définie dans [Leb15b]) qui a le mérite de posséder une expression explicite en termes de la fonction de corrélation à deux points.

Enfin, une forme (très faible) de la conjecture de cristallisation serait la suivante :

Conjecture 2. *Les minimiseurs de $\mathbb{W}^{\mathrm{elec}}$ sont d'entropie relative spécifique infinie.*

Bien entendu, le processus ponctuel aléatoire associé à un réseau est d'entropie infinie. Montrer que $\mathsf{ent}[P|\mathbf{\Pi}^1] = +\infty$ pour tout minimiseur de $\mathbb{W}^{\mathrm{elec}}$ serait une indication de la nature "ordonnée" desdits minimiseurs.

Chapitre 2

Grandes déviations pour les champs empiriques

Ce chapitre est constitué de l'article "Large Deviation Principle for Empirical Fields of Log and Riesz gases" [LS15] écrit avec S. Serfaty.

Dans ce chapitre, on fait une confusion terminologique entre un processus ponctuel et sa loi. Ici, un élément de $\mathcal{P}(\mathcal{X})$ est appelé *point process* alors qu'il s'agit techniquement de *la loi* d'un processus ponctuel (un *random point process*, ou *law of a point process*). La terminologie et les notations sont cohérentes dans tout le chapitre, mais diffèrent de celles des chapitres suivants.

Contents

2.1 Introduction

2.1.1 General setting and main results

We consider the Hamiltonian of a system of N points in the Euclidean space $\mathbb{R}^d$ ($d \geq 1$) interacting via logarithmic, Coulomb or Riesz pairwise interactions, in a potential V:

$$\mathcal{H}_N(x_1, \dots, x_N) := \sum_{1 \leq i \neq j \leq N} g(x_i - x_j) + N \sum_{i=1}^{N} V(x_i), \quad x_i \in \mathbb{R}^d,$$

where the interaction kernel is given by either

$$g(x) = -\log|x|, \quad \text{in dimension } d = 1, \tag{2.1.1}$$

or

$$g(x) = -\log|x|, \quad \text{in dimension } d = 2, \tag{2.1.2}$$

or in general dimension

$$g(x) = \frac{1}{|x|^s}, \quad \max(0, d-2) \leq s < d. \tag{2.1.3}$$

Whenever the parameter s appears, it will be with the convention that s is taken to mean 0 if we are in the cases (2.1.1) or (2.1.2). The potential V is a confining potential, growing fast enough at infinity, on which we shall make assumptions later.

We are interested in proving a Large Deviation Principle (LDP) for the Gibbs measure associated to this Hamiltonian

$$d\mathbb{P}_{N,\beta}(x_1, \dots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\frac{\beta}{2} N^{-\frac{s}{d}} \mathcal{H}_N(x_1, \dots, x_N)} dx_1 \dots dx_N, \tag{2.1.4}$$

where $\beta > 0$ is a constant that represents an inverse temperature, and the temperature scaling $\beta N^{-s/d}$ (understood with the convention $s = 0$ in cases (2.1.1)–(2.1.2)) is chosen to obtain non-trivial results.

In the case (2.1.1), this Gibbs measure corresponds to a "1D log-gas" system, also called a "β-ensemble". As is well known, particular instances of these occur in random matrix theory, for example when $\beta = 1, 2, 4$ with a quadratic potential V (with the GOE, GUE, GSE ensembles) and they have been intensively studied. In the case (2.1.2) it corresponds to a two-dimensional log-gas or Coulomb gas or "one-component plasma", a particular instance being the Ginibre

ensemble of random matrices obtained with the choice $\beta = 2$ and V quadratic. For a general presentation of these we refer to the textbooks [Meh04, For10, AGZ10] and the foundational papers [Wig55, Dys62] where the connection between the law of the eigenvalues of random matrices and Coulomb gases was first noticed. A related version, with particles of opposite signs, also called classical Coulomb gas is also a fundamental model of statistical mechanics, cf. the review [Spe97] and references therein.

The case $d \geq 3$ and $s = d - 2$ corresponds to a higher dimensional Coulomb gas, which can be seen as a toy (classical) model for matter. The study of these was pioneered e.g. in [PS72, JLM93, LL69, LN75].

Finally, the case (2.1.3) can be seen as a generalization of the Coulomb case with more general Riesz interactions. By extension, we may call such a system a Riesz gas. Motivations for studying Riesz gases are numerous in the physics literature, see for instance [Maz11, BBDR05]: they can also correspond to physically meaningful particle systems, such as systems with Coulomb interaction constrained to a lower-dimensional subspace. Another important motivation for studying such systems is the topic of approximation theory. We refer to the forthcoming monograph of Borodachev-Hardin-Saff [BHS], the review papers [SK97, BHS12] and references therein. In that context such systems are mostly studied on the d-dimensional sphere or torus.

In all cases of interactions, the ensembles governed by the law (2.1.4) are considered as difficult systems in statistical mechanics because the interactions they contain are truly long-range, and the points are not constrained to a lattice. As always in statistical mechanics, one would like to understand if there are phase-transitions for particular values of the (inverse) temperature β. For such systems, one may expect what physicists call a liquid for small β, and a crystal for large β, cf. for instance [HM13]. In the case of the two-dimensional Coulomb gas (or one-component plasma) there are in fact important controversies in the physic communities (see for instance [Sti98]) as to whether there is a finite β for which the system cristallizes, and what its value is. This crystallization phenomenon has only been justified numerically, the first instance seems to be [BST66]. The exact definition of crystallization matters a lot of course, the one taken by physicists is that of non-decay of the two-point correlation function, a rather weak criterion. One consequence of the results we prove here will be that there is no finite temperature of cristallization with the strict definition of the configuration being a crystal. In other words crystallization in that sense can happen only in the limit $\beta \to \infty$. In one dimension, the result is complete thanks to the result of [Leb15c, Leb15b]: we will see that the crystallization happens if and only if β is infinite.

Such systems naturally exhibit two lengthscales: a mesoscopic (or macroscopic) scale corresponding to the scale of confinement of the potential V – here 1 – at which one can study the average (or mean-field) distributions of the points, and a microscopic scale corresponding to the interparticle distance – here $N^{-1/d}$ – at which one can study the "local laws" for the distributions of points. Of course, crystallization is a phenomenon that happens at the microscopic or local scale.

Our approach in this paper is in line with the approaches of [SS15a] for the case (2.1.1), [SS15b] for the case (2.1.2), [RS15] for the Coulomb cases, and [PS15] for the general Riesz case. As in those previous papers, it allows to treat the case of arbitrary β and quite general V. As in [PS15], it also allows to treat all cases (2.1.1)–(2.1.2)–(2.1.3) in one unified approach.

Prior to these works, the case (2.1.1) is certainly the one that has been most intensively studied and for general values of β and general V's. This culminated with very detailed results in the most recent papers which obtain on the one hand very precise asymptotic expansions of the partition function [BG13b, BG13a, Shc13, Shc14, BFG13] and on the other hand complete characterizations of the point processes at the microscopic level, including spacing between the points [VV09, BEY14, BEY12]. The case (2.1.2) has been studied for general V in the particular

case $\beta = 2$, which allows to use determinantal representations, and characterize the limiting processes at the microscopic level [Gin65, BS09]. Central Limit Theorems for fluctuations were also obtained [Joh98, RV07, AHM11, AHM15]. The case (2.1.2) without temperature (formally $\beta = \infty$) is also well understood with rigidity results on the number of points in microscopic boxes [AOC12, RNS15]. There was however little on the case of general β (away from the determinantal case) for (2.1.2) or for any β with the Riesz interaction kernel.

It is well-known since [Cho58] (see [ST97] for the logarithmic case, or [Ser15, Chap. 2] for a simple proof in the general case) that to leading order, under suitable assumptions on V, and if $s < d$ in (2.1.3), we have

$$\min \mathcal{H}_N = N^2 \mathcal{I}(\mu_V) + o(N^2) \tag{2.1.5}$$

in the limit $N \to \infty$, where

$$\mathcal{I}(\mu) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(x-y)\, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}^d} V(x)\, d\mu(x) \tag{2.1.6}$$

is the mean-field energy functional defined for Radon measures μ, and the so-called *equilibrium measure* μ_V is the minimizer of $\mathcal{I}$ in the space of probability measures on $\mathbb{R}^d$, denoted $\mathcal{P}(\mathbb{R}^d)$. This is true only for $s < d$, which is the condition for (2.1.6) to make sense and to have a minimizer. We will always assume that μ_V is a measure with a Hölder continuous density on its support, we abuse notation by denoting its density $\mu_V(x)$ and we also assume that its support Σ is a compact set with a nice boundary. We allow for several connected components of Σ (also called the multi-cut regime in the logarithmic case of dimension 1). The detailed assumptions are listed in Section 2.2.1.

An LDP for the law of the "empirical measure" $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ under the Gibbs measure

$$\frac{1}{Z_{N,\beta}} e^{-\frac{\beta}{2}\mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N$$

(i.e. (2.1.4) but with a different temperature scaling) also holds: there exists an LDP at speed N^2 with rate function $\beta/2(\mathcal{I} - \mathcal{I}(\mu_V))$. This was shown in [HP00, BAG97] (for the case (2.1.1)), [BAZ98] (for the case (2.1.2) for $\beta = 2$), [CGZ14] for a more general setting including the Riesz one, see also [Ser15, Chap. 2] for a simple presentation.

This settles in some sense the understanding of the leading order macroscopic behavior of these systems: at finite temperature, all empirical measures $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ ressemble the equilibrium measure μ_V, except with exponentially small probability.

On the other hand, the behavior of the Hamiltonian $\mathcal{H}_N$ and of its minimizers has been understood at the next order and at the microscopic scale where the points become well-separated, i.e. $N^{-1/d}$. First, it was remarked in [SS15b] (for the case (2.1.2)), [SS15a] (for the case (2.1.1)), [RS15] (for all the Coulomb cases), and [PS15] (for the general situation) that $\mathcal{H}_N$ can be exactly split into the sum of a constant leading order term and a typically next order term, as

$$\mathcal{H}_N(x_1,\dots,x_N) = N^2\mathcal{I}(\mu_V) + 2N\sum_{i=1}^N \zeta(x_i) + N^{1+s/d} w_N(x_1,\dots,x_N) \tag{2.1.7}$$

in the case (2.1.3) and respectively

$$\mathcal{H}_N(x_1,\dots,x_N) = N^2\mathcal{I}(\mu_V) - \frac{N}{d}\log N + 2N\sum_{i=1}^N \zeta(x_i) + N w_N(x_1,\dots,x_N) \tag{2.1.8}$$

in the cases (2.1.1)–(2.1.2), where w_N will be defined in (2.2.22), and ζ is a function depending only on V, which is nonnegative and vanishes exactly in a set that we denote ω and which

contains Σ (precise definitions will be given in (2.2.4) in Section 2.2.1). It was shown in [SS15b, SS15a, RS15, PS15] that the object w_N has a limit $\mathcal{W}$ as $N \to \infty$ called the "renormalized energy", which is expressed in terms of the potential generated by the limits of configurations blown-up at the scale $N^{1/d}$. Its precise definition is given in Section 2.2.3. As a consequence, minimizers of $\mathcal{H}_N$ converge, after blow-up, to minimizers of $\mathcal{W}$. It is expected (but this remains at the level of a conjecture except in dimension 1), that at least in low dimensions, the minimum of $\mathcal{W}$ is achieved at simple (Bravais) lattice configurations, i.e. minimizers of $\mathcal{W}$ are expected to be crystalline and ressemble perfect lattices. This settled in [SS15b, SS15a, RS15, PS15] the analysis of the microscopic behavior of minimizers in the formal case $\beta = \infty$ by connecting $\mathcal{H}_N$ to $\mathcal{W}$ and minimizers of $\mathcal{H}_N$ to the cristallization question of minimizing $\mathcal{W}$. The information obtained this way on $\mathcal{H}_N$ also allowed to deduce information on the case with temperature i.e. on $\mathbb{P}_{N,\beta}$: an asymptotic expansion of the logarithm of the partition function $Z_{N,\beta}$ and a qualitative description of the limit of $\mathbb{P}_{N,\beta}$, which become sharp only as $\beta \to \infty$, and hints at a crystallization phenomenon. In dimension 1, the crystallization was rigorously established in [SS15a], using the result of [Leb15c].

Our goal here is to obtain a complete LDP that lies at this next order and is valid for all β. It describes the configurations after blow-up at the microscopic scale around points in the support Σ of the equilibrium measure μ_V and gives a rate function on the random point processes obtained via the blown-up limits. Equivalently it is an LDP "at the process level" also called "type-III LDP", cf. for example [RAS09]. For general reference on large deviations one may see e.g. [DZ10]. The idea of using large deviations methods for such systems already appeared in [BBDR05] where results of the same flavor but at a more formal level are presented.

a. Preliminary notation.

Before giving a statement, let us introduce some notation. We denote by $\mathcal{X}$ the set of locally finite (not necessarily simple) point configurations in $\mathbb{R}^d$, or equivalently the set of purely atomic Radon measures giving an integer mass to singletons, cf. [DVJ88]. The topology of vague convergence induces a topology on $\mathcal{X}$. A point process is then defined to be a probability measure on $\mathcal{X}$, i.e. an element of $\mathcal{P}(\mathcal{X})$, cf. [DVJ88]. We can then see configurations $(x_1, \dots, x_N)$ as elements of the space $\mathcal{X}$ of discrete (finite or infinite) point configurations in $\mathbb{R}^d$. When starting from an N-uple of points $(x_1, \dots, x_N)$, we first rescale the associated finite configuration $\sum_{i=1}^N \delta_{x_i}$ by a factor $N^{1/d}$ and then define the map

$$\begin{array}{rcl} i_N : (\mathbb{R}^d)^N & \to & \mathcal{P}(\Sigma \times \mathcal{X}) \\ (x_1, \dots, x_N) & \mapsto & \displaystyle\int_\Sigma \delta_{(x, \theta_{N^{1/d}x} \cdot (\sum_{i=1}^N \delta_{N^{1/d} x_i}))} dx \end{array} \tag{2.1.9}$$

where θ_λ denotes the action of translation by λ and δ is the Dirac mass.

The space $\mathcal{P}(\Sigma \times \mathcal{X})$ is defined as the space of "tagged point processes", where we keep as a tag the point $x \in \Sigma$ around which the configuration was blown up. It is equipped with the topology of weak convergence of measures on $\Sigma \times \mathcal{X}$ (the topology is discussed further in Section 2.2.4). If $\bar{P}$ is a tagged point process we will always assume that the first marginal of $\bar{P}$ is the normalized Lebesgue measure on Σ. We will generally denote with bars the quantities that correspond to tagged processes, and without bars the quantities that correspond to non-tagged processes. We denote by $\mathcal{P}_s(\mathcal{X})$ the set of translation-invariant, or stationary point processes. We also call stationary a tagged point process $\bar{P}$ such that the disintegration measure $\bar{P}^x$ (cf. [AGS08, Section 5.3] for a definition) is stationary for (Lebesgue-)a.e. $x \in \Sigma$ and denote by $\mathcal{P}_s(\Sigma \times \mathcal{X})$ the set of stationary tagged point processes.

In [PS15] and previous articles, a renormalized energy $\mathcal{W}$ was defined at the level of the potentials generated by a point configuration. In the particular case (2.1.1), it can be interpreted

as the L^2 norm of the Stieltjes transform, properly normalized (cf. [SS15a]). This energy may be "projected down" to a renormalized energy $\mathbb{W}$ defined on point configurations themselves (all definitions will be recast more completely below in Section 2.2.3). One can then extend it as an energy on point processes $P \in \mathcal{P}(\mathcal{X})$ by

$$\widetilde{\mathbb{W}}_m(P) := \int \mathbb{W}_m(\mathcal{C})\, dP(\mathcal{C}) \tag{2.1.10}$$

where we keep as index m the intensity of the point process, or equivalently the background density. We then define the renormalized energy of a tagged point process as

$$\overline{\mathbb{W}}_{\mu_V}(\bar{P}) := \frac{1}{c_{d,s}} \int_\Sigma \widetilde{\mathbb{W}}_{\mu_V(x)}(\bar{P}^x) dx \tag{2.1.11}$$

where $c_{d,s}$ is a constant depending only on d, s.

Next, we define a specific relative entropy as the infinite-volume limit of the usual relative entropy with respect to some reference measure. Below C_N denotes the hypercube of sidelength N, $[-N/2, N/2]^d$ and $|U|$ denote the Lebesgue measure (or volume) of a set U.

Definition 2.1.1. *Let P be a stationary point process on $\mathbb{R}^d$. The relative specific entropy $\mathsf{ent}[P|\mathbf{\Pi}^1]$ of P with respect to $\mathbf{\Pi}^1$, the Poisson point process of uniform intensity 1, is given by*

$$\mathsf{ent}[P|\mathbf{\Pi}^1] := \lim_{N\to\infty} \frac{1}{|C_N|} \mathrm{Ent}\left(P_{|C_N}|\mathbf{\Pi}^1_{|C_N}\right) \tag{2.1.12}$$

where $P_{|C_N}$ denotes the process induced on (the point configurations in) C_N, and $\mathrm{Ent}(\cdot|\cdot)$ denotes the usual relative entropy (or Kullbak-Leibler divergence) of two probability measures defined on the same probability space.

We take the appropriate sign convention for the entropy so that $\mathsf{ent} \geq 0$ i.e. if μ, ν are two probability measures defined on the same space we let

$$\mathrm{Ent}\,(\mu|\nu) := \int \log \frac{d\mu}{d\nu} d\mu$$

if μ is absolutely continuous with respect to ν and $+\infty$ otherwise. It is known (see e.g. [RAS09]) that the limit (2.1.12) exists for all stationary processes, hence the relative specific entropy is well-defined, and also that the functional $P \mapsto \mathsf{ent}[P|\mathbf{\Pi}^1]$ is affine lower semi-continous and that its sub-level sets are compact.

We end this section by recalling the definition of LDP.

Definition 2.1.2. *A sequence $\{\mu_N\}_N$ of probability measures on a metric space X is said to satisfy a Large Deviation Principle (LDP) at speed r_N with rate function $I : X \to [0, +\infty]$ if the following holds for any $A \subset X$*

$$-\inf_{\mathring{A}} I \leq \liminf_{N\to\infty} \frac{1}{r_N} \log \mu_N(A) \leq \limsup_{N\to\infty} \frac{1}{r_N} \log \mu_N(A) \leq -\inf_{\overline{A}} I,$$

where $\mathring{A}$ (resp. $\overline{A}$) denotes the interior (resp. the closure) of A. The functional I is said to be a "good rate function" if it is lower semi-continuous and has compact sub-level sets.

We refer to [DZ10] for a detailed treatment of the theory of large deviations and to [RAS09] for an introduction to the applications of LDP's in the statistical physics setting.

b. Main result and consequences.

We may now state our main LDP result.

Theorem 9 (Large Deviation Principle)**.** *Assume V satisfies the assumptions of Section 2.2.1. Let $\overline{\mathfrak{P}}_{N,\beta}$ be the random tagged empirical field associated to the Gibbs measure by pushing forward $\mathbb{P}_{N,\beta}$ by the map* (2.1.9)*. Then for any $\beta > 0$, the sequence $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ satisfies a large deviation principle at speed N with good rate function $\beta(\overline{\mathcal{F}}_\beta - \inf \overline{\mathcal{F}}_\beta)$ where*

$$\overline{\mathcal{F}}_\beta(\bar{P}) := \frac{1}{2}\overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \frac{1}{\beta}\left(\int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]\,dx + 1 - |\Sigma|\right). \tag{2.1.13}$$

A first consequence of the LDP is that in the limit $N \to \infty$, the Gibbs measure (more precisely the limit of $\overline{\mathfrak{P}}_{N,\beta}$) concentrates on minimizers of $\overline{\mathcal{F}}_\beta$. Also, it is easy to see that $\bar{P}$ minimizes $\overline{\mathcal{F}}_\beta$ if and only if its disintegration measures $\bar{P}^x$ minimize for a.e. $x \in \Sigma$ the non averaged rate function

$$\mathcal{F}_\beta(P) := \frac{1}{2c_{d,s}}\widetilde{\mathbb{W}}_{\mu_V(x)}(P) + \frac{1}{\beta}\mathsf{ent}[P|\mathbf{\Pi}^1]. \tag{2.1.14}$$

Identifying the minimizers of either $\mathcal{F}_\beta$ or $\overline{\mathcal{F}}_\beta$ is a hard question in general, even if one knew what the minimizers of $\mathbb{W}$ are. However, one readily sees the effect of the temperature: in the minimization there is a competition between the term $\overline{\mathbb{W}}_{\mu_V}$ or $\widetilde{\mathbb{W}}_{\mu_V(x)}$ based on the renormalized energy, and which is thus expected to favor crystalline (hence very ordered) configurations, and the entropy term which to the contrary favors disorder. The temperature determines the relative weight of these two competing effects: as $\beta \to 0$ (i.e. temperature gets large) the entropy term dominates and configurations can be expected to behave like a Poisson point process, while as $\beta \to \infty$ (i.e. temperature gets very small), the renormalized energy dominates, and configurations can be expected to crystallize. In particular, as we will observe later, our result implies that crystallization, in the strict sense of configurations being cristalline, should not be expected to happen at a finite (fixed) β because cristalline configurations give rise to an infinite entropy. Thus the cristallization expected in the physics literature can only be a weaker form of crystallization such as a transition to slower decaying or non-decaying correlation functions.

Our result naturally raises two questions: the first is to understand better $\overline{\mathbb{W}}$ and its minimizers and the second is to better understand the specific relative entropy, about which not much seems to be known in general.

In the particular case of (2.1.1) with a quadratic potential $V(x) = x^2$, the equilibrium measure is known to be the semi-circular law whose density is given by

$$x \mapsto \frac{1}{2\pi}\mathbf{1}_{[-2,2]}\sqrt{4-x^2},$$

and the limiting point process at the microscopic level around a point $x \in [-2,2]$ (let us emphasize that in this case there is *no* averaging over translations) is identified for any $\beta > 0$ in [VV09] to be the "sine-β" point process, which we will denote by $\mathrm{Sine}_\beta(x)$ (so that $\mathrm{Sine}_\beta(x)$ has intensity $\frac{1}{2\pi}\sqrt{4-x^2}$). They are all equal in law up to rescaling and we denote by Sine_β the corresponding process with intensity 1. It is also proven to be the limit of the β-circular ensemble [Nak14]. A corollary of our result is then a new characterization of these processes:

Corollary 2.1.3 (Sine-beta process)**.** *For any $\beta > 0$, the tagged point process*

$$\overline{\mathrm{Sine}_\beta} := \int_{[-2,2]} \delta_{(x,\mathrm{Sine}_\beta(x))}$$

minimizes the rate function $\overline{\mathcal{F}}_\beta$ *among tagged point processes in* $\mathcal{P}(\Sigma \times \mathcal{X})$. *The point process* Sine_β *minimizes*

$$\mathcal{F}_\beta(P) = \frac{1}{4\pi}\widetilde{\mathbb{W}}_1(P) + \frac{1}{\beta}\mathsf{ent}[P|\mathbf{\Pi}^1]$$

among stationary point processes of intensity 1 *in* $\mathbb{R}$.

The one other case in which the limiting Gibbsian point process is identified is the case (2.1.2) with V quadratic, which gives rise to the so-called "Ginibre point process" [Gin65, BS09]. In this case we also obtain the new characterization:

Corollary 2.1.4 (Ginibre process)**.** *The Ginibre point process minimizes*

$$\mathcal{F}_2(P) = \frac{1}{4\pi}\widetilde{\mathbb{W}}_1(P) + \frac{1}{2}\mathsf{ent}[P|\mathbf{\Pi}^1]$$

among stationary point processes of intensity 1 *in* $\mathbb{R}^2$.

Corollaries 2.1.3 and 2.1.4 are proven in Section 2.4.3.

As mentioned above, the infimum $\inf \overline{\mathcal{F}}_\beta$ is unknown in general and its determination seems to be a difficult problem. However we know exactly how $\overline{\mathcal{F}}_\beta$ depends on μ_V hence on V, because we know how the $\overline{\mathbb{W}}$ and entropy terms scale in terms of the equilibrium measure density (which is the same as the point process intensity). For any $m > 0$, we let σ_m be the map which rescales a point configuration by the factor $m^{1/d}$, i.e. turns a configuration of density m into one of intensity 1. Then we may consider $\bar{P}'$ the push-forward of $\bar{P}$ by the map on $\Sigma \times \mathcal{X}$

$$(x, \mathcal{C}) \mapsto (x, \sigma_{\mu_V(x)}\mathcal{C}).$$

In the case (2.1.3), the rescaling yields

$$\overline{\mathcal{F}}_\beta(\bar{P}) = \frac{1}{2c_{d,s}}\int_\Sigma \widetilde{\mathbb{W}}_1(\bar{P}'^x)\mu_V(x)^{1+s/d}\,dx + \frac{1}{\beta}\left(\int_\Sigma \mathsf{ent}[\bar{P}'^x|\mathbf{\Pi}^1]\mu_V(x)\,dx + \int_\Sigma \mu_V(x)\log\mu_V(x)\,dx\right). \quad (2.1.15)$$

In the cases (2.1.1)–(2.1.2), the rescaling yields

$$\overline{\mathcal{F}}_\beta(\bar{P}) = \frac{1}{2}\left(\frac{1}{c_{d,s}}\int_\Sigma \widetilde{\mathbb{W}}_1(\bar{P}'^x)\mu_V(x)\,dx - \frac{1}{d}\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx\right) + \frac{1}{\beta}\left(\int_\Sigma \mathsf{ent}[\bar{P}'^x|\mathbf{\Pi}^1]\,\mu_V(x)\,dx + \int_\Sigma \mu_V(x)\log\mu_V(x)\,dx\right)$$

and in these particular cases, the terms recombine into

$$\overline{\mathcal{F}}_\beta(\bar{P}) = \int_\Sigma \left(\frac{1}{2c_{d,s}}\widetilde{\mathbb{W}}_1(\bar{P}'^x) + \frac{1}{\beta}\mathsf{ent}[\bar{P}'^x|\mathbf{\Pi}^1]\right)\mu_V(x)\,dx + \left(\frac{1}{\beta} - \frac{1}{2d}\right)\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx. \quad (2.1.16)$$

There has been a lot of interest recently in proving "universality results" for such systems, i.e. proving that their microscopic behavior is independent of V, hence of μ_V. Such results have been obtained in the cases (2.1.1) in [BEY14, BEY12, BFG13], etc. In the above formulae, the terms

not involving μ_V are independent of μ_V and V, hence universal. In the cases (2.1.1)–(2.1.2), since μ_V is a probability measure, one can deduce from (2.1.16) that

$$\boxed{\min \overline{\mathcal{F}}_\beta = \min\left(\frac{1}{2c_{d,s}}\widetilde{\mathbb{W}}_1 + \frac{1}{\beta}\mathsf{ent}[\cdot|\mathbf{\Pi}^1]\right) + \left(\frac{1}{\beta} - \frac{1}{2d}\right)\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx.}$$

Hence the dependence of $\min \overline{\mathcal{F}}_\beta$ in μ_V is just an additive constant which happens to vanish when $\beta = 2$ in dimension 1 and $\beta = 4$ in dimension 2. This is in agreement with the universality known in these cases: minimizers of $\overline{\mathcal{F}}_\beta$ are independent of V hence universal.

In contrast, in (2.1.15) μ_V comes as a multiplicative weight and in the minimization of $\overline{\mathcal{F}}_\beta$ the relative weights of the energy $\widetilde{\mathbb{W}}_1$ and of the entropy depend on μ_V : this can be seen as creating an effective temperature $\beta\mu_V^{s/d}$. Hence the minimizers of $\overline{\mathcal{F}}_\beta$ will not be universal, and this indicates that universality, in the sense previously used, fails in higher dimensional Coulomb cases or in Riesz cases. In other words, universality seems to be directly tied with the logarithmic nature of the interaction. We note that no positive or negative prediction in that direction seemed to have been proposed.

A byproduct of the LDP is naturally the existence of a thermodynamic limit for these systems, i.e. an asymptotic term in N in the expansion of $\log Z_{N,\beta}$, which in view of the above discussion is given by :

Corollary 2.1.5 (Thermodynamic limit)**.** *Under the same assumptions, we have, as $N \to \infty$,*

$$\log Z_{N,\beta} = -\frac{\beta N^{2-\frac{s}{d}}}{2}\mathcal{I}(\mu_V) - N\beta \min \overline{\mathcal{F}}_\beta + o((\beta+1)N) \tag{2.1.17}$$

in the cases (2.1.3)*; and in the cases* (2.1.1)–(2.1.2)

$$\log Z_{N,\beta} = -\frac{\beta N^2}{2}\mathcal{I}(\mu_V) + \frac{\beta N}{2d}\log N - N\beta \min \overline{\mathcal{F}}_\beta + o((\beta+1)N)$$

or more explicitly

$$\begin{aligned}\log Z_{N,\beta} = -\frac{\beta N^2}{2}\mathcal{I}(\mu_V) + \frac{\beta N}{2d}\log N - N\beta \min\left(\frac{1}{2c_{d,s}}\widetilde{\mathbb{W}}_1 + \frac{1}{\beta}\mathsf{ent}[\cdot|\mathbf{\Pi}^1]\right)\\ - N\beta\left(\frac{1}{\beta} - \frac{1}{2d}\right)\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx + o((\beta+1)N).\end{aligned} \tag{2.1.18}$$

Here the $o(1)$ tend to zero as $N \to \infty$ independently of β.

This provides an asymptotic expansion of the free energy (i.e. $-\frac{1}{\beta}\log Z_{N,\beta}$) up to order N, where in view of (2.1.13), the order N term itself has the structure of a free energy.

The existence of such a thermodynamic limit had been known for a long time for the two and three dimensional Coulomb cases [LN75, SM76, PS72]. Our formulae are to be compared with the recent results of [Shc13, Shc14, BG13b, BG13a, BFG13] in the dimension 1 logarithmic case. These authors obtain asymptotic expansions of $\log Z_{N,\beta}$ to much lower orders than this, however they make quite strong assumptions on the regularity of the potential V, and sometimes the coefficients are not easy to explicitly compute. Since in this setting (2.1.1), $\log Z_{N,\beta}$ is known explicitly for $V(x) = x^2$ via Selberg integrals, by comparing to (2.1.18) this allows to identify the value of $\mathcal{F}_\beta(\mathrm{Sine}_\beta) = \min \mathcal{F}_\beta$ (where $\mathcal{F}_\beta$ is defined in (2.1.14)), and then to immediately deduce the explicit coefficients in the expansion of $\log Z_{N,\beta}$ up to order N for general V (and

the difference in the order N coefficient only involves the difference in $\int \mu_V \log \mu_V$). In the case (2.1.2) our result can also be compared to the formal result of [ZW06].

In both logarithmic cases, we recover in (2.1.18) the cancellation of the order N term when $\beta = 4$ in dimension 2 and $\beta = 2$ in dimension 1 that was first observed in [Dys62, Part II, section II] and [ZW06], and when this happens then, V_1 and V_2 being two potentials satisfying our assumptions, we obtain

$$\log Z_{N,\beta}(V_2) - \log Z_{N,\beta}(V_1) = -\frac{\beta N^2}{2}(\mathcal{I}(\mu_{V_2}) - \mathcal{I}(\mu_{V_1})) + o((\beta+1)N)$$

in agreement with the well-known fact [BIZ80, EM03] that expansions of $\log Z_{N,\beta}$ corresponding to different potentials V then differ by an expansion in even powers of N only.

Finally, in the general case of Riesz gases (2.1.3), our result (2.1.17) seems to be the first rigorous one of its kind.

2.1.2 Proof outline

Using the splitting formula (2.1.7)–(2.1.8), one can factor out the constant terms from the Hamiltonian and the partition function, and reduce to studying only

$$d\mathbb{P}_{N,\beta}(x_1,\dots,x_N) = \frac{1}{K_{N,\beta}} e^{-\frac{\beta N}{2} w_N(x_1,\dots,x_N)} e^{-N\beta \sum_{i=1}^N \zeta(x_i)}\, dx_1 \dots dx_N \tag{2.1.19}$$

where

$$K_{N,\beta} = Z_{N,\beta} e^{\frac{1}{2}\beta N^{2-s/d}\mathcal{I}(\mu_V)} e^{-\frac{1}{2}\beta \frac{N}{d}\log N}$$

(here the second multiplicative term exists only in the cases (2.1.1)–(2.1.2)). It is already proven in [PS15, Theorem 6.] (note the different normalization of temperature there) that

$$|\log K_{N,\beta}| \le C_\beta \beta N \tag{2.1.20}$$

with C_β bounded on any interval $[\beta_0, +\infty)$ with $\beta_0 > 0$. We next define the reference measure $\mathbb{Q}_{N,\beta}$ as the probability measure on $(\mathbb{R}^d)^N$ with density

$$d\mathbb{Q}_{N,\beta}(x_1,\dots,x_N) := \frac{e^{-N\beta\sum_{i=1}^N \zeta(x_i)}}{\left(\int_{\mathbb{R}^d} e^{-N\beta\zeta(x)}\,dx\right)^N} dx_1 \dots dx_N. \tag{2.1.21}$$

The effect of ζ is that of confining the points to the set ω containing the support Σ of the equilibrium measure. Thus, one can think of $\mathbb{Q}_{N,\beta}$ as being essentially the N times tensor product of the normalized Lebesgue measure on ω, and of (2.1.19) as being formally

$$\frac{1}{K_{N,\beta}} e^{-\frac{\beta N}{2} w_N(x_1,\dots,x_N)} \prod_{i=1}^N \mathbf{1}_\omega(x_i) dx_i.$$

To prove an LDP, the standard method consists in evaluating the logarithm of the probability $\mathbb{P}_{N,\beta}(B(\bar{P},\varepsilon))$, where $\bar{P}$ is a given tagged point process, element of $\mathcal{P}(\Sigma\times\mathcal{X})$ (recall $(\mathbb{R}^d)^N$ embeds into this space via (2.1.9)) and $B(\bar{P},\varepsilon)$ is a ball of small radius ε around it, for a distance that metrizes the weak topology we are working with.
Since

$$\mathbb{P}_{N,\beta}(B(\bar{P},\varepsilon)) \simeq \frac{1}{K_{N,\beta}} \int_{i_N(x_1,\dots,x_N)\in B(\bar{P},\varepsilon)} e^{-\frac{\beta N}{2} w_N(x_1,\dots,x_N)} \prod_{i=1}^N \mathbf{1}_\omega(x_i) dx_i$$

we may formally write

$$\lim_{\varepsilon\to 0} \log \mathbb{P}_{N,\beta}(B(\bar{P},\varepsilon)) = -\log K_{N,\beta} - \frac{\beta N}{2} w_N(\bar{P}) + \lim_{\varepsilon\to 0} \log\Big(|\{(x_1,\dots,x_N)\in\omega^N, i_N(x_1,\dots,x_N)\in B(\bar{P},\varepsilon)\}|\Big). \quad (2.1.22)$$

Extracting this way the exponential of a function is the idea of the Varadhan integral lemma (cf. [DZ10, Theorem 4.3.1]), and works when the function is continuous. In similar contexts to ours, this is used e.g. in [Geo93, GZ93].

In (2.1.22) the term in the second line is the logarithm of the volume of point configurations whose associated "empirical field" is close to $\bar{P}$. By classical large deviations theorems, such a quantity is expected to be the entropy of $\bar{P}$. More precisely since we are dealing with blown-up configurations, or empirical fields, we need to use a relative specific entropy as defined above (cf. also [RAS09, Geo93]) as opposed to a usual entropy.

The most problematic term in (2.1.22) is the second one in the right-hand side, $w_N(\bar{P})$, which really makes no sense. The idea is that it should be close to $\overline{\mathbb{W}}(\bar{P})$ which is the well-defined quantity appearing in the rate function (2.1.13). If we were dealing with a continuous function of $\bar{P}$ then the replacement of $w_N(\bar{P})$ by $\overline{\mathbb{W}}(\bar{P})$ would be fine. However there are three difficulties here:

1. w_N depends on N and we need to take the limit $N\to\infty$,
2. this limit cannot be uniform because the quantities that define w_N becomes infinite when two points approach each other,
3. w_N is not adapted to the topology that we are working with, which is a weak topology which retains only local information on the point configurations, while w_N contains long-range interactions and does not depend only on local data of the points but on the whole configuration.

Thus, the approach outlined in (2.1.22) cannot work directly. Instead, we have to look again at the whole ball $B(\bar{P},\varepsilon)$ and to show that in that ball there is a logarithmically large enough volume of configurations for which we can replace w_N by $\overline{\mathbb{W}}(\bar{P})$. This will give a lower bound on $\log \mathbb{P}_{N,\beta}(B(\bar{P},\varepsilon))$ and the upper bound is in fact much easier to deduce from the previously known results of [PS15]. The second obstacle above, related to the discontinuity of the Hamiltonian near the diagonals of $(\mathbb{R}^d)^N$, is similar to the difficulty encountered in [BG99]. It is handled differently though, by controlling the difference between w_N and a version of it where the singularities are truncated at some small level η. This works out precisely because the renormalized energies are defined as limits as $\eta\to 0$ of quantities truncated at the level η. By controlling this difference thanks to the tools of [PS15], we are able to show that it is small often enough, i.e. the volume of the configurations where it is small is logarithmically large enough.

The third point above, the fact that the total energy is nonlocal in the data of the configuration, creates the most delicate difficulty. The way we circumvent it is via the "screening procedure" developed in [SS12, SS15b, SS15a, RS15, PS15]. Each configuration generates a potential, denoted H, and an "electric field" $E=\nabla H$, and the energy really corresponds to the (renormalized) integral of $|E|^2$. We show that thanks to the screening, we can always modify a bit each configuration so as to make the energy that it generates additive (hence local) in space, while not moving the configuration too far from $\bar{P}$ and not losing too much logarithmic volume in phase-space. This will be detailed in Sections 2.5 and 2.6.

In the end our result is a consequence of two intermediate results.

The first one is a large deviation result for the "reference" empirical field i.e. for the measure $\bar{\mathfrak{Q}}_{N,\beta}$, defined as the push-forward of $\mathbb{Q}_{N,\beta}$ by i_N, cf. (2.1.9) and (2.1.21). We let

$$c_{\omega,\Sigma} := \log|\omega| - |\Sigma| + 1, \quad (2.1.23)$$

where ω is the zero-set of ζ, as mentioned above.

Proposition 2.1.6. *For any $A \subset \mathcal{P}_s(\Sigma \times \mathcal{X})$, we have*

$$-\inf_{\mathring{A}\cap\mathcal{P}_{s,1}} \int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx - c_{\omega,\Sigma} \leq \liminf_{N\to\infty} \frac{1}{N}\log \bar{\mathfrak{Q}}_{N,\beta}(A)$$
$$\leq \limsup_{N\to\infty} \frac{1}{N}\log \bar{\mathfrak{Q}}_{N,\beta}(A) \leq -\inf_{\bar{P}\in\bar{A}} \int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx - c_{\omega,\Sigma}. \quad (2.1.24)$$

In (2.1.24) and in the rest of the paper, if A is a set of (tagged) configurations, $\mathring{A}$ denotes the interior of A and $\bar{A}$ denotes the closure of A. The meaning of the (technical) restriction $\mathring{A}\cap\mathcal{P}_{s,1}$ will be precised later, let us say that $\mathcal{P}_{s,1}(\Sigma\times\mathcal{X})$ denotes the set of tagged point process of total intensity 1 (i.e. there is an average number of 1 point by unit volume, see Section 2.2.4).

Quantities obtained by averaging a given configuration over translations as in (2.1.9), are called "empirical fields". The first large deviations principles for empirical fields seem to be stated in [Var88], [Föl88] and the relative specific entropy was then formalized by Föllmer and Orey [FO88] in the non-interacting discrete case (see also [RAS09] for another approach), by Georgii [Geo93] in the interacting discrete case and Georgii and Zessin [GZ93] in the interacting continuous case. In that light, the result of this proposition is not too surprising, however our setting differs from the one of [GZ93] in that the reference measure $\mathbb{Q}_{N,\beta}$ is not the restriction of a Poisson point process to a hypercube but somehow only approximates a Bernoulli point process on some domain ω - which is not a hypercube - with the possibility of some points falling outside ω. Moreover we want to study large deviations for tagged point processes (let us emphasize that our use of "tags" is not the same as the "marks" in [GZ93]) which requires an additional approximation argument. The proof of these successive adaptations to our context occupies Section 2.7.

Let us say a word about the choice of topology on $\mathcal{X}$. It is well known that large deviation principles hold for empirical fields after endowing $\mathcal{X}$ with a strong topology, namely the τ-topology (the initial topology on $\mathcal{X}$ associated to the maps $\mathcal{C}\mapsto f(\mathcal{C})$ for any bounded measurable function f which is local in the sense of (2.2.30)), see e.g. [Geo93], [RAS09]. Altough we expect both Proposition 2.1.6 and Theorem 9 to hold with this stronger topology, with essentially the same proof, we do not pursue this generality here. Let us here emphasize that even when restating Proposition 2.1.6 in the τ-topology our main theorem does not follow from an application of Varadhan's integral lemma, because w_N is neither bounded nor local.

Proposition 2.1.6 is then complemented by the following result, which essentially yields the main theorem:

Proposition 2.1.7. *Let $\bar{P}\in\mathcal{P}_{s,1}(\Sigma\times\mathcal{X})$. For all $\delta_1,\delta_2>0$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\log\bar{\mathfrak{Q}}_{N,\beta}\left(B(\bar{P},\delta_1)\cap T_{N,\delta_2}(\bar{P})\right) \geq -\int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx - c_{\omega,\Sigma}, \quad (2.1.25)$$

where $T_{N,\delta_2}(\bar{P})$ denotes a set of point processes obtained from N-uples of points $(x_1,\dots,x_N)$ by $i_N(x_1,\dots,x_N)$ (where i_N is defined in (2.1.9)) which satisfy

$$w_N(x_1,\dots,x_N) \leq \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \delta_2.$$

This proposition is the hard part of the proof. To obtain it we need to show that the set $T_{N,\delta_2}(\bar{P})$ has enough volume in phase-space. This relies on taking arbitrary configurations in $B(\bar{P},\delta_1)$ and showing that a large enough fraction of them (in the sense of volume) can be screened and modified to generate a small energy truncation error, as alluded to above.

2.1.3 Open questions and further study

a. Crystallization and phase transitions.

Let us observe the following :

Lemma 2.1.8. *Let Γ be a point configuration periodic with respect to some lattice $\Lambda \in \mathbb{R}^d$ and let P_Γ be the associated stationary point process defined by*

$$P_\Gamma := \fint_\Lambda \delta_{\theta_x \cdot \Gamma} dx.$$

Then the specific relative entropy $\mathsf{ent}[P_\Gamma|\mathbf{\Pi}^1]$ is equal to $+\infty$.

Proof. It is in fact easy to see that for any integer N the point process induced by P_Γ in the hypercube C_N is absolutely singular with respect to the Poisson point process $\mathbf{\Pi}^1_{|C_N}$ hence the usual relative entropy $\mathrm{Ent}[P_{\Gamma|C_N}|\mathbf{\Pi}^1_{|C_N}]$ is infinite. Thus by definition we also have $\mathsf{ent}[P_\Gamma|\mathbf{\Pi}^1] = +\infty$. □

It follows from Lemma 2.1.8 that if β is finite, the minimizer of $\mathcal{F}_\beta$ cannot be a periodic point process and in particular it cannot be the point process associated to some lattice (or crystal). Hence there is no crystallization in the strong sense i.e. the particles cannot concentrate on an exact lattice. However some weaker crystallization could occur at finite β e.g. if the connected two-point correlation function $\rho_2 - 1$ of minimizers of $\mathcal{F}_\beta$ decays more slowly to 0 as β gets larger or ceases to be in L^1 for β greater than some β_c. Hints towards such a transition in the behavior of $\rho_2 - 1$ for the one-dimensional log-gas may be found in [For93] where an explicit formula for the two-point correlation function is computed for the limiting point processes associated to the β-Circular Ensemble (which according to [Nak14] turn out to also be Sine_β).

Such a change in the long-distance behavior of ρ_2 would not necessarily imply a first-order phase transition i.e. a singularity in the first derivative of $\beta \mapsto \min \mathcal{F}_\beta$, as would be implied e.g. by the existence of two minimizers of $\mathcal{F}_\beta$ with different energies. The existence of a first-order phase transition in the two-dimensional logarithmic case (also called the two-dimensional one-component plasma) is discussed in the physics literature, see e.g. [Sti98]. On the other hand, it might be that for some β there exists several minimizers of $\mathcal{F}_\beta$ with the same energy and the physical implications of such a situation is unclear to us. Let us note that uniqueness of the minimizers (or at least the fact that they all have the same energy, hence the same entropy) would for example allow to retrieve as a straightforward corollary of our LDP the equiparitition property shown in [BMSS13] for β-models.

In the following paragraph we collect some open questions, stemming from the ones discussed above.

b. Open questions.

— Is the minimum of $\mathcal{F}_\beta$ unique? Let us observe that the specific relative entropy $\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$ is affine, hence so is $\mathcal{F}_\beta$ and no easy "strict convexity" argument seems to hold. Do at least all the minimizers of $\mathcal{F}_\beta$ share the same energy and entropy?

— Can the variational characterization of the Sine_β, Ginibre and other limiting point processes be used to provide more information on these processes?

— Does crystallization hold in a weak sense, e.g. at the level of a change in the large-distance behaviour of the two-point correlation function of minimizers of $\mathcal{F}_\beta$ when β crosses some critical value?

— Can we characterize the minima (minimum?) of $\widetilde{\mathbb{W}}_1$ for $d \geq 2$? Can we at least prove that any minimizer of $\widetilde{\mathbb{W}}_1$ has infinite specific relative entropy, which would be a first hint towards their conjectural "ordered" nature?
— Is there a limit to the Gibbsian point process $\mathbf{P}_{N,\beta}$, defined as the push-forward of $\mathbb{P}_{N,\beta}$ by $(x_1, \dots, x_N) \mapsto \sum_{i=1}^N \delta_{N^{1/d}x_i}$ and of their translates $\theta_x \cdot \mathbf{P}_{N,\beta}$ for x in the "bulk" (the interior of Σ) ? In the cases where the existence of a limit is known, can we find a purely "energy based" proof? Can we at least prove that any limit point of $\mathbf{P}_{N,\beta}$ is translation invariant? Can we at least prove the mere existence of limit points for a general class of V, s, d?
— Can one somehow use the next-order information on $Z_{N,\beta}$ of Corollary 2.1.5 to prove a central limit theorem for the fluctuations?

c. Further study.

In order to deduce further consequences of the LDP, it is more convenient to express the renormalized energy of a point process in terms of its two-point correlation functions. This is inspired by [BS13] and is the object of [Leb15b]. This approach allows one to obtain further qualitative information, as the convergence of minimizers of $\mathcal{F}_\beta$ to a Poisson point process in the limit $\beta \to 0$, thus retrieving results of [AD14] in the special case of sine-beta processes.

The result of Theorem 9 characterizes the microscopic behavior in an averaged way, the average scale being macroscopic (roughly speaking, one can deduce from it the microscopic point processes obtained by averaging over $B(x, \varepsilon)$ for any $x \in \Sigma$ and $\varepsilon > 0$), whereas *local laws* consist in replacing the macroscopic average as in (2.1.9) by averages at microscopic scales $\varepsilon = N^{-\delta}$ for $0 < \delta < \frac{1}{d}$. In [Leb15a] the techniques of the present paper are pushed further in order to derive a LDP concerning such finer scale observables.

The rest of the paper is organized as follows: Section 2.2 contains our assumptions, the definitions of the renormalized energy and of the specific relative entropy, as well as some important notation. In Section 2.3 we present some preliminary results on the renormalized energy. Section 2.4 contains the proofs of the main results and corollaries, assuming the results of Propositions 2.1.6 and 2.1.7. In Section 2.5 we recall the screening result and describe the procedure to screen random point configurations. We also describe the regularization procedure. In Section 2.6 we complete the proof of Proposition 2.1.7 by showing that given a random point configuration we can often enough screen it and regularize it to have the right energy. In Section 2.7 we prove Proposition 2.1.6. In Section 2.8, we collect miscellaneous additional proofs.

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2.2 Assumptions and main definitions

2.2.1 Our assumptions

We now start to describe more precisely the setting in which we work, which is identical to that of [PS15].

We first place assumptions on V that ensure the existence of the equilibrium μ_V from standard potential theory:

$$V \text{ is lower semi-continuous (l.s.c.) and bounded below}$$
$$\{x : V(x) < \infty\} \text{ has positive } g\text{-capacity}$$
$$\lim_{|x|\to\infty} V(x) = +\infty, \quad \text{resp. } \lim_{|x|\to\infty} \frac{V(x)}{2} - \log|x| = +\infty \text{ in cases (2.1.1)} - \text{(2.1.2)}$$

The following theorem, due to Frostman [Fro35] (cf. also [ST97]) then gives the existence and characterization of the equilibrium measure (let us recall that the energy functional was defined in (2.1.6)):

Theorem 10 (Frostman)**.** *Assume that V satisfies (2.2.1)–(2.2.2), then there exists a unique minimizer $\mu_V \in \mathcal{P}(\mathbb{R}^d)$ of $\mathcal{I}$ and $\mathcal{I}(\mu_V)$ is finite. Moreover the following properties hold:*
— *$\Sigma := \mathrm{supp}(\mu_V)$ is bounded and has positive g-capacity,*
— *for $c := \mathcal{I}(\mu_V) - \int \frac{V}{2} d\mu_V$ and $H^{\mu_V}(x) := \int g(x-y) d\mu_V(y)$ there holds*

$$\begin{cases} H^{\mu_V} + \frac{V}{2} \geq c & \text{quasi everywhere (q.e.)}, \\ H^{\mu_V} + \frac{V}{2} = c & \text{q.e. on } \Sigma. \end{cases}$$

We now define the function ζ that appeared above:

$$\zeta := H^{\mu_V} + \tfrac{V}{2} - c \geq 0. \tag{2.2.4}$$

We let ω be the zero set of ζ and by Theorem 10 we have

$$\Sigma \subset \omega := \{\zeta = 0\}.$$

The function H^{μ_V} is the solution to a classical obstacle problem in the Coulomb case, respectively a fractional obstacle problem in the other cases (cf. [CSS08]). The set ω then corresponds to the *contact set* or *coincidence set* of the obstacle problem, and Σ is the set where the obstacle is "active", sometimes called the *droplet*.

We will assume that μ_V is really a d-dimensional measure (i.e. Σ is a nice d-dimensional set), with a density, and we need to assume that this density (that we still denote μ_V) is bounded and sufficiently regular on its support. More precisely, we make the following assumptions (which are technical and could certainly be somewhat relaxed):

$$\partial\Sigma \text{ is } C^1$$
$$\mu_V \text{ has a density which is } C^{0,\kappa} \text{ in } \Sigma,$$
$$\exists c_1, c_2, \overline{m} > 0 \text{ s.t. } c_1 \mathrm{dist}(x, \partial\Sigma)^\alpha \leq \mu_V(x) \leq \min(c_2 \mathrm{dist}(x, \partial\Sigma)^\alpha, \overline{m}) < \infty \text{ in } \Sigma,$$

with the conditions

$$0 < \kappa \leq 1, \qquad 0 \leq \alpha \leq \frac{2\kappa d}{2d - s}. \tag{2.2.8}$$

Of course if $\alpha < 1$ one should take $\kappa = \alpha$, and if $\alpha \geq 1$, one should take $\kappa = 1$ and $\alpha \leq \frac{2d}{d-s}$. These assumptions are meant to include the case of the semi-circle law $\frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(x)$ arising for a quadratic potential in the setting (2.1.1). We also know that in the Coulomb cases, a quadratic potential gives rise to an equilibrium measure which is a multiple of a characteristic function of a ball, also covered by our assumptions with $\alpha = 0$. Finally, in the Riesz case, it was noticed in [CGZ14, Corollary 1.4] that any compactly supported radial profile can be obtained as the equilibrium measure associated to some potential. Our assumptions are thus never empty.

The last assumption is that there exists $\beta_1 > 0$ such that

$$\begin{cases} \int e^{-\beta_1 V(x)/2}\, dx < \infty & \text{in the case (2.1.3)} \\ \int e^{-\beta_1 (\frac{V(x)}{2} - \log|x|)}\, dx < \infty & \text{in the cases (2.1.1)–(2.1.2),} \end{cases} \tag{2.2.9}$$

It is a standard assumption ensuring the existence of the partition function.

2.2.2 The extension representation for the fractional Laplacian

In the next two sections, we recall elements from [PS15]. Our method of proof relies on expressing the interaction part of the Hamiltonian as a quadratic integral of the potential generated by the point configuration via

$$g * \sum_i \delta_{x_i}$$

(where $*$ denotes the convolution product) and expanding this integral interaction to next order in N. Outside of the Coulomb case, the Riesz kernel g is not the convolution kernel of a local operator, but rather of a fractional Laplacian. However, according to Caffarelli and Silvestre [CS07], when $d-2 < s < d$, this fractional Laplacian nonlocal operator can be transformed into a local but inhomogeneous operator of the form $\operatorname{div}(|y|^\gamma \nabla \cdot)$ by adding one space variable $y \in \mathbb{R}$ to the space $\mathbb{R}^d$. We refer to [PS15] for more details. In what follows, k will denote the dimension extension. We will take $k=0$ in all the Coulomb cases, i.e. $s = d-2$ and $d \geq 3$ or (2.1.2). In all other cases, we will need to take $k=1$. Points in the space $\mathbb{R}^d$ will be denoted by x, and points in the extended space $\mathbb{R}^{d+k}$ by X, with $X=(x,y)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$. We will often identify $\mathbb{R}^d \times \{0\}$ and $\mathbb{R}^d$.

If γ is chosen such that

$$d-2+k+\gamma = s\,, \tag{2.2.10}$$

then, given a measure μ on $\mathbb{R}^d$, the potential $H^\mu(x)$ generated by μ defined in $\mathbb{R}^d$ by

$$H^\mu(x) = g * \mu(x) = \int_{\mathbb{R}^d} \frac{1}{|x-x'|^s}\, d\mu(x')$$

can be extended to a function $H^\mu(X)$ on $\mathbb{R}^{d+k}$ defined by

$$H^\mu(X) = \int_{\mathbb{R}^d} \frac{1}{|X-(x',0)|^s}\, d\mu(x')$$

and this function satisfies

$$-\operatorname{div}(|y|^\gamma \nabla H^\mu) = c_{d,s}\mu\delta_{\mathbb{R}^d} \tag{2.2.11}$$

where by $\delta_{\mathbb{R}^d}$ we mean the uniform measure on $\mathbb{R}^d \times \{0\}$ i.e. $\mu\delta_{\mathbb{R}^d}$ acts on test functions φ by

$$\int_{\mathbb{R}^{d+k}} \varphi(X) d(\mu\delta_{\mathbb{R}^d})(X) = \int_{\mathbb{R}^d} \varphi(x,0)\, d\mu(x),$$

and

$$c_{d,s} = \begin{cases} 2s\, \frac{2\pi^{\frac{d}{2}}\Gamma\left(\frac{s+2-d}{2}\right)}{\Gamma\left(\frac{s+2}{2}\right)} & \text{for } s > \max(0, d-2)\ , \\ (d-2)\frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)} & \text{for } s = d-2 > 0\ , \\ 2\pi & \text{in cases (2.1.1), (2.1.2)}\ . \end{cases}$$

In particular $g(X) = |X|^{-s}$ seen as a function of $\mathbb{R}^{d+k}$ satisfies

$$-\operatorname{div}(|y|^\gamma \nabla g) = c_{d,s}\delta_0. \tag{2.2.12}$$

In order to recover the Coulomb cases, it suffices to take $k=\gamma=0$, in which case we retrieve the fact that g is a multiple of the fundamental solution of the Laplacian. If $s > d-2$ we take $k=1$ and γ satisfying (2.2.10). In the case (2.1.1), we note that $g(x) = -\log|x|$ appears as the $y=0$ restriction of $-\log|X|$, which is (up to a factor 2π) the fundamental solution to the Laplacian operator in dimension $d+k=2$. In this case, we may thus choose $k=1$ and $\gamma=0$, $c_{d,s} = c_{1,0} = 2\pi$, and the potential $H^\mu = g*\mu$ still satisfies (2.2.11), while g still satisfies (2.2.12).

To summarize, we will take

$$\begin{aligned} \text{in the case } \max(0,d-2)<s<d, &\quad \text{then } k=1,\ \gamma=s-d+2-k\,,\\ \text{in the case (2.1.1)}, &\quad \text{then } k=1,\ \gamma=0\,,\\ \text{in the case (2.1.2) or } d\geq 3, s=d-2, &\quad \text{then } k=0,\ \gamma=0\,. \end{aligned}$$

We note that the formula (2.2.10) always remains formally true when taking the convention that $s=0$ in the case $g(x)=-\log|x|$, and we also note that the assumption $d-2\leq s<d$ implies that in all cases $\gamma\in(-1,1)$.

2.2.3 The renormalized energy for electric fields

In this section, we recall the definition from [PS15]. First, let us define the truncated Riesz (or logarithmic) kernel as follows: for $1>\eta>0$ and $X\in\mathbb{R}^{d+k}$, let

$$f_\eta(X)=(g(X)-g(\eta))_+\,. \tag{2.2.16}$$

We note that the function f_η vanishes outside of $B(0,\eta)\subset\mathbb{R}^{d+k}$ and satisfies that

$$\delta_0^{(\eta)}:=\frac{1}{c_{d,s}}\mathrm{div}\,(|y|^\gamma\nabla f_\eta)+\delta_0$$

is a positive measure supported on $\partial B(0,\eta)$, and which is such that for any test-function φ,

$$\int\varphi\delta_0^{(\eta)}=\frac{1}{c_{d,s}}\int_{\partial B(0,\eta)}\varphi(X)|y|^\gamma g'(\eta).$$

One can thus check that $\delta_0^{(\eta)}$ is a positive measure of mass 1, and we may write

$$-\mathrm{div}\,(|y|^\gamma\nabla f_\eta)=c_{d,s}(\delta_0-\delta_0^{(\eta)})\quad\text{in }\mathbb{R}^{d+k}.$$

We will also denote by $\delta_p^{(\eta)}$ the measure $\delta_0^{(\eta)}(X-p)$, for $p\in\mathbb{R}^d\times\{0\}$. Again, we note that this includes the cases (2.2.14)–(2.2.15). In the Coulomb cases, i.e. when $k=0$, then $\delta_0^{(\eta)}$ is simply the normalized surface measure on $\partial B(0,\eta)$.

We are now in a position to define the renormalized energy for a finite configuration of points, i.e. the quantity w_N appearing in (2.1.7)–(2.1.8). It is defined via the gradient of the potential generated by the point configuration, embedded into the extended space $\mathbb{R}^{d+k}$. More precisely, for a configuration of points $(x_1,\dots,x_N)$, we introduce the potential H_N generated by the points and the "background charge" $N\mu_V$:

$$H_N=g*\left(\sum_{i=1}^N\delta_{x_i}-N\mu_V\delta_{\mathbb{R}^d}\right). \tag{2.2.17}$$

We also introduce the blown-up configuration $(x'_1,\dots,x'_N)=(N^{1/d}x_1,\dots,N^{1/d}x_N)$, the blown-up equilibrium measure of density $\mu'_V(x')=\mu_V(N^{-1/d}x')$, and the blown-up potential

$$H'_N=g*\left(\sum_{i=1}^N\delta_{x'_i}-\mu'_V\delta_{\mathbb{R}^d}\right). \tag{2.2.18}$$

In view of the discussion of Section 2.2.2, H_N and H'_N can be viewed as functions on $\mathbb{R}^{d+k}$ satisfying the relations

$$-\mathrm{div}\,(|y|^\gamma\nabla H_N)=c_{d,s}\left(\sum_{i=1}^N\delta_{x_i}-N\mu_V\delta_{\mathbb{R}^d}\right)\qquad-\mathrm{div}\,(|y|^\gamma\nabla H'_N)=c_{d,s}\left(\sum_{i=1}^N\delta_{x'_i}-\mu'_V\delta_{\mathbb{R}^d}\right)$$

The gradients of these potentials are called "electric fields" and denoted E, by analogy with the Coulomb case in which they correspond to the physical electric field generated by the points viewed as singular charges.

In the sequel, we will always identify a point configuration with the sum (possibly with multiplicity) of Dirac masses that it generates, i.e. $\mathcal{C}$ will mean a sum of Dirac masses with integer weights, as well as a point configuration.

For any electric field E solving a relation of the form

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\Big(\mathcal{C} - m(x)\delta_{\mathbb{R}^d}\Big) \quad \text{in } \mathbb{R}^{d+k},$$

where $\mathcal{C}$ is some locally finite point configuration in $\mathbb{R}^d \times \{0\}$ (identified with $\mathbb{R}^d$) we define

$$E_\eta := E - \nabla f_\eta * \mathcal{C} \tag{2.2.19}$$

where $*$ denotes again the convolution product i.e. $(\nabla f_\eta * \mathcal{C})(x) = \sum_{p\in\mathcal{C}} \nabla f_\eta(x-p)$. Let us note that this convolution product is well-defined because f_η is supported on $B(0,\eta)$ and $\mathcal{C}$ is locally finite. If E happens to be the gradient of a function H, then we will also denote

$$H_\eta := H - f_\eta * \mathcal{C}. \tag{2.2.20}$$

This corresponds to "truncating off" the infinite peak in the potential around each point of the configuration:

Remark 2.2.1. *If $H = g*\Big(\sum_{i=1}^N \delta_{x_i} - m(x)\delta_{\mathbb{R}^d}\Big)$ then the transformation from H to H_η amounts to truncating the kernel g, but only for the Dirac part of the r.h.s. Indeed, letting $g_\eta(x) = \min(g(x), g(\eta))$ be the truncated kernel, we have*

$$H_\eta = g_\eta * \left(\sum_{i=1}^N \delta_{x_i}\right) - g * (m\delta_{\mathbb{R}^d}).$$

We may then define the truncated versions of H_N and H'_N as in (2.2.20), in particular

$$H'_{N,\eta} = H_{N,\eta} - \sum_{i=1}^N f_\eta(x - x'_i). \tag{2.2.21}$$

With this notation, we let

$$w_N(x_1,\dots,x_N) := \frac{1}{Nc_{d,s}} \lim_{\eta\to 0} \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H'_{N,\eta}|^2 - Nc_{d,s}g(\eta)\right). \tag{2.2.22}$$

It is proven in [PS15] that this limit exists and that with this definition, the exact relations (2.1.7)–(2.1.8) hold. With the presence of the factor $\frac{1}{N}$ the quantity w_N is expected to be typically of order 1.

The renormalized energy of an infinite configuration of points (already at the blown-up scale) is defined in a similar way, via an electric field which is the gradient of a potential associated to the configuration. Note that while for a finite configuration of N points, we may find a unique potential generated by it via (2.2.17), for an infinite configuration there is no canonical choice of such a potential (one may always add the gradient of a function satisfying $-\mathrm{div}\,(|y|^\gamma \nabla H) = 0$. This explains the need for a definition based on the electric field, and a definition down at the level of points.

Definition 2.2.2 (Admissible vector fields)**.** *Given a number $m \geq 0$, we define the class $\mathcal{A}_m$ to be the class of gradient vector fields $E = \nabla H$ that satisfy*

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\Big(\mathcal{C} - m\delta_{\mathbb{R}^d}\Big) \quad \text{in } \mathbb{R}^{d+k}$$

where $\mathcal{C}$ is a point configuration in $\mathbb{R}^d \times \{0\}$.

This class corresponds to vector fields that will be limits of those generated by the original configuration $(x_1, \dots, x_N)$ after blow-up at the scale $N^{1/d}$ near the point x, where $m = \mu_V(x)$ can be understood as the local density of points.

We are now in a position to define the renormalized energy. In the definition, C_R denotes as before the hypercube $[-R/2, R/2]^d$.

Definition 2.2.3 (Renormalized energy)**.** *For any $m > 0$, $\nabla H \in \mathcal{A}_m$ and $0 < \eta < 1$, we define*

$$\mathcal{W}_\eta(\nabla H) = \limsup_{R\to\infty} \left(\frac{1}{R^d} \int_{C_R\times\mathbb{R}^k} |y|^\gamma |\nabla H_\eta|^2 - m c_{d,s} g(\eta) \right)$$

where H_η is as in (2.2.20)*, and*

$$\mathcal{W}(\nabla H) = \lim_{\eta\to 0} \mathcal{W}_\eta(\nabla H). \tag{2.2.23}$$

Let us observe that the value of the parameter m is implicit in the notation $\mathcal{W}(\nabla H)$. In fact for any given $E = \nabla H$, there exists at most one $m > 0$ such that E is in $\mathcal{A}_m$ hence there is in fact no ambiguity.

Definition 2.2.4 (Scaling on E)**.** *We define the following "scaling" map allowing us to pass bijectively from an electric field in $\mathcal{A}_m$ to an electric field in $\mathcal{A}_1$. We let*

$$\sigma_m E := m^{-\frac{s+1}{d}} E(\cdot\, m^{-1/d}) \tag{2.2.24}$$

By scaling, we may then always reduce to studying the class $\mathcal{A}_1$, indeed, if $E \in \mathcal{A}_m$, then $\sigma_m E \in \mathcal{A}_1$ and

$$\mathcal{W}_\eta(E) = m^{1+s/d} \mathcal{W}_{\eta m^{1/d}}(\sigma_m E) \qquad \mathcal{W}(E) = m^{1+s/d} \mathcal{W}(\sigma_m E) \tag{2.2.25}$$

in the case (2.1.3), and respectively

$$\mathcal{W}_\eta(E) = m\left(\mathcal{W}_{m\eta}(\sigma_m E) - \frac{2\pi}{d}\log m \right) \qquad \mathcal{W}(E) = m\left(\mathcal{W}(\sigma_m E) - \frac{2\pi}{d}\log m \right) \tag{2.2.26}$$

in the cases (2.1.1)–(2.1.2).

The name renormalized energy (originating in Bethuel-Brezis-Hélein [BBH94] in the context of two-dimensional Ginzburg-Landau vortices) reflects the fact that $\int |y|^\gamma |\nabla H|^2$ which is infinite, is computed in a renormalized way by first changing H into H_η and then removing the appropriate divergent part $c_{d,s} g(\eta)$ per point.

It is proven in [PS15] that the limit in (2.2.23) exists, $\{\mathcal{W}_\eta\}_{\eta<1}$ are uniformly bounded below on $\mathcal{A}_1$ by a finite constant depending only on s and d, and $\mathcal{W}$ and $\mathcal{W}_\eta$ have a minimizer over the class $\mathcal{A}_1$. We can also note that $\mathcal{W}$ does not feel compact perturbations of the points in $\mathcal{C}$. As already mentioned the questions of identifying $\min_{\mathcal{A}_1} \mathcal{W}$ is open, and we expect some (Bravais) lattice configuration to achieve the minimum, at least in low dimension. In [SS12] it is proven that in the case (2.1.2), $\mathcal{W}$ achieves its minimum over lattice configurations of volume 1 at the triangular lattice. The same result is extended to the general case (2.1.3) in dimension

2 in [PS15]. In dimension 1, the minimum of $\mathcal{W}$ is known in the case (2.1.1): it is the value obtained at the lattice $\mathbb{Z}$ [SS15a], cf. [Leb15c] for uniqueness.

We let $\mathcal{A} = \cup_{m>0}\mathcal{A}_m$ be the class of all m-admissible gradient vector fields for any $m > 0$ (let us note that in the definition of $\mathcal{A}$ the union over $m > 0$ is in fact disjoint). It is observed in [PS15] that $\mathcal{A}$ is contained in $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ for any $p < p_{\max} := \min(2, \frac{2}{\gamma+1}, \frac{d+k}{s+1})$ (note that they blow up exactly like $1/|X|^{s+1}$ near each point of $\mathcal{C}$). These spaces $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ are endowed with the strong local topology. We note that $\mathcal{A}$ is a Borel subset of $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ for $p < p_{\max}$ and that $\mathcal{W} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is measurable.

2.2.4 Point configurations and point processes

In this section we introduce or recall the notation that we will use throughout the paper.

a. General.

If (X, d_X) is a metric space we endow the space $\mathcal{P}(X)$ of Borel probability measures on X with the Dudley distance:

$$d_{\mathcal{P}(X)}(P_1, P_2) = \sup\left\{\int F(dP_1 - dP_2)|\ F \in \mathrm{Lip}_1(X)\right\} \tag{2.2.27}$$

where $\mathrm{Lip}_1(X)$ denotes the set of functions $F : X \to \mathcal{C}$ that are 1-Lipschitz with respect to d_X and such that $\|F\|_\infty \leq 1$. It is well-known that the distance $d_{\mathcal{P}(X)}$ metrizes the topology of weak convergence on $\mathcal{P}(X)$. If $P \in \mathcal{P}(X)$ is a probability measure and $f : X \to \mathbb{R}^d$ a measurable function, we denote by $\mathbf{E}_P[f]$ the expectation of f under P.

b. Point configurations.

If A is a Borel set of $\mathbb{R}^d$ we denote by $\mathcal{X}(A)$ the set of locally finite point configurations in A or equivalently the set of non-negative, purely atomic Radon measures on A giving an integer mass to singletons (see [DVJ88]). As mentioned before, we will write $\mathcal{C}$ for $\sum_{p\in\mathcal{C}} \delta_p$.

We endow the set $\mathcal{X} := \mathcal{X}(\mathbb{R}^d)$ (and the sets $\mathcal{X}(A)$ for A Borel) with the topology induced by the topology of weak convergence of Radon measure (also known as vague convergence or convergence against compactly supported continuous functions). If B is a compact subset of $\mathbb{R}^d$ we endow $\mathcal{X}(B)$ with the following distance:

$$d_{\mathcal{X}(B)}(\mathcal{C}, \mathcal{C}') := \sup\left\{\int F(d\mathcal{C} - d\mathcal{C}')|\ F \in \mathrm{Lip}_1(B)\right\}. \tag{2.2.28}$$

The total mass $\mathcal{C}(B)$ of $\mathcal{C}$ on B corresponds to the number of points of the point configuration in B and when $\mathcal{C}(B) = \mathcal{C}'(B)$ the distance $d_{\mathcal{X}(B)}$ coincides with the "minimal connection" distance.

We endow $\mathcal{X} := \mathcal{X}(\mathbb{R}^d)$ with the following distance:

$$d_{\mathcal{X}}(\mathcal{C}, \mathcal{C}') := \sum_{k\geq 1} \frac{1}{2^k}\left(\frac{d_{\mathcal{X}(C_k)}(\mathcal{C}, \mathcal{C}')}{(\mathcal{C}(C_k) + \mathcal{C}'(C_k)) \vee 1} \wedge 1\right). \tag{2.2.29}$$

We denote by $\mathrm{Lip}_1(\mathcal{X})$ the set of all functions $F : \mathcal{X} \to \mathcal{C}$ that are 1-Lipschitz with respect to $d_{\mathcal{X}}$ and such that $\|F\|_\infty \leq 1$. We say that a measurable function $f : \mathcal{X} \to \mathcal{C}$ is local when

$$f(\mathcal{C}) = f(\mathcal{C} \cap C_k) \text{ for all } \mathcal{C} \in \mathcal{X} \tag{2.2.30}$$

for some integer k. We denote by $\mathrm{Loc}_k(\mathcal{X})$ the set of functions that satisfies (2.2.30) for a fixed integer k and we let $\mathrm{Loc}(\mathcal{X}) := \cup_{k\geq 1}\mathrm{Loc}_k(\mathcal{X})$.

Lemma 2.2.5. *The following properties hold:*

1. *The topological space $\mathcal{X}$ is Polish.*
2. *The distances $d_{\mathcal{X}(B)}$ and $d_{\mathcal{X}}$ are actual distances compatible with the respective topologies on $\mathcal{X}(B)$ and $\mathcal{X}$.*
3. *Any bounded continuous function $F : \mathcal{X} \to \mathcal{C}$ can be approximated by a sequence of bounded local functions. Moreover the approximation is uniform on the set of 1-Lipschitz functions in that for any $\delta > 0$ there exists an integer k such any function $F \in \mathrm{Lip}_1(\mathcal{X})$ is δ-close (in sup-norm) to some local function $f \in \mathrm{Loc}_k(\mathcal{X})$.*

Lemma 2.2.5 is proven in Section 2.8.

c. Translations, volume, compactness.

The additive group $\mathbb{R}^d$ acts on $\mathcal{X}$ by translations $\{\theta_t\}_{t\in\mathbb{R}^d}$: if $\mathcal{C} = \{x_i, i \in I\} \in \mathcal{X}$ we let

$$\theta_t \cdot \mathcal{C} := \{x_i - t, i \in I\}.$$

We will use the same notation for the action of $\mathbb{R}^d$ on Borel sets of $\mathbb{R}^d$: if A is Borel and $t \in \mathbb{R}^d$, we denote by $\theta_t \cdot A$ the translation of A by the vector $-t$.

For any integer N we identify a configuration $\mathcal{C}$ that has N points with all the N-uples of points in $\mathbb{R}^d$ which correspond to $\mathcal{C}$ and if A is a set of configurations with N points we denote by $\mathbf{Leb}^{\otimes N}(A)$ the Lebesgue measure of the corresponding subset of $(\mathbb{R}^d)^N$.

If $\mathcal{C}$ is a point configuration, $x \in \mathbb{R}^d$ and $R > 0$ we denote by $\mathcal{N}(x, R)(\mathcal{C})$ the number of point of $\mathcal{C}$ in C_R. The following lemma is elementary:

Lemma 2.2.6 (Compactness in $\mathcal{X}$)**.** *Let $C : \mathbb{R} \to \mathbb{R}^+$ be an arbitrary function, then the following set is compact in $\mathcal{X}$:*

$$\{\mathcal{C} \in \mathcal{X} \mid \mathcal{N}(x, R)(\mathcal{C}) \leq C(R) \text{ for all } R > 0\}.$$

Proof. It follows from the compactness of the hypercubes C_R for all $R > 0$ (hence of their powers C_R^n) and from the definition (2.2.29) of the distance on $\mathcal{X}$, together with a diagonal extraction procedure in order to extract a subsequence converging on each C_k for $k \geq 1$. □

d. Point processes.

A point process is a probability measure on $\mathcal{X}$, a tagged point process is a probability measure on $\Lambda \times \mathcal{X}$ where Λ is some Borel set of $\mathbb{R}^d$ with non-empty interior. Usually Λ will be Σ (the support of the equilibrium measure μ).

When Λ is fixed, we shall always assume that the first marginal of a tagged point process $\bar{P}$ is the normalized Lebesgue measure on Λ hence we may consider the disintegration measures $\{\bar{P}^x\}_{x\in\Lambda}$ of $\bar{P}$ (for a definition see [AGS08, Section 5.3]), such that for any measurable function F on $\Lambda \times \mathcal{X}$ we have

$$\mathbf{E}_{\bar{P}}[F] = \fint_{\Lambda} \mathbf{E}_{\bar{P}^x}[F(x, \cdot)]dx.$$

We denote by $\mathcal{P}_s(\mathcal{X})$ the set of translation-invariant (or stationary) point processes. We also call stationary a tagged point process such that the disintegration measure $\bar{P}^x$ is stationary for (Lebesgue-)a.e. $x \in \Lambda$ and denote by $\mathcal{P}_s(\Lambda \times \mathcal{X})$ the set of stationary tagged point processes.

Let P be a point process. If there exists a measurable function $\rho_{1,P}$ such that for any function $\varphi \in C_c^0(\mathbb{R}^d)$ we have

$$\mathbf{E}_P\left[\sum_{x\in\mathcal{C}} \varphi(x)\right] = \int_{\mathbb{R}^d} \rho_{1,P}(x)\varphi(x)dx, \tag{2.2.31}$$

then we say that $\rho_{1,P}$ is the one-point correlation function (or "intensity") of the point process P. For $m \geq 0$ we say that a point process P is of intensity m when the function $\rho_{1,P}$ of (2.2.31) exists and satisfies $\rho_{1,P} \equiv m$.

We will denote by $\mathcal{P}_{s,1}(\mathcal{X})$ the set of stationary point processes of intensity 1 and by $\mathcal{P}_{s,1}(\Lambda \times \mathcal{X})$ the set of stationary tagged point processes (with space coordinate taken in Λ) such that the integral on $x \in \Lambda$ of the intensity of the disintegration measure $\bar{P}^x$ (which is by assumption a stationary point process) is 1.

We define the following "scaling" map allowing us to pass bijectively from a point process of intensity m to a point process of intensity 1.

$$\sigma_m P := \text{ the push-forward of } P \text{ by } \mathcal{C} \mapsto m^{1/d}\mathcal{C}. \tag{2.2.32}$$

Let us conclude this paragraph with a remark on the notion of convergence of point processes used here.

Remark 2.2.7. *We endow $\mathcal{P}(\mathcal{X})$ with the usual topology of weak convergence of probability measures (for the Borel σ-algebra on $\mathcal{X}$). This induces by definition a notion of convergence that corresponds to the weak convergence of probability distributions on $\mathcal{X}$. Another natural topology on $\mathcal{P}(\mathcal{X})$ is "convergence of the finite distributions" [DVJ08, Section 11.1] - sometimes also called the "convergence with respect to vague topology for the counting measure of the point process". The latter might seem weaker than the former, however the two notions of convergence coincide as stated in [DVJ08, Theorem 11.1.VII].*

e. Electric field processes.

An electric field process is an element of $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k}))$ where $p < p_{\max}$, concentrated on $\mathcal{A}$. It will usually be denoted by P^{elec}. We say that P^{elec} is stationary when it is law-invariant under the (push-forward by) translations $\theta_x \cdot E = E(\cdot - x)$ for any $x \in \mathbb{R}^d \subset \mathbb{R}^d \times \{0\}^k$. A tagged electric field process is an element of $\mathcal{P}(\Sigma \times L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k}))$ whose first marginal is the normalized Lebesgue measure on Σ and whose disintegration slices are electric field processes. It will be denoted by $\bar{P}^{\mathrm{elec}}$. We say that a tagged electric field process $\bar{P}^{\mathrm{elec}}$ is stationary if for a.e. $x \in \Sigma$, the disintegration measure $\bar{P}^{\mathrm{elec},x}$ is stationary.

f. Application of the stationarity.

We end this section with an elementary lemma exposing a consequence of the stationarity assumptions which we will make a constant use of.

Lemma 2.2.8. *For any P stationary (point or electric) process, resp. $\bar{P}$ stationary (point or electric) tagged process, for every $T, R > 0$, for any Φ scalar nonnegative function of the point configuration or electric field X, we have*

$$\mathbf{E}_P\left[\fint_{C_T \times \mathbb{R}^k} \Phi(X(x))\, dx\right] = \mathbf{E}_P\left[\fint_{C_R \times \mathbb{R}^k} \Phi(X(x))\right].$$

Moreover $\mathbf{E}_P\left[\lim_{R\to\infty} \fint_{C_R\times\mathbb{R}^k} \Phi(X(x))\right]$ exists and coincides with $\mathbf{E}_P\left[\fint_{C_T\times\mathbb{R}^k} \Phi(X(x))\, dx\right]$ for any $T > 0$.

Proof. The multiparameter ergodic theorem (cf. [Bec81]) ensures that for any $T > 0$

$$\mathbf{E}_P\left[\fint_{C_T\times\mathbb{R}^k}\Phi(X(x))\,dx\right] = \mathbf{E}_P\left[\lim_{R\to\infty}\frac{1}{R^d}\int_{C_R}\fint_{C_T\times\mathbb{R}^k}\Phi(X(\lambda+x))\,dx\,d\lambda\right]$$
$$= \mathbf{E}_P\left[\lim_{R\to\infty}\frac{1}{R^d}\fint_{C_R}\Phi(X(x))\right]$$

where we used Fubini's theorem and the fact that $\mathbf{1}_{C_{R-T}} \le \mathbf{1}_{C_R} * \mathbf{1}_{C_T} \le \mathbf{1}_{C_{R+T}}$ and Φ nonnegative. The result follows. □

2.2.5 The renormalized energy for point configurations and processes

a. The case of electric field processes.

We may now define the renormalized energy for random electric fields (in all the rest of the paper we take $p < p_{\max}$).

Definition 2.2.9. *If $P^{\rm elec}$ is an electric field process, we let*

$$\widetilde{\mathcal{W}}_\eta(P^{\rm elec}) := \int \mathcal{W}_\eta(E)dP^{\rm elec}(E) \qquad \widetilde{\mathcal{W}}(P^{\rm elec}) := \int \mathcal{W}(E)dP^{\rm elec}(E)$$

whenever the expressions in the right-hand side make sense.
We also define

$$\overline{\mathcal{W}}_{\mu_V(x)}(\bar{P}^{\rm elec}) := \fint_\Sigma \widetilde{\mathcal{W}}(\bar{P}^{{\rm elec},x})dx, \tag{2.2.33}$$

whenever $\bar{P}^{\rm elec}$ is a tagged electric field process such that for a.e. $x \in \Sigma$, the disintegration measure $\bar{P}^{{\rm elec},x}$ is concentrated on $\mathcal{A}_{\mu_V(x)}$ (otherwise we set $\overline{\mathcal{W}}_{\mu_V(x)}(\bar{P}^{\rm elec}) = +\infty$).

b. The case of point configurations and point processes.

For any $m > 0$ and for any admissible gradient vector field $E \in \mathcal{A}_m$ we let

$$\mathrm{Conf}_m(E) := \frac{-\mathrm{div}\,(|y|^\gamma E)}{c_{d,s}} + m\delta_{\mathbb{R}^d} \tag{2.2.34}$$

be the underlying point configuration. For any $E \in \mathcal{A}$ there is exactly one value of $m > 0$ such that $E \in \mathcal{A}_m$ and we let $\mathrm{Conf}(E) := \mathrm{Conf}_m(E)$ for the suitable value of m, this defines a map $\mathcal{A} \to \mathcal{X}$ and we denote by $\mathcal{X}^o \subset \mathcal{X}$ its image i.e. the set of point configurations $\mathcal{C}$ for which there exists at least one admissible gradient vector field E such that $\mathrm{Conf}(E) = \mathcal{C}$. It is clear that the maps $\mathrm{Conf}_m : \mathcal{A}_m \to \mathcal{X}$ and $\mathrm{Conf} : \mathcal{A} \to \mathcal{X}$ are measurable. Let us note that the fiber of Conf at any $\mathcal{C} \in \mathcal{X}^o$ is always infinite, if E is in the fiber of $\mathcal{C}$ we can simply add to E the gradient of any function satisfying $\mathrm{div}\,(|y|^\gamma \nabla H) = 0$ on $\mathbb{R}^{d+k}$ and by doing so we recover exactly the fiber of $\mathcal{C}$.

We may then define the renormalized energy of a point configuration/process by means of the renormalized energy of electric field/processes in the fiber of Conf.

Definition 2.2.10. *If $\mathcal{C}$ is a point configuration and $m > 0$ we let*

$$\mathbb{W}_m(\mathcal{C}) := \inf\{\mathcal{W}(E) \mid E \in \mathcal{A}_m, \mathrm{Conf}_m(E) = \mathcal{C}\}$$

with the convention $\inf(\varnothing) = +\infty$ *(hence* $\mathbb{W}(\mathcal{C}) = +\infty$ *when* $\mathcal{C} \in \mathcal{X}\backslash\mathcal{X}^o$*).*
If P *is a point process and* $m > 0$ *we let as in* (2.1.10)

$$\widetilde{\mathbb{W}}_m(P) := \int \mathbb{W}_m(\mathcal{C}) dP(\mathcal{C}).$$

If $\bar{P} \in \mathcal{P}(\Sigma \times \mathcal{X})$ *is a tagged point process we let as in* (2.1.11)

$$\overline{\mathbb{W}}_{\mu_V}(\bar{P}) := \frac{1}{c_{d,s}} \int_\Sigma \widetilde{\mathbb{W}}_{\mu_V(x)}(\bar{P}^x) dx.$$

In Section 2.8, we will prove the following :

Lemma 2.2.11. *If* $k = 0$*, two* E*'s in* $\mathcal{A}_m$ *such that* $\mathrm{Conf}_m(E) = \mathcal{C}$ *and* $\mathcal{W}(E)$ *is finite differ by a constant vector field, and if* $k = 1$*, an* $E \in \mathcal{A}_m$ *such that* $\mathrm{Conf}_m(E) = \mathcal{C}$ *and* $\mathcal{W}(E)$ *is finite is unique. In all cases, the* inf *in the definition of* $\mathbb{W}_m$ *is a uniquely achieved minimum.*

The following lemma, proven in Section 2.8, is stated for point processes of intensity 1 and probability P^{elec} concentrated on $\mathcal{A}_1$ and is easily extended to any intensity $m > 0$ and class $\mathcal{A}_m$ by the scaling map (2.2.24) and the scaling relations (2.2.25), (2.2.26).

Lemma 2.2.12. *Let* P *be a stationary point process such that* $\widetilde{\mathbb{W}}_1(P)$ *is finite. Then there exists at least one stationary probability measure* P^{elec} *concentrated on* $\mathcal{A}_1$ *such that the push-forward of* P^{elec} *by* Conf_1 *is* P *and* $\widetilde{\mathcal{W}}(P^{\mathrm{elec}}) < +\infty$*. Moreover we have*

$$\widetilde{\mathbb{W}}_1(P) = \min\{\widetilde{\mathcal{W}}(P^{\mathrm{elec}}) \mid P^{\mathrm{elec}} \text{ stationary and } \mathrm{Conf}_1 \# P^{\mathrm{elec}} = P\}, \tag{2.2.35}$$

where $\mathrm{Conf}_1 \# P^{\mathrm{elec}}$ *denotes the push-forward of* P^{elec} *by* Conf_1*.*

The identity (2.2.35) extends readily not only to point processes P such that $\widetilde{\mathbb{W}}_m(P)$ is finite for some $m > 0$ (with of course Conf_m instead of Conf_1) but also to the context of tagged point processes, by applying the result of Lemma 2.2.12 to each disintegration $\bar{P}^x$ ($x \in \Sigma$) of a tagged point process $\bar{P}$.

2.3 Preliminaries on the energy

In this section we recall a few facts about the renormalized energies from [PS15,SS15b,SS15a, RS15] and we deduce a few new properties which will be crucial for us.

2.3.1 Splitting and lower bound estimates

Here we recall how $\overline{\mathbb{W}}$ is related to the Hamiltonian $\mathcal{H}_N$. The connection originates in the exact "splitting formula" mentioned in the introduction in (2.1.7)–(2.1.8), where ζ is as in (2.2.4) and w_N is as in (2.2.22). For a proof of this formula in our situation, see [PS15]. Once this is established, one needs to analyze the limit as $N \to \infty$ of w_N. This was done in the previous works, and we will make repeated use of the following lower bound, which is an immediate consequence of [PS15, Proposition 5.2] (in [PS15] it was given in terms of the electric field process, but it can be "projected down" at the level of the point processes via the map (2.2.34) and Definition 2.2.10):

Lemma 2.3.1. *Assume* V *satisfies* (2.2.1)*,* (2.2.2)*,* (2.2.3)*, and that* μ_V *is a measure with a density which is bounded and almost everywhere (a.e.) continuous. For any* N*, let* $x_1, \dots, x_N \in \mathbb{R}^d$ *and define* $\bar{P}_{\nu_N} = i_N(\{x_1, \dots, x_N\})$ *as in* (2.1.9)*. Assume that* $w_N(x_1, \dots, x_N) \leq C$ *for some*

C independent of N. Then up to extraction of a subsequence, $\bar{P}_{\nu_N}$ converges weakly in the sense of probability measures to a measure $\bar{P} \in \mathcal{P}(\Sigma \times \mathcal{X})$ which is stationary and

$$\liminf_{N\to\infty} N^{-1-\frac{s}{d}} \Big(\mathcal{H}_N(x_1,\dots,x_N) - N^2\mathcal{I}(\mu_V)\Big) \geq \overline{\mathbb{W}}_{\mu_V}(\bar{P}) \text{ in case (2.1.3)}$$

respectively

$$\liminf_{N\to\infty} \frac{1}{N}\left(\mathcal{H}_N(x_1,\dots,x_N) - N^2\mathcal{I}(\mu_V) + \frac{N}{d}\log N\right) \geq \overline{\mathbb{W}}_{\mu_V}(\bar{P}) \text{ in cases (2.1.1)} - -(2.1.2),$$

or equivalently

$$\liminf_{N\to\infty} w_N(x_1,\dots,x_N) \geq \overline{\mathbb{W}}_{\mu_V}(\bar{P}). \tag{2.3.1}$$

This of course gives a formal lower bound for the energy part in the rate function (or an LDP upper bound), the difficulty will be in obtaining the corresponding upper bound (respectively an LDP lower bound).

Remark 2.3.2. *The relation* (2.3.1) *constitutes the* Γ-lim inf *(or lower bound) part of* Γ*-convergence. The result of Proposition 2.1.7 implies in particular that given any $\bar{P} \in \mathcal{P}_{s,1}(\Sigma\times\mathcal{X})$ which has a finite entropy (average of the relative specific entropy), and $\delta_1,\delta_2 > 0$, there exists (for any N large enough) $(x_1,\dots,x_N)$ such that $i_N(x_1,\dots,x_N) \in B(\bar{P},\delta_1)$ and $w_N(x_1,\dots,x_N) \leq \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \delta_2$. Taking δ_1,δ_2 tending to 0 as $N\to\infty$, we obtain the upper bound (or* Γ*-limsup) part of* Γ*-convergence, for stationary $\bar{P}$ with finite entropy. In fact a careful inspection of the proof of Proposition 2.1.7 shows that we may remove the assumption on the entropy. Indeed for any $\bar{P}$ stationary such that $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$ is finite (otherwise there is nothing to prove), starting with a sequence of N-tuples $\{X_N\}_N = \{(x_1,\dots,x_N)_N\}_N$ such that $i_N(X_N)$ is in $B(\bar{P},\delta_1)$ for N large enough (which can always be constructed by sampling $\bar{P}$ on large hypercubes) we may, by first regularizing X_N and then applying the screening procedure, get another sequence $\{X'_N\}_N$ such that $i_N(X'_N)$ is eventually in $B(\bar{P},2\delta_1)$ and $\limsup_{N\to\infty} w_N(X'_N) \leq \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \delta_2$. Together with a standard diagonal argument this implies the full* Γ*-convergence of w_N to $\overline{\mathbb{W}}_{\mu_V}$.*

On the other hand, it was proven in [SS15b, SS15a] that in the logarithmic cases, the Γ*-limit of w_N is the analogue of $\overline{\mathbb{W}}$ but defined from a different variant of the renormalized energy (with the same notation $\mathbb{W}$). This allows to check that the two variants of the renormalized energy coincide over stationary point processes thus answering a question raised in [RS15].*

2.3.2 Almost monotonicity of the energy and truncation error

The following is an immediate consequence of [PS15, Lemma 2.3], using the monotonicity of g. It shows the almost monotone character of the limit defining w_N, and provides at the same time an estimate of the error made when truncating the potentials at a level η.

Lemma 2.3.3. *For any $x_1,\dots,x_N \in \mathbb{R}^d$, letting $H'_{N,\eta}$ be as in* (2.2.21)*, for any $1/2 > \eta > \tau > 0$ we have*

$$\begin{aligned} -CN\|\mu_V\|_{L^\infty}\eta^{\frac{d-s}{2}} \leq \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H'_{N,\tau}|^2 - Nc_{d,s}g(\tau)\right) - \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H'_{N,\eta}|^2 - Nc_{d,s}g(\eta)\right) \\ \leq CN\|\mu_V\|_{L^\infty}\eta^{\frac{d-s}{2}} + c_{d,s}\sum_{i\neq j, |x_i-x_j|\leq 2\eta} g(|x_i-x_j|). \end{aligned}$$

where C depends only on d and s. In particular, sending $\tau \to 0$ yields that for all $\eta < 1/2$,

$$o_\eta(1)N \leq Nc_{d,s}w_N(x_1,\dots,x_N) - \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H'_{N,\eta}|^2 - Nc_{d,s}g(\eta)\right) \leq o_\eta(1)N + c_{d,s}\sum_{i\neq j, |x_i-x_j|\leq 2\eta} g(|x_i-x_j|). \quad (2.3.2)$$

where the term $o_\eta(1)$ goes to 0 when $\eta \to 0$ and is independent of the configuration.

Let us note that a lower estimate on the error of a similar form can also be obtained for finite N, however we will rather need such a lower estimate on the limiting renormalized energy, which will allow us to control the interaction due to close points by the truncation error $\mathcal{W} - \mathcal{W}_\eta$.

Lemma 2.3.4. *Let $E \in \mathcal{A}_m$ be such that $\mathcal{W}(E) < +\infty$. For any $1 > \eta > 0$ we have*

$$c_{d,s}\limsup_{\tau\to 0}\limsup_{R\to\infty}\frac{1}{R^d}\sum_{p\neq q\in\mathcal{C}\cap K_R, |p-q|\leq\eta}(g(|p-q|+\tau)-g(\eta))_+ \leq \mathcal{W}(E) - \mathcal{W}_\eta(E) + Cm^2\eta^{\frac{d-s}{2}}$$

where C depends only on s and d.

Proof. In [PS15, (2.29)] it is proven that for $0 < \tau < \eta < 1$, we have

$$-Cm^2\eta^{\frac{d-s}{2}} + \limsup_{R\to\infty}\frac{c_{d,s}}{R^d}\sum_{p\neq q\in\mathcal{C}\cap K_{R-3}}(g(|p-q|+\tau)-g(\eta))_+ \leq \mathcal{W}_\tau(E) - \mathcal{W}_\eta(E).$$

It then suffices to let $\tau \to 0$ to conclude. □

The next important property of w_N and $\widetilde{\mathbb{W}}$ is that they control the number of points and their "discrepancies" (i.e. the difference between the number of points in a ball and the integral of the equilibrium measure over that ball), as well as the electric fields themselves.

2.3.3 Coerciveness of the energy

In view of the monotonicity properties of $\mathcal{W}$ and w_N, an upper bound on the renormalized energy of an electric field E translates into a bound on E_η in L^2 with weight $|y|^\gamma$, for any η small. This in turns easily implies a bound on E in L^p spaces according to the following lemma:

Lemma 2.3.5. *Let K be a compact set with piecewise C^1 boundary and let E be a vector field satisfying a relation of the form*

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\Big(\mathcal{C} - \mu\delta_{\mathbb{R}^d}\Big) \text{ in } K\times\mathbb{R}^k$$

where $\mathcal{C}$ is a point configuration in K and μ is a bounded measure on C_R, and let E_η be given by (2.2.19).

For any $0 < \eta < 1$, for any $p < p_{\max}$, we have

$$\|E\|_{L^p(K\times\mathbb{R}^k)} \leq C\left(\int_{K\times\mathbb{R}^k}|y|^\gamma|E_\eta|^2\right)^{1/2} + C_{p,\eta,d,s}\mathcal{C}(K).$$

with a constant $C_{p,\eta,d,s}$ depending only on p,η,d,s such that $C_{p,\eta,d,s}\to 0$ when $\eta\to 0$ and the other parameters are fixed, and a constant C depending on K,p,s,d.

Proof. Using Hölder's inequality, we note that $L^2_{|y|^\gamma}(K)$ embeds continuously into $L^p(K)$ for $1 < p < \min(2, \frac{2}{\gamma+1}) \leq p_{\max}$. The lemma follows from observing that

$$\|E\|_{L^p(K\times\mathbb{R}^k)} \leq \|E_\eta\|_{L^p(K\times\mathbb{R}^k)} + \|\nabla f_\eta\|_{L^p}\mathcal{C}(K).,$$

which follows from Minkowski inequality and the definition of E_η. □

We may now state a compactness result for electric fields.

Lemma 2.3.6. *For any compact set $K \subset \mathbb{R}^d$ with piecewise C^1 boundary, let $\{E_n\}_n$ be a sequence of vector fields in $L^p(K, \mathbb{R}^{d+k})$ such that*

$$-\mathrm{div}\,(|y|^\gamma E_n) = c_{d,s}\,(\mathcal{C}_n - \mu_n\delta_{\mathbb{R}^d}) \ \text{ in } K\times\mathbb{R}^k \tag{2.3.3}$$

for a certain sequence $\{\mathcal{C}_n\}_n$ of point configurations in K and $\{\mu_n\}_n$ of bounded functions on K. Assume that $\{\mathcal{C}_n\}_n$ converges to $\mathcal{C}$ and that $\{\mu_n\}_n$ converges to μ in $L^\infty(K)$. For any $\eta > 0$, if $\int_{K\times\mathbb{R}^k} |y|^\gamma |E_{n,\eta}|^2$ is bounded uniformly in n, then there exists a vector field E satisfying

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C} - \mu\delta_{\mathbb{R}^d}) \ \text{ in } K\times\mathbb{R}^k$$

and such that for any $\eta > 0$, $z \in [0, +\infty]$

$$\int_{K\times[-z,z]^k} |y|^\gamma |E_\eta|^2 \leq \liminf_{n+\infty} \int_{K\times[-z,z]^k} |y|^\gamma |E_{n,\eta}|^2 \tag{2.3.4}$$

and in the case $k = 1$, for any $z > 0$

$$\int_{K\times(\mathbb{R}\setminus(-z,z))} |y|^\gamma |E|^2 \leq \liminf_{n+\infty} \int_{K\times(\mathbb{R}\setminus(-z,z))} |y|^\gamma |E_n|^2. \tag{2.3.5}$$

Proof. The sequence $\{E_{n,\eta}\}_n$ is bounded in $L^2_{|y|^\gamma}(K\times\mathbb{R}^k, \mathbb{R}^{d+k})$ hence we may find a vector field E such that up to extraction the sequence $\{E_n\}_n$ converges weakly to E in $L^2_{|y|^\gamma}(K\times\mathbb{R}^k, \mathbb{R}^{d+k})$. By Lemma 2.3.5, the convergence is also in L^p_{loc} for $p < p_{\max}$ hence in the sense of distributions, and we may take the limit in (2.3.3). Lower semi-continuity as in (2.3.4) and (2.3.5) is then a consequence of the weak convergence. □

Finally we state a compactness result for sequences of stationary electric processes with bounded energy.

Lemma 2.3.7. *Let $\{P_n^{\mathrm{elec}}\}_n$ be a sequence of stationary electric processes concentrated on $\mathcal{A}_1$ such that $\{\widetilde{\mathcal{W}}(P_n^{\mathrm{elec}})\}_n$ is bounded. Then, up to extraction, the sequence $\{P^{\mathrm{elec}}\}_n$ converges weakly to a stationary electric process P^{elec} concentrated on $\mathcal{A}_1$ such that*

$$\widetilde{\mathcal{W}}(P^{\mathrm{elec}}) \leq \liminf_{n\to\infty} \widetilde{\mathcal{W}}(P_n^{\mathrm{elec}}). \tag{2.3.6}$$

Proof. Up to extraction we may assume that the $\liminf_{n\to\infty}$ in (2.3.6) is actually a $\lim_{n\to\infty}$. It is clear that any weak limit point of $\{P_n^{\mathrm{elec}}\}_n$ is stationary and concentrated on $\mathcal{A}_1$. In view of Lemma 2.2.8 we have for any $R > 0$

$$\widetilde{\mathcal{W}}(P_n^{\mathrm{elec}}) = \mathbf{E}_{P_n^{\mathrm{elec}}}\left[\fint_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2 \right] - c_{d,s} g(\eta),$$

but the function $E \mapsto \fint_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2$ is weakly lower semi-continuous as observed in the proof of Lemma 2.3.6, thus if $P^{\rm elec}$ is a weak limit point we have

$$\lim_{n\to\infty} \widetilde{\mathcal{W}}(P_n^{\rm elec}) \geq \widetilde{\mathcal{W}}(P^{\rm elec}).$$

Therefore it remains to show that there exists a converging subsequence. Using Lemma 2.2.8 as above and the boundedness of $\{\widetilde{\mathcal{W}}(P_n^{\rm elec})\}_n$ we write for any $R, \eta > 0$

$$\mathbf{E}_{P_n^{\rm elec}}\left[\fint_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2\right] - c_{d,s}g(\eta) \leq C.$$

From Lemma 2.3.5, we deduce that $\{\int \|E\|_{L^p(C_R)} dP_n^{\rm elec}\}_n$ is also bounded for any $p < p_{\max}$, and this for any hypercube C_R ($R > 0$). Using for example Lemma 2.1 in [SS12] and the coerciveness of the L^p norm, we deduce as in [SS12, Section 2, Step 1] that this implies existence of weak limit points for $\{P_n^{\rm elec}\}_n$. □

2.3.4 Discrepancy estimates

In this section we give estimates to control the discrepancy between the number of points in a domain and the expected number of points according to the background intensity, in terms of the energy. These estimates show that local non-neutrality of the configurations has an energy cost, which in turn implies that stationary point processes of finite energy must have small discrepancies. We then apply these considerations to coercivity properties of $\mathcal{W}$.

The first estimate is based on the following energy lower bound, proven in [PS15, Lemma 2.2] (there it is stated for balls, but the proof for cubes is identical). We let $C_R(x)$ be the hypercube of center x and sidelength R in $\mathbb{R}^d$ and we denote by $\mathcal{D}(x,R)$ the discrepancy between the number of points in $C_R(x)$ and its expected value

$$D(x,R) := \int_{C_R(x)} d\mathcal{C} - \int_{C_R(x)} \mu(y)\,dy.$$

Lemma 2.3.8. *[PS15, Lemma 2.2] Assume E satisfies a relation of the form*

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\Big(\mathcal{C} - \mu\delta_{\mathbb{R}^d}\Big)$$

in some subset $U \subset \mathbb{R}^{d+k}$ for some $\mu \in L^\infty(U)$, with $\mathcal{C}$ a point configuration, and let E_η be associated as in (2.2.19). Then for any $0 < \eta < 1$, $R > 2$ and $x \in \mathbb{R}^d \times \{0\}$, denoting $\tilde{C}_R(x) = C_R(x) \times [-R/2, R/2]$, if $\tilde{C}_{2R}(x) \subset U$ we have

$$\int_{\tilde{C}_{2R}(x)} |y|^\gamma |E_\eta|^2 \geq C\frac{\mathcal{D}(x,R)^2}{R^s} \min\left(1, \frac{\mathcal{D}(x,R)}{R^d}\right), \tag{2.3.7}$$

for some C depending only on d, s and $\|\mu\|_{L^\infty}$.

For finite point configurations we get as a consequence of the previous lemma:

Lemma 2.3.9. *For any integer N and any $x_1, \dots, x_N \in \mathbb{R}^d$, we let w_N be as in (2.2.22) and $\bar{P}_{\nu_N}$ be $i_N(x_1 \dots, x_N)$ as in (2.1.9). Let us define $\overline{\mathcal{D}}_N(R)$ as*[1]

$$\overline{\mathcal{D}}_N(R)(x,\mathcal{C}) := \int_{C_R} d\mathcal{C} - \int_{C_R(N^{1/d}x)} d\mu'_V,$$

1. This is the correct object when dealing with $\bar{P}_{\nu_N}$ because the configurations have been translated.

We have

$$\mathbf{E}_{\bar{P}_{\nu_N}}\left[\overline{\mathcal{D}}_N(R)^2\min\left(1,\frac{\overline{\mathcal{D}}_N(R)}{R^d}\right)\right]\leq R^{d+s}\left(C_1+C_2w_N(x_1,\dots,x_N)\right)$$

with C_1, C_2 positive constants depending only on d and s.

Proof. Using the definition of P_{ν_N} and (2.3.7) we get (applying Fubini's identity in the first line) for any fixed $0<\eta<1$, with $E_N=\nabla H'_N$ is the electric field generated by the configuration (where H'_N is as in (2.2.18)):

$$\begin{aligned}\mathbf{E}_{\bar{P}_{\nu_N}}\left[\overline{\mathcal{D}}_N(R)^2\min\left(1,\frac{\overline{\mathcal{D}}_N(R)}{R^d}\right)\right]&\leq\frac{CR^s}{N}\int_{N^{1/d_\Sigma}}\int_{\theta_x\cdot\tilde{C}_{2R}}|y|^\gamma|E_{N,\eta}|^2\\&\leq\frac{CR^{d+s}}{N}\int_{\mathbb{R}^{d+k}}|y|^\gamma|E_\eta|^2\leq\frac{CR^{d+s}}{N}\left(\int_{\mathbb{R}^{d+k}}|y|^\gamma|E_{N,\eta}|^2-Nc_{d,s}g(\eta)+Nc_{d,s}g(\eta)\right)\end{aligned}$$

and we conclude by using (2.3.2). □

In the following we specialize to stationary point processes of intensity 1 but the corresponding result for a different intensity is easily deduced by scaling. We denote as previously by $\mathcal{N}(x,R):\mathcal{X}\mapsto\mathbb{N}$ the number of points of a configuration in the hypercube $C_R(x)$ and by $\mathcal{D}(x,R)$ the discrepancy $\mathcal{D}(x,R)=\mathcal{N}(x,R)-R^d$. We note that in fact by stationarity their laws do not depend on x.

Lemma 2.3.10. *Let P be a stationary point process such that $\widetilde{\mathbb{W}}_1(P)$ is finite. Then P has intensity 1 i.e. $\mathbf{E}_P[\mathcal{N}(0,R)]=R^d$ for all $R>0$. Moreover for any $R>1$ it holds*

$$\mathbf{E}_P\left[\mathcal{D}^2(0,R)\right]\leq C(C+\widetilde{\mathbb{W}}_1(P))R^{d+s}=o(R^{2d}) \tag{2.3.8}$$

with C a positive constant depending only on d and s. This implies for $R>1$

$$\mathbf{E}_P\left[\mathcal{N}^2(0,R)\right]\leq R^{2d}+C(C+\widetilde{\mathbb{W}}_1(P))R^{d+s}. \tag{2.3.9}$$

Proof. The first point of the lemma is an easy consequence of the second one, indeed from (2.3.8) we get using Jensen's inequality that $\mathbf{E}_P[\mathcal{D}(x,R)]=o(R^d)$. On the other hand the stationarity assumption implies that $\mathbf{E}_P[\mathcal{D}(0,R)]=R^d\mathbf{E}_P[\mathcal{D}(0,1)]$ (for any $R>0$) hence in fact $\mathbf{E}_P[\mathcal{D}(0,R)]=0$ for any $R>0$ which implies that P has intensity 1. We now turn to proving (2.3.8).

From Lemma 2.2.12 we know that we may find an electric process $P^{\rm elec}$ concentrated on $\mathcal{A}_1$ such that the push-forward of $P^{\rm elec}$ by Conf_1 is P, and satisfying $\widetilde{\mathcal{W}}(P^{\rm elec})=\widetilde{\mathbb{W}}_1(P)$. Set $\eta_0=\frac{1}{4}$. By the monotonicity property (2.3.4) we see that

$$\int\mathcal{W}_{\eta_0}(E)dP^{\rm elec}(E)\leq\widetilde{\mathcal{W}}(P^{\rm elec})+C\leq\widetilde{\mathbb{W}}_1(P)+C$$

with a constant C depending only on d,s.

In the case $k=1$, by stationarity and the definition of $\mathcal{W}$ we see that

$$\mathbf{E}_{P^{\rm elec}}\left[\int_{C_1\times\mathbb{R}^k}|y|^\gamma|E_{\eta_0}|^2\right]=\int\mathcal{W}_{\eta_0}(E)dP^{\rm elec}(E)+c_{d,s}g(\eta_0)\leq\widetilde{\mathcal{W}}(P^{\rm elec})+C,$$

with a constant C depending only on d,s. Hence for any $R>0$ we may find $T\in(R,2R)$ such that

$$\mathbf{E}_{P^{\rm elec}}\left[\int_{C_1\times\{-T,T\}}|y|^\gamma|E_{\eta_0}|^2\right]\leq\frac{1}{R}\left(\widetilde{\mathcal{W}}(P^{\rm elec})+C\right)=\frac{1}{R}\left(\widetilde{\mathbb{W}}_1(P)+C\right). \tag{2.3.10}$$

Letting $\check{C}_R$ be the hyperrectangle $C_R \times [-T,T]^k$ we have

$$\int_{\partial \check{C}_R} |y|^\gamma E_{\eta_0} \cdot \vec{\nu} = \int_{\check{C}_R} -\mathrm{div}\left(|y|^\gamma E_{\eta_0}\right) = c_{d,s}(\mathcal{D}(0,R) + r_{\eta_0}), \tag{2.3.11}$$

where the point configuration is implicitly $\mathrm{Conf}_1(E)$ and where the error term r_{η_0} is bounded by n_{η_0}, the number of points of $\mathrm{Conf}_1(E)$ in an η_0-neighborhood of ∂C_R. We may see the η_0-neighborhood of ∂C_R as included in a disjoint union of $O(R^{d-1})$ hypercubes of sidelength 1 and by stationarity we have

$$\mathbf{E}_P[n_{\eta_0}^2] \leq CR^{2d-2}\mathbf{E}_P\left[\mathcal{N}(0,1)^2\right]. \tag{2.3.12}$$

Taking the expectation of (2.3.11) against P^{elec} and using elementary inequalities and (2.3.12) we get

$$\mathbf{E}_P\left[\mathcal{D}(0,R)^2\right] \leq C\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{\partial \check{C}_R} |y|^\gamma |E_{\eta_0}|^2\right]\left(\int_{\partial \check{C}_R} |y|^\gamma\right) + CR^{2d-2}\mathbf{E}_P\left[\mathcal{N}(0,1)^2\right]. \tag{2.3.13}$$

In the case $k=0$ we have

$$\int_{\partial \check{C}_R} |y|^\gamma = CR^{d-1} = CR^{s+1}, \tag{2.3.14}$$

whereas in the case $k=1$, recalling that $\gamma = s+2-d-k$ we easily compute that

$$\int_{\partial \tilde{C}_R} |y|^\gamma \leq CR^{d-1}\int_0^T |y|^\gamma + CR^d T^\gamma \leq CR^{d-1}R^{s+3-d-1} + CR^{d+s+2-d-1} = CR^{s+1}. \tag{2.3.15}$$

We may also split $\partial \check{C}_R$ as the disjoint union of

1. $2d$ lateral faces of the type $[-R/2,-R/2]\times \ldots \{\pm R/2\} \times \cdots \times [-R/2,R/2]\times[-T,T]^k$,
2. 0 (if $k=0$) or 2 (if $k=1$) faces of the type $C_R \times \{\pm T\}^k$.

For each of the $2d$ faces of the first type we may write using the stationarity of P^{elec}

$$\begin{aligned}\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{[-\frac{R}{2},\frac{R}{2}]\times\cdots\{\pm\frac{R}{2}\}\times\cdots\times[-\frac{R}{2},\frac{R}{2}]\times[-T,T]^k} |y|^\gamma |E_{\eta_0}|^2\right] &= \frac{1}{R}\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_R\times[-T,T]^k} |y|^\gamma |E_{\eta_0}|^2\right] \\ \leq \frac{1}{R}\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_R\times\mathbb{R}^k} |y|^\gamma |E_{\eta_0}|^2\right] &= \frac{CR^d}{R}\left(\widetilde{\mathcal{W}}(P^{\mathrm{elec}}) + C\right) \leq CR^{d-1}\left(\widetilde{\mathbb{W}}_1(P) + C\right),\end{aligned} \tag{2.3.16}$$

whereas for the second type of faces we have, using (2.3.10) and the stationarity of P^{elec}

$$\begin{aligned}\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_R\times\{-T,T\}^k} |y|^\gamma |E_{\eta_0}|^2\right] &= R^d\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_1\times\{-T,T\}^k} |y|^\gamma |E_{\eta_0}|^2\right] \\ &\leq CR^{d-1}(\widetilde{\mathbb{W}}_1(P) + C)\end{aligned} \tag{2.3.17}$$

Inserting (2.3.14) (if $k=0$) or (2.3.15) (if $k=1$), (2.3.16) and (2.3.17) (if $k=1$) into (2.3.13) we obtain

$$\mathbf{E}_P\left[\mathcal{D}(0,R)^2\right] \leq C\left(\widetilde{\mathbb{W}}_1(P) + C\right)R^{d-1+s+1} + CR^{2d-2}\mathbf{E}\left[\mathcal{N}(0,1)^2\right]. \tag{2.3.18}$$

The fact that $\mathbf{E}\left[\mathcal{N}(0,1)^2\right]$ is itself bounded by $C\left(\widetilde{\mathbb{W}}_1(P) + C\right)$ can be deduced from the previous discrepancy estimate (2.3.7). Let us denote by ψ_R the function $\psi_R : x \mapsto x^2 \min(1, \frac{x}{R^d})$, dividing (2.3.7) by R^d we see that for any $E \in \mathcal{A}_1$ it holds, for R large enough,

$$\frac{1}{R^{d+s}}\psi_R(\mathcal{D}(0,R)) \leq \frac{C}{R^d}\int_{\tilde{C}_{2R}(x)} |y|^\gamma |E_{\eta_0}|^2$$

with the notation $\tilde{C}_{2R}$ as in Lemma 2.3.8. Taking as before the expectation under $P^{\rm elec}$ it yields

$$\frac{1}{R^{d+s}}\mathbf{E}_P[\psi_R(\mathcal{D}(0,R))] \leq C\mathbf{E}_{P^{\rm elec}}\left[\frac{1}{R^d}\int_{\tilde{C}_{2R}(x)}|y|^\gamma|E_{\eta_0}|^2\right]$$

By stationarity and the definition of $\mathcal{W}(E)$ we have

$$\mathbf{E}_{P^{\rm elec}}\left[\frac{1}{R^d}\int_{\tilde{C}_{2R}(0)}|y|^\gamma|E_{\eta_0}|^2\right] \leq C(\widetilde{\mathcal{W}}_{\eta_0}(P^{\rm elec})+C)$$

hence we get

$$\frac{1}{R^{d+s}}\mathbf{E}_P[\psi_R(\mathcal{D}(0,R))] \leq C(\widetilde{\mathbb{W}}_1(P)+C)$$

which gives a less accurate bound on the discrepancy than (2.3.8) but allows one to bound $\mathbf{E}\left[\mathcal{N}(0,1)^2\right]$ by $C\left(\widetilde{\mathbb{W}}_1(P)+C\right)$. Finally we get (2.3.8) from (2.3.18), and the bound (2.3.9) follows easily from (2.3.8). □

In particular, we observe that in the two-dimensional Coulomb case (2.1.2) the bound (2.3.8) yields

$$\mathbf{E}_P\left[\mathcal{D}(0,R)^2\right] \leq C(\widetilde{\mathbb{W}}_1(P)+C)R^2$$

hence the variance of the number of points for a point process of finite renormalized energy is comparable to that of a Poisson point process. It is unclear to us whether this estimate is sharp or not. In the one-dimensional log-gas case however, we have

Remark 2.3.11. *In the case* (2.1.1)*, if P be a stationary point process of finite renormalized energy then we have*

$$\liminf_{R\to\infty}\frac{1}{R}\mathbf{E}_P[\mathcal{D}(0,R)^2] = 0, \tag{2.3.19}$$

in particular the Poisson point process $\mathbf{\Pi}^1$ has infinite renormalized energy for $d=1, s=0$.

Proof. We follow the same line as in the proof of Lemma 2.3.10. We replace (2.3.10) by the following observation: since $\int_{C_1\times\mathbb{R}}|y|^\gamma|E_{\eta_0}|^2$ is finite we have

$$\liminf_{T\to\infty}(T\log T)\int_{C_1\times\{-T,T\}}|y|^\gamma|E_{\eta_0}|^2 = 0.$$

In particular we might find an increasing sequence $\{T_k\}_k$ with $\lim_{k\to\infty}T_k=+\infty$ and

$$\lim_{k\to\infty}(T_k\log T_k)\int_{C_1\times\{-T_k,T_k\}}|y|^\gamma|E_{\eta_0}|^2 = 0.$$

Setting $R_k = T_k\sqrt{\log T_k}$ and keeping the same notation as in the previous proof we see that for k large enough we have

$$\mathbf{E}_P[\mathcal{D}(0,R_k)^2] \leq CT_k(1+\frac{R_k}{T_k\log T_k}) = o(R_k),$$

which proves (2.3.19). □

The discrepancy estimate (2.3.9) gives a uniform bound on the discrepancy in terms of the renormalized energy. The next lemma allows to control the number of points on small scales (in a more precise but non-uniform way) and is based instead on Lemma 2.3.4.

Lemma 2.3.12. *Let $\bar{P}^{\rm elec} \in \mathcal{P}_s(\Sigma \times L^p_{\rm loc}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k}))$ be such that $\overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\rm elec}) < +\infty$, and let $\bar{P}$ the underlying tagged point process i.e. the push-forward of $\bar{P}^{\rm elec}$ by $(x, E) \mapsto (x, {\rm Conf}_{\mu_V(x)}E)$. Then for any $0 < \tau < \eta^2/2$ and $\eta < 1$, we have*

$$\frac{g(2\tau)}{\tau^d}\mathbf{E}_{\bar{P}}[(\mathcal{N}(0,\tau)^2 - 1)_+] + \mathbf{E}_{\bar{P}}\left[\sum_{p\neq q\in\mathcal{C}\cap C_1, |p-q|\leq \eta^2/2} g(|p-q|)\right] \leq C\left(\overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\rm elec}) - \int_\Sigma \widetilde{\mathcal{W}}_\eta(\bar{P}^{{\rm elec},x})dx\right) + C\eta^{d-s}, \quad (2.3.20)$$

for some constant $C > 0$ depending only on s, d.

Proof. We integrate the left-hand side in the result of Lemma 2.3.4 with respect to $\bar{P}^{{\rm elec},x}$ and obtain, by monotone convergence in τ and stationarity (cf. Lemma 2.2.8), that it is equal to

$$c_{d,s}\limsup_{\tau\to 0}\mathbf{E}_{\bar{P}^{{\rm elec},x}}\left[\limsup_{R\to\infty}\sum_{p\neq q\in\mathcal{C}\cap C_R}(g(|p-q|+\tau) - g(\eta))_+\right] = c_{d,s}\limsup_{\tau\to 0}\mathbf{E}_{\bar{P}^x}\left[\sum_{p\neq q\in\mathcal{C}\cap C_1}(g(|p-q|+\tau) - g(\eta))_+\right].$$

Using again the monotone convergence theorem in $\tau \to 0$, this is equal to

$$c_{d,s}\mathbf{E}_{\bar{P}^x}\left[\sum_{p\neq q\in\mathcal{C}\cap C_1}(g(|p-q| - g(\eta))_+\right].$$

Now we note that in all cases (2.1.1)–(2.1.2)–(2.1.3), there exists $C > 0$ depending only on s and d such that if $\tau < \eta^2/2$,

$$\sum_{p\neq q\in\mathcal{C}\cap C_1}(g(|p-q| - g(\eta))_+ \geq C\sum_{p\neq q\in\mathcal{C}\cap C_1, |p-q|<\eta^2/2} g(|p-q|)$$
$$\geq C\sum_{\vec{i}\in C_1\cap\tau\mathbb{Z}^d}\mathcal{N}(\vec{i},\tau)(\mathcal{C})(\mathcal{N}(\vec{i},\tau)(\mathcal{C}) - 1)g(2\tau) \geq \frac{C}{2}g(2\tau)\sum_{\vec{i}\in C_1\cap\tau\mathbb{Z}^d}((\mathcal{N}(\vec{i},\tau)^2(\mathcal{C}) - 1)_+$$

where we denote by $(\mathcal{N}(\vec{i},\tau)(\mathcal{C})$ the number of points of the configuration $\mathcal{C}$ in the hypercube of center $\vec{i}$ and sidelength, with $\vec{i} \in \tau\mathbb{Z}^d$ whose edges are parallel to the axes of $\mathbb{Z}^d$ and of sidelength τ.

Using stationarity again, we find that the expectation of this quantity is bounded below by a constant times the left-hand side in (2.3.20), and the result then follows from Lemma 2.3.4 integrated against $\bar{P}^{{\rm elec},x}$ and then against the normalized Lebesgue measure on Σ. □

2.3.5 Minimality of the local energy

As already mentioned, given a configuration $\mathcal{C}$ in a compact set K and an underlying (bounded, measurable) density μ on K, there exist many electric vector fields that are compatible with the configuration i.e. such that $-{\rm div}\,(|y|^\gamma E) = c_{d,s}(\mathcal{C} - \mu\delta_{\mathbb{R}^d})$. Indeed to any such vector field one may add any solution of $-{\rm div}\,(|y|^\gamma E) = 0$.

Since the configuration in a given compact set is finite there is however a natural choice, which we call the "local field", given by

$$E^{\rm loc} := \nabla H^{\rm loc}, \text{ with } H^{\rm loc} := c_{d,s}g * (\mathcal{C} - \mu\delta_{\mathbb{R}^d}\mathbf{1}_K). \quad (2.3.21)$$

The following lemma shows that among all possible electric fields for a finite point configurations, the local electric field defined by (2.3.21) has a smaller energy than any "screened" electric field. The reason is that $E^{\rm loc}$ is an $L^2_{|y|^\gamma}$ orthogonal projection of any generic compatible E onto gradients, and the projection decreases the $L^2_{|y|^\gamma}$ norm.

Lemma 2.3.13. *Let μ be a bounded measurable function on a compact set (with piecewise C^1 boundary) $K \subset \mathbb{R}^d$, $\mathcal{C}$ a point configuration and $E^{\rm loc}$ the local electric field as in* (2.3.21)*. Let $E \in L^p_{\rm loc}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ be a vector field satisfying*

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C} - \mu\delta_{\mathbb{R}^d}) & \text{in } K \times \mathbb{R}^k \\ E \cdot \vec{\nu} = 0 & \text{on } \partial K \times \mathbb{R}^k. \end{cases} \tag{2.3.22}$$

Then, for any $0 < \eta < 1$ we have

$$\int_{\mathbb{R}^{d+k}} |y|^\gamma |E^{\rm loc}_\eta|^2 \le \int_{K\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2. \tag{2.3.23}$$

Proof. First we note that we may extend E by 0 outside of K and since $E \cdot \vec{\nu}$ is continuous across ∂K, no divergence is created there, and E satisfies

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}(\mathcal{C} - \mu\delta_{\mathbb{R}^d}) = -\mathrm{div}\,(|y|^\gamma E^{\rm loc}) \quad \text{in } \mathbb{R}^{d+k}. \tag{2.3.24}$$

Second, we notice that if (2.3.22) holds we must have $\mathcal{C}(K) = \int_K \mu$, i.e. there is global neutrality of the charges in K. This global neutrality implies that $H^{\rm loc}$ as defined in (2.3.21) decays like $|\nabla g|$ i.e. like $|x|^{-s-1}$ as $|x| \to \infty$ in $\mathbb{R}^{d+k}$ and $E^{\rm loc}$ decreases like $|x|^{-s-2}$ (with the convention $s = 0$ in the cases (2.1.1)–(2.1.2)). If the right-hand side of (2.3.23) is infinite, then there is nothing to prove. If it is finite, given $M > 1$, and letting χ_M be a smooth nonnegative function equal to 1 in $C_M \times [-M, M]^k$ and 0 at distance ≥ 1 from $C_M \times [-M, M]$, we may write

$$\begin{aligned} \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E_\eta|^2 &= \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E_\eta - E^{\rm loc}_\eta|^2 + \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E^{\rm loc}_\eta|^2 \\ &\qquad + 2\int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma (E_\eta - E^{\rm loc}_\eta)\cdot E^{\rm loc}_\eta \\ &\ge \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E^{\rm loc}_\eta|^2 + 2\int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma (E_\eta - E^{\rm loc}_\eta)\cdot(\nabla H^{\rm loc}_\eta) \\ &= \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E^{\rm loc}_\eta|^2 + 2\int_{\mathbb{R}^{d+k}} H^{\rm loc}_\eta |y|^\gamma (E_\eta - E^{\rm loc}_\eta)\cdot \nabla\chi_M \end{aligned}$$

where we integrated by parts and used (2.3.24) to remove one of the terms. Letting $M \to \infty$, the last term tends to 0 by finiteness of the right-hand side of (2.3.23) and decay properties of $H^{\rm loc}$ and $E^{\rm loc}$, and we obtain the result. □

2.4 Proof of the main results

In this section, we give the proof of Theorem 9 and its corollaries, assuming the results of Propositions 2.1.6 and 2.1.7, whose proof will occupy the main part of the paper.

2.4.1 Exponential tightness and goodness of the rate function

Lemma 2.4.1. *The following holds:*

— *For any $\beta > 0$, the sequences $\{\mathfrak{P}_{N,\beta}\}_N$ and $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ are exponentially tight.*

— *The functionals $\widetilde{\mathbb{W}}_1$ (resp. $\overline{\mathbb{W}}_{\mu_V}$) are lower semi-continuous over stationary point processes (resp. tagged point processes), bounded below, and have compact sub-level sets.*
— *The functionals $\mathcal{F}_\beta$ and $\overline{\mathcal{F}}_\beta$ are good rate functions.*

Proof. The exponential tightness of $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ is an easy consequence of the fact that the total number of points in $N^{1/d}\Sigma$ is bounded by N. Indeed it implies that for $\overline{\mathfrak{P}}_{N,\beta}$-a.e. tagged point process $\bar{P}_N$ and for any $R > 0$

$$\mathbf{E}_{\bar{P}_N}\left[\mathcal{N}(0,R)\right] \leq CR^d$$

with a constant C depending only on V, d. Let us fix two increasing sequences $\{R_k\}_k$ and $\{M_k\}_k$ going to ∞. Markov's inequality implies that for any $N, k \geq 1$ for $\overline{\mathfrak{P}}_{N,\beta}$-a.e. tagged point process $\bar{P}_N$ we have

$$\bar{P}_N\left(\mathcal{N}(0,R_k) \geq M_k R_k^d\right) \leq \frac{C}{M_k}.$$

Using Lemma 2.2.6 and a simple extraction argument we see that

$$K := \bigcap_{k=1}^{+\infty} \left\{ \bar{P}, \bar{P}\left(\mathcal{N}(0,R_k) \geq M_k R^d\right) \leq \frac{C}{M_k} \right\}$$

is a compact set in $\mathcal{P}(\Sigma \times \mathcal{X})$ which contains $\overline{\mathfrak{P}}_{N,\beta}$-a.e. tagged point process $\bar{P}_N$.

We may now be more precise in the description of a "typical" (up to very large deviations) limit point under $\overline{\mathfrak{P}}_{N,\beta}$. Inserting the formula (2.1.7)–(2.1.8) into (2.1.19), using the definition (2.1.19) and the control (2.1.20) on $K_{N,\beta}$, we obtain that for any β larger than any fixed $\beta_0 > 0$ and any $M > 0$

$$\begin{aligned}\mathbb{P}_{N,\beta}(w_N^{-1}([M,+\infty]) &\leq \frac{1}{K_{N,\beta}} e^{-\frac{1}{2}\beta MN} \int e^{-N\beta\sum_{i=1}^N \zeta(x_i)}\, dx_1 \dots dx_N \\ &\leq e^{C_\beta \beta N - \frac{1}{2}\beta MN} \left(\int_{\mathbb{R}^d} e^{-N\beta\zeta(x)}\, dx\right)^N\end{aligned}$$

Thanks to assumption (2.2.9), for N large enough the function $e^{-N\beta\zeta}$ is dominated in L^1 (indeed ζ behaves like $g(x) + \frac{1}{2}V - c$ as $|x| \to \infty$), and by the dominated convergence theorem we have

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} e^{-N\beta\zeta(x)}\, dx = |\{\zeta = 0\}| = |\omega|, \tag{2.4.1}$$

hence we find

$$\frac{1}{N}\log \mathbb{P}_{N,\beta}(w_N^{-1}([M,+\infty]) \leq -\frac{1}{2}\beta(M - C) \tag{2.4.2}$$

for some constant C depending only on d, s, V and β_0. Let us define

$$K_N^M = i_N\left(w_N^{-1}([-\infty, M))\right) \text{ and } K^M := \cup_{N\geq 1} K_N^M.$$

Equation (2.4.2) implies that $\frac{1}{N}\log \overline{\mathfrak{P}}_{N,\beta}((K^M)^c) \leq -\frac{1}{2}\beta(M-C)$. On the other hand, Lemma 2.3.1 shows that any limit point of a sequence of finite configurations with bounded energy is stationary and has finite energy $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$. This allows to restrict ourselves to studying the large deviations around tagged point processes $\bar{P}$ which are stationary and with finite energy. In particular, as a consequence of Lemma 2.3.10, $\bar{P}^x$ has intensity $\mu_V(x)$ for Lebesgue-a.e. $x \in \Sigma$.

Let us next prove the lower semi-continuity of $\widetilde{\mathbb{W}}_1$ (resp. $\overline{\mathbb{W}}_{\mu_V}$) on the space of stationary (resp. tagged stationary) point processes. Let $\{P_n\}_n$ be a sequence of stationary point processes converging to $P \in \mathcal{P}_s(\mathcal{X})$. We may assume that $\liminf_{n\to\infty} \widetilde{\mathbb{W}}_1(P_n) < +\infty$ otherwise

there is nothing to prove, and up to extraction we may also assume that $\liminf_{n\to\infty} \widetilde{\mathbb{W}}_1(P_n) = \lim_{n\to\infty} \widetilde{\mathbb{W}}_1(P_n)$. By Lemma 2.2.12 there exists for each n a stationary electric process P_n^{elec} whose push-forward by Conf_1 is equal to P_n and such that $\widetilde{\mathcal{W}}(P_n^{\mathrm{elec}}) = \widetilde{\mathbb{W}}_1(P_n)$. The sequence $\{\widetilde{\mathcal{W}}(P_n^{\mathrm{elec}})\}$ is bounded, which together with Lemma 2.3.7 implies that up to extraction we have $P_n^{\mathrm{elec}} \to P^{\mathrm{elec}}$ for some electric process P^{elec} which is also stationary and it is easy to see that P^{elec} satisfies $\mathrm{Conf}_1 \# P^{\mathrm{elec}} = P$.

Moreover we know from Lemma 2.3.7 that $\liminf_{n\to\infty} \widetilde{\mathcal{W}}(P_n^{\mathrm{elec}}) \geq \widetilde{\mathcal{W}}(P^{\mathrm{elec}})$ but by assumption we have $\widetilde{\mathcal{W}}(P_n^{\mathrm{elec}}) = \widetilde{\mathbb{W}}_1(P_n)$ and by definition we have $\widetilde{\mathbb{W}}_1(P) \leq \widetilde{\mathcal{W}}(P^{\mathrm{elec}})$, hence

$$\widetilde{\mathbb{W}}_1(P) \leq \liminf_{n\to\infty} \widetilde{\mathbb{W}}_1(P_n)$$

which implies the lower semi-continuity of $\widetilde{\mathbb{W}}_1$. The lower semi-continuity of $\overline{\mathbb{W}}_{\mu_V}$ is a straightforward consequence. The fact that both are bounded below follows from the same fact known for $\mathcal{W}$.

To prove the compactness of sub-level sets for $\widetilde{\mathbb{W}}_1$ and $\overline{\mathbb{W}}_{\mu_V}$, they key point is to see that Lemma 2.3.10 implies uniform integrability of $\mathcal{N}(0,R)$ against point processes living on any sub-level set of the energy functional. Then using the compactness result of Lemma 2.2.6 we see that every sequence of point processes in a sub-level set is tight, hence the sub-level sets (being closed by lower semi-continuity) are compact.

Finally, it is known that the specific relative entropy ent is a good rate function (see e.g. [RAS09]), which also implies that $\overline{\mathsf{ent}}$ is and thus $\mathcal{F}_\beta$ and $\overline{\mathcal{F}}_\beta$ are good rate functions as the sum of two good rate functions. □

Goodness of the rate function implies in particular the existence of minimizers for $\mathcal{F}_\beta$ and $\overline{\mathcal{F}}_\beta$.

2.4.2 Proof of Theorem 9 and Corollary 2.1.5

From Propositions 2.1.6 and 2.1.7, the proof of Theorem 9 is standard.

Let $\bar{P}$ be in $\mathcal{P}_s(\Sigma\times\mathcal{X})$. Using the notation of (2.1.19) and (2.1.21), we have for any $\delta_1,\delta_2>0$

$$\begin{aligned}
&\overline{\mathfrak{P}}_{N,\beta}\left(B(\bar{P},\delta_1)\right)\\
&\quad= \frac{1}{K_{N,\beta}}\left(\int_{\mathbb{R}} e^{-N\beta\zeta}\right)^N \int_{i_N(x_1,\dots,x_N)\in B(\bar{P},\delta_1)} \exp(-\frac{N}{2}\beta w_N(x_1,\dots,x_N))d\mathbb{Q}_{N,\beta}(x_1,\dots,x_N)\\
&\qquad\qquad\geq \frac{1}{K_{N,\beta}}\left(\int_{\mathbb{R}} e^{-N\beta\zeta}\right)^N \exp(-\frac{\beta}{2}(\overline{\mathbb{W}}(\bar{P})+\delta_2))\\
&\qquad\qquad\times \mathbb{Q}_{N,\beta}\Big(i_N(x_1,\dots,x_N)\in B(\bar{P},\delta_1) \text{ and } w_N(x_1,\dots,x_N)\leq \overline{\mathbb{W}}(\bar{P})+\delta_2\Big). \quad (2.4.3)
\end{aligned}$$

Consequently for any $\delta_1,\delta_2>0$ we get

$$\begin{aligned}
\frac{1}{N}\log\overline{\mathfrak{P}}_{N,\beta}\left(B(\bar{P},\delta_1)\right) \geq \frac{1}{N}\log\bar{\mathfrak{Q}}_{N,\beta}\Big(B(\bar{P},\delta_1) \text{ and } w_N(x_1,\dots,x_N)\leq\overline{\mathbb{W}}(\bar{P})+\delta_2\Big)\\
-\frac{1}{N}\log K_{N,\beta} - \frac{\beta}{2}(\overline{\mathbb{W}}(\bar{P})+\delta_2) + \log\int_{\mathbb{R}} e^{-N\beta\zeta}.
\end{aligned}$$

Applying Proposition 2.1.7 and using (2.1.23) and (2.4.1), we get for any $\delta_2>0$

$$\begin{aligned}
&\lim_{\delta_1\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\overline{\mathfrak{P}}_{N,\beta}\left(B(\bar{P},r)\right) + \frac{1}{N}\log K_{N,\beta}\\
&\qquad\geq -\int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx - (\log|\omega| - |\Sigma| + 1) + \log|\omega| - \frac{\beta}{2}(\overline{\mathbb{W}}(\bar{P})+\delta_2). \quad (2.4.4)
\end{aligned}$$

More precisely Proposition 2.1.7 is only stated for $\bar{P} \in \mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$ (with the restriction that the "global" intensity of $\bar{P}$ is 1) but if $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$ is finite we know from Lemma 2.3.10 that $\bar{P}$ is indeed in $\mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$ because $\bar{P}^x$ must be a.e. of intensity $\mu_V(x)$, and otherwise (2.4.4) holds trivially since the right-hand side is $-\infty$. Now by sending $\delta_2 \to 0$, we obtain

$$\lim_{\delta_1 \to 0} \liminf_{N \to \infty} \frac{1}{N} \log \overline{\mathfrak{P}}_{N,\beta}\left(B(\bar{P}, \delta_1)\right) + \frac{1}{N} \log K_{N,\beta} \geq -\int_\Sigma \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx - \frac{\beta}{2} \overline{\mathbb{W}}_{\mu_V}(\bar{P}) - (1 - |\Sigma|). \tag{2.4.5}$$

On the other hand, returning to (2.4.3), we have by lower semi-continuity of $\overline{\mathbb{W}}_{\mu_V}$ over stationary processes as proven in Lemma 2.4.1 and by Lemma 2.3.1 and Proposition 2.1.6,

$$\limsup_{\delta_1 \to 0} \lim_{N \to \infty} \frac{1}{N} \log \overline{\mathfrak{P}}_{N,\beta}\left(B(\bar{P}, \delta_1)\right) + \frac{1}{N} \log K_{N,\beta} \leq -\int_\Sigma \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx - \frac{\beta}{2} \overline{\mathbb{W}}_{\mu_V}(\bar{P}) - (1 - |\Sigma|). \tag{2.4.6}$$

The exponential tightness proven in Lemma 2.4.1 allows one to pass from a weak formulation as in (2.4.5), (2.4.6) i.e. a large deviation inequality around a fixed $\bar{P}$ to any subset $A \subset \mathcal{P}(\Sigma \times \mathcal{X})$. We then get

$$\begin{aligned} -\inf_{\bar{P} \in \mathring{A}} \left(-\int_\Sigma \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx + \frac{\beta}{2} \overline{\mathbb{W}}_{\mu_V}(\bar{P}) \right) - (1 - |\Sigma|) \\ \leq \liminf_{N \to \infty} \frac{1}{N} \left(\log \overline{\mathfrak{P}}_{N,\beta}(A) + \log K_{N,\beta} \right) \leq \limsup_{N \to \infty} \frac{1}{N} \left(\log \overline{\mathfrak{P}}_{N,\beta}(A) + \log K_{N,\beta} \right) \\ \leq -\inf_{\bar{P} \in \bar{A}} \left(\int_\Sigma \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx + \frac{\beta}{2} \overline{\mathbb{W}}_{\mu_V}(\bar{P}) \right) - (1 - |\Sigma|). \end{aligned} \tag{2.4.7}$$

Hence taking $A = \mathcal{P}(\Sigma \times \mathcal{X})$ we see that $\lim_{N \to \infty} \frac{1}{N} \log K_{N,\beta}$ exists and

$$\lim_{N \to \infty} \frac{1}{N} \log K_{N,\beta} = -\beta \inf \overline{\mathcal{F}}_\beta. \tag{2.4.8}$$

Finally, inserting (2.4.8) into (2.4.7) yields Theorem 9.

Combining (2.1.19) and (2.4.8) we immediately get the main results of Corollary 2.1.5. For (2.1.18) we use the following scaling result:

Lemma 2.4.2. *For any $m > 0$ and P in $\mathcal{P}_s(\mathcal{X})$ of intensity m we have*

$$\mathsf{ent}[P | \mathbf{\Pi}^1] = m\, \mathsf{ent}[(\sigma_m P) | \mathbf{\Pi}^1] + 1 - m + m \log m. \tag{2.4.9}$$

Proof. Let us recall that the usual relative entropy $\mathrm{Ent}[\mu|\nu]$, where μ and ν are two probability measures on the same probability space is defined as $\int \log(\frac{d\mu}{d\nu}) d\mu$ if μ is absolutely continuous with respect to ν and $+\infty$ otherwise). By a change of variable $\mathcal{C} \mapsto m^{1/d}\mathcal{C}$ (as in the definition (2.2.32)) we get

$$\begin{aligned} \mathrm{Ent}[(\sigma_m P)_{|C_N} | \mathbf{\Pi}^1_{|C_N}] &= \int_{\mathcal{C} \in \mathcal{X}(C_N)} \log \left[\frac{d(\sigma_m P)_{|C_N}(\mathcal{C})}{d\mathbf{\Pi}^1_{|C_N}(\mathcal{C})} \right] d(\sigma_m P)_{|C_N}(\mathcal{C}) \\ &= \int_{\mathcal{C} \in \mathcal{X}(C_{m^{-1}N})} \log \left[\frac{dP_{|C_{m^{-1}N}}(\mathcal{C})}{d\mathbf{\Pi}^1_{|C_N}(m^{1/d}\mathcal{C})} \right] dP_{|C_{m^{-1}N}}(\mathcal{C}). \end{aligned}$$

hence

$$\mathrm{Ent}[(\sigma_m P)_{|C_N} | \mathbf{\Pi}^1_{|C_N}] = \mathrm{Ent}[P_{|C_{mN}} | \mathbf{\Pi}^1_{|C_{mN}}] + \int_{\mathcal{C} \in \mathcal{X}(C_{m^{-1}N})} \log \left[\frac{\mathbf{\Pi}^1_{|C_{m^{-1}N}}(\mathcal{C})}{\mathbf{\Pi}^1_{|C_N}(m^{1/d}\mathcal{C})} \right] dP_{|C_{m^{-1}N}}(\mathcal{C}),$$

thus we are left to compute the quotient of the densities $\frac{d\mathbf{\Pi}^1_{|C_{m^{-1}N}}(\mathcal{C})}{d\mathbf{\Pi}^1_{|C_N}(m^{1/d}\mathcal{C})}$. But the density of a Poisson point process depends only on the number of point of the configuration and if $\mathcal{C}$ is a point configuration with k points in $\mathcal{X}(C_{m^{-1}N})$, we have

$$\frac{d\mathbf{\Pi}^1_{|C_{m^{-1}N}}(\mathcal{C})}{d\mathbf{\Pi}^1_{|C_N}(m^{1/d}\mathcal{C})} = \frac{\frac{e^{-(m^{-1})N}(m^{-1}N)^k}{k!}}{\frac{e^{-N}(N)^k}{k!}} = e^{-(m^{-1}-1)N-k\log m}.$$

Since P has intensity m, the average number of points of a configuration under P in $\mathcal{X}(m^{-1}N)$ is N hence we get

$$\mathrm{Ent}[(\sigma_m P)_{|C_N}|\mathbf{\Pi}^1_{|C_N}] = \mathrm{Ent}[P_{|C_{m^{-1}N}}|\mathbf{\Pi}^1_{|C_{m^{-1}N}}] - (m^{-1}-1)N - N\log m.$$

Dividing the previous identity by N and taking the limit $N\to\infty$ yields, by definition of ent

$$\mathsf{ent}[(\sigma_m P)|\mathbf{\Pi}^1] = \frac{1}{m}\mathsf{ent}[P|\mathbf{\Pi}^1] - m^{-1} + 1 - \log m.$$

Consequently, if P is of intensity m, (2.4.9) holds. □

From Lemma 2.4.2 we observe that if $\bar{P}$ is a stationary tagged point process such that $\bar{P}^x$ has intensity $\mu_V(x)$ for Lebesgue-a.e. $x\in\Sigma$, then

$$\int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx = \int_\Sigma \mu_V(x)\mathsf{ent}[(\sigma_{\mu_V(x)}\bar{P}^x)|\mathbf{\Pi}^1]dx + |\Sigma| - 1 + \int_\Sigma \mu_V(x)\log\mu_V(x)$$

which yields the scaling for the entropy term in (2.1.15) and (2.1.16), moreover the term $(|\Sigma|-1)$ cancels with that of (2.4.7).

2.4.3 Properties of the limit Gibbs measure

The large deviation principle of Theorem 9 deals with the empirical fields associated to the Gibbs measure $\mathbb{P}_{N,\beta}$, when averaging the random configurations over translations in the support of the equilibrium measure. A natural question is to ask about the behaviour of the Gibbsian point process itself, that is the push-forward $\mathbf{P}_{N,\beta}$ of $\mathbb{P}_{N,\beta}$ by the map $(x_1,\dots,x_N)\mapsto \sum_{i=1}^N \delta_{N^{1/d}x_i}$. The mere existence of limit points for $\{\mathbf{P}_{N,\beta}\}_N$ is unclear in general. Since we are not averaging over translations, we cannot use the discrepancy estimates as in Lemma 2.3.10 to bound the number of points in a given compact set.

In this section we recall some known results of convergence of the (non-averaged) Gibbsian point process and connect them with the minimization of the LDP rate function.

a. Sine-beta processes

In [VV09] Valko and Virag define a family indexed by $\beta>0$ of point processes called "Sine-β processes" (by analogy with the usual sine-kernel, or Dyson sine process, known for $\beta=2$) and prove the convergence of the Gibbsian point process associated to the (random) eigenvalues of β-matrix models to the Sine-β process.

For $x\in[-2,2]$ let us denote by $\mathrm{Sine}_\beta(x)$ the Sine-β process of [VV09] rescaled to have intensity $\frac{1}{2\pi}\sqrt{4-x^2}$. For any $\beta>0$, let $\mathbf{P}_{N,\beta}$ be the point process induced by pushing-forward the Gibbs measure corresponding to the case $d=1$, $s=0$ and $V(x)=x^2$ by the map $(x_1,\dots,x_N)\mapsto\sum_{i=1}^N\delta_{Nx_i}$. The following is an immediate consequence of [VV09, Theorem1]:

$$\text{For any } x\in(-2,2), \text{ for any } \beta>0, \text{ we have } \theta_{Nx}\cdot\mathbf{P}_{N,\beta}\Longrightarrow \mathrm{Sine}_\beta(x). \tag{2.4.10}$$

The convergence $\Longrightarrow$ is proven in [VV09] "in law with respect to vague topology for the counting measure of the point process" which coincides with the notion of convergence used in this paper as explained in Remark 2.2.7.

We now give a proof of Corollary 2.1.3 i.e. the minimality of $\overline{\mathcal{F}}_\beta$ at $\overline{\mathrm{Sine}_\beta}$.

Proof. Let $F : \Sigma \times \mathcal{X} \to \mathcal{C}$ be a bounded continuous function. By definition we have

$$\mathbf{E}_{\overline{\mathfrak{P}}_{N,\beta}}\left[\int F(x,\mathcal{C})\,dP(x,\mathcal{C})\right] = \mathbf{E}_{\mathbb{P}_{N,\beta}}\left[\int_{[-2,2]} F(x,\theta_{Nx}\cdot\mathcal{C})dx\right] = \int_{[-2,2]} \mathbf{E}_{\mathbf{P}_{N,\beta}}\left[F(x,\theta_{Nx}\cdot\mathcal{C})\right]dx$$

From (2.4.10) we know that the sequence of functions $\{x \mapsto \mathbf{E}_{\mathbf{P}_{N,\beta}}\left[F(x,\theta_{Nx}\cdot\mathcal{C})\right]\}_N$ converges almost-everywhere on $[-2,2]$ to $x \mapsto \mathbf{E}_{\mathrm{Sine}_\beta(x)}[F(x,\mathcal{C})]$. Since F is bounded the dominated convergence theorem implies that

$$\lim_{N\to\infty} \mathbf{E}_{\overline{\mathfrak{P}}_{N,\beta}}\left[\int F(x,\mathcal{C})\,dP(x,\mathcal{C})\right] = \int_{[-2,2]} \mathbf{E}_{\mathrm{Sine}_\beta(x)}[F(x,\mathcal{C})] = \mathbf{E}_{\overline{\mathrm{Sine}_\beta}}[F(x,\mathcal{C})].$$

Since this is true for any bounded continuous function on $\Sigma \times \mathcal{X}$ we get that the sequence of tagged point processes $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ converges to $\overline{\mathrm{Sine}_\beta}$, but the large deviation principle implies that if $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ converges the limit must be a minimizer of $\overline{\mathcal{F}}_\beta$.

The fact that the point process Sine_β itself minimizes $\mathcal{F}_\beta$ among stationary point processes of intensity 1 is then an easy consequence by scaling. □

b. The Ginibre process

In dimension $d = 2$, little is known about the asymptotic behaviour of the Gibbsian point processes except for the particular value $\beta = 2$. Again, we let $\mathbf{P}_{N,\beta}$ be the push-forward of $\mathbb{P}_{N,\beta}$ by the map $(x_1,\dots,x_N) \mapsto \sum_{i=1}^N \delta_{\sqrt{N}x_i}$

The following was proven by Ginibre [Gin65] (see also e.g. [HKPV09])

Proposition 2.4.3 (Ginibre)**.** *The point process $\mathbf{P}_{N,\beta}$ for $\beta = 2$ and $V(x) = |x|^2/2$ is determinantal with kernel*

$$K_N(x,y) := \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \sum_{l=0}^{N-1} \frac{(x_i\bar{x_j})^l}{l!}$$

and has a limit Gin_2 *(called the Ginibre point process) which is the determinantal point process on $\mathbb{R}^2$ with kernel*

$$K_\infty(x,y) := \frac{1}{\pi} e^{-\frac{1}{2}|x|^2-\frac{1}{2}|y|^2+x\bar{y}}.$$

More recently Gin_2 has been identified as the limit of $\mathbf{P}_{N,\beta}$ for a wider class of potentials V – still at inverse temperature $\beta = 2$ – in [AHM11, Proposition 7.4.]. The convergence is proven for any potential V of class C^∞ (satisfying the growth conditions (2.2.3)) such that $\Delta V(0) = 1$ using a determinantal expression of $\mathbf{P}_{N,\beta}$.

The large deviation principle of Theorem 9 together with translation-invariance properties of the Ginibre ensemble imply Corollary 2.1.4.

We will rely on the following translation-invariance property, whose proof we postpone to Section 2.8.4.

Lemma 2.4.4. *Let $k \geq 0$ and f in Loc_k - see definition near (2.2.30). For all $\varepsilon > 0$, all integer $n \geq 0$ and all $u_n \in \mathbb{R}^d$ such that $C_k \cup (u_n + C_k) \subset B(0,\sqrt{(1-\varepsilon)n})$ we have*

$$\mathbf{E}_{\mathbf{P}_{n,2}}[f] - \mathbf{E}_{\mathbf{P}_{n,2}}[f(\theta_{u_n}\cdot)] = O\left(\exp(-\frac{\varepsilon^2}{2}n)\right) \quad \text{as } n \to \infty \tag{2.4.11}$$

uniformly on the choice of u_n.

We may now give a proof of Corollary 2.1.4.

Proof. Let F be a bounded continuous function in $\mathrm{Loc}_k(\mathcal{X})$. We have

$$\mathbf{E}_{\mathfrak{P}_{N,2}}\left[\int F(\mathcal{C})\,dP(x,\mathcal{C})\right] = \frac{1}{\pi N}\int_{B(0,\sqrt{N})} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x\cdot\mathcal{C})\right]\,dx. \tag{2.4.12}$$

Let us denote by $A_{N,\varepsilon}$ the set

$$A_{N,\varepsilon} = \{x\in B(0,\sqrt{N}), (C_k\cup C_k+x)\subset B(0,\sqrt{(1-\varepsilon)N})\}.$$

Since k is fixed, we have $|A_{N,\varepsilon}|\sim\pi(1-\varepsilon)N$ as $N\to\infty$ and since F is bounded we have

$$\left|\frac{1}{\pi N}\int_{B(0,\sqrt{N})} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x\cdot\mathcal{C})\right]\,dx - \frac{1}{\pi N}\int_{A_{N,\varepsilon}} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x\cdot\mathcal{C})\right]\,dx\right| = O(\varepsilon). \tag{2.4.13}$$

From Lemma 2.4.4 we have

$$\mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x\cdot\mathcal{C})\right] = \mathbf{E}_{\mathbf{P}_{N,2}}[F] + O(\exp(-\frac{\varepsilon^2}{2}N))$$

uniformly for $x\in A_{N,\varepsilon}$, so that

$$\frac{1}{\pi N}\int_{A_{N,\varepsilon}} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x\cdot\mathcal{C})\right]\,dx = (1-\varepsilon)\mathbf{E}_{\mathbf{P}_{N,2}}[F(\mathcal{C})] + o(1).$$

But we know (from Proposition 2.4.3) that $\mathbf{E}_{\mathbf{P}_{N,2}}[F(\mathcal{C})]$ converges to $\mathbf{E}_{\mathrm{Gin}_2}[F(\mathcal{C})]$. Hence we have

$$\lim_{N\to\infty}\frac{1}{\pi N}\int_{A_{N,\varepsilon}} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x\cdot\mathcal{C})\right]\,dx = (1-\varepsilon)\mathbf{E}_{\mathrm{Gin}_2}[F(\mathcal{C})]. \tag{2.4.14}$$

Combining (2.4.12), (2.4.13), (2.4.14) and letting $\varepsilon\to 0$ we obtain

$$\lim_{N\to\infty}\mathbf{E}_{\mathfrak{P}_{N,2}}\left[\int F(\mathcal{C})\,dP(x,\mathcal{C})\right] = \mathbf{E}_{\mathrm{Gin}_2}[F(\mathcal{C})] \tag{2.4.15}$$

for all continuous bounded local functions F. By Lemma 2.2.5 we know that local functions are dense in $\mathrm{Lip}_1(\mathcal{X})$, hence (2.4.15) is valid for any Lipschitz function F. This implies that $\mathfrak{P}_{N,2}$ converges to Gin_2, but the Large Deviation Principle of Theorem 9 implies that if $\mathfrak{P}_{N,2}$ has a limit it must be a minimizer of $\mathcal{F}_2$. □

2.5 Screening and regularization

In this section we enter the core of the proof, i.e. we describe important ingredients for the proof of Proposition 2.1.7, which rely on previous work, in particular the screening procedure introduced in [SS12, SS15b, RS15, PS15]. The goal of this section is to introduce two operations on point configurations (say, in a given hypercube C_R) which we may roughly describe this way:

1. The screening procedure Φ^{scr} takes "good" (also called "screenable") configurations and replace them by "better" configurations which are well-balanced (the number of points matches the volume) and for which there is a corresponding electric field supported in C_R with controlled energy. If the screening procedure encounters a "bad" configurations, it replaces it by "standard" configurations (at the cost of a loss of information).
2. The regularization procedure Φ^{reg} takes a configuration and separates all the pair of points which are closer than a certain threshold τ.

2.5.1 The screening procedure

When we get to the next section, we will want to construct point configurations by elementary blocks (hyperrectangles) and compute their energy additively in these blocks. One of the technical tricks borrowed from the original works above is that this may be done by gluing together electric fields whose normal components agree on the boundaries. More precisely, assume that space is partitioned into hyperrectangles $K \in \mathcal{K}$. We would like to construct a vector field E_K in each K such that

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma E_K) = c_{d,s}\,(\mathcal{C}_K - \mu_V'\delta_{\mathbb{R}^d}) & \text{in } K\times\mathbb{R}^k \\ E_K\cdot\vec{\nu} = 0 & \text{on } \partial(K\times\mathbb{R}^k) \end{cases} \tag{2.5.1}$$

(where $\vec{\nu}$ is the outer unit normal to K) for some discrete set of points $\mathcal{C}_K \subset K$, and with

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |(E_K)_\eta|^2$$

well controlled (recall the definition (2.2.19)). Integrating the relation (2.5.1), we see that a compatibility condition must be satisfied in order for this equation to be solvable, i.e. that

$$\int_K d\mathcal{C} = \int_K d\mu_V' \tag{2.5.2}$$

in particular the partition must be made so that $\int_K d\mu_V'$ are integers.

When solving (2.5.1), we may take E_K to be a gradient, but we do not require it. Once the relations (2.5.1) are satisfied on each K, we may paste together the vector fields E_K into a unique vector field E, and the discrete sets of points $\mathcal{C}_K$ into a configuration $\mathcal{C}$. By (2.5.2) the cardinality of $\mathcal{C}$ will be equal to $\int_{\mathbb{R}^d} d\mu_V'$, which is exactly N. We will thus have obtained a configuration of N points, whose energy we will try to evaluate. The important fact is that the enforcement of the boundary condition $E_K\cdot\vec{\nu}=0$ on each boundary ensures that

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C} - \mu_V'\delta_{\mathbb{R}^d}) \quad \text{in } \mathbb{R}^{d+k}$$

holds globally. Indeed, a vector field which is discontinuous across an interface has a distributional divergence concentrated on the interface equal to the jump of the normal derivative, i.e. here there is no extra divergence created across these interfaces. Even if the E_K's were gradients, the global E is in general no longer a gradient. This does not matter however, since the energy of the true electric field ∇H generated by the configuration $\mathcal{C}$ (and the background $-\mu_V'\delta_{\mathbb{R}^d}$) is necessarily smaller than that of E as seen in Lemma 2.3.13. This way

$$\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H_{N,\eta}'|^2 \le \sum_K \int_{K\times\mathbb{R}^k} |y|^\gamma |(E_K)_\eta|^2$$

and the energy has indeed become additive over the cells. This shows that to compute the w_N (recall (2.2.22)) associated with the configuration of N points $\mathcal{C}$, we may indeed relax the gradient constraint and evaluate the energy of the electric fields E_K constructed in each K. This explains why we need to find ways of obtaining vector fields E_K satisfying (2.5.1).

These vector fields will themselves be constructed from a given point configuration in each cell K, sampled at random via the law $\mathbb{Q}_{N,\beta}$ and two cases will occur. The first case occurs when the configuration in the cell K has an energy which is not too large, this "not too large energy" will be characterized by the fact that there exists a vector field E in the cell K such that $-\frac{1}{c_{d,s}}\mathrm{div}\,(|y|^\gamma E) + \mu_V'\delta_{\mathbb{R}^d}$ coincides with the configuration of points (i.e. the first equation

in (2.5.1) is verified, but not necessarily the second), and whose $\int_K |y|^\gamma |E|^2$ is not too large (in a way that will be specified below). Such configurations will be called "screenable". Indeed, for them, the result of [PS15] ensures that we may modify the configuration in a thin layer near ∂K, and modify the vector field E a little bit as well, so that (2.5.1) is satisfied, and that the energy has not been changed very much. The second case is the case where there exists no such E of reasonable energy in the cell K. In that case the configuration is not screenable in the cell K, we will completely discard it and replace it by an artificial (frozen) configuration (typically a perturbation of a periodic one) whose energy is well controlled, but which has nothing to do with the original configuration. This will not matter in the end, because we will be able to show that such bad cells are rare for a typical configuration.

An important task will be later to estimate the volume in the space of configurations of the modified configurations that we obtain this way. In fact, what we need to produce above is not just one configuration, but a family of them whose volume is not too small.

a. A preliminary construction

The first lemma we state below concerns the construction of families of "artificial" configurations whose energy is well controlled. This will be used in two different ways: to fill up an empty space with points during the screening procedure, and also in the next section in order to replace "bad configurations" for which the screening procedure fails to apply.

Lemma 2.5.1. *Let $0 < \underline{m} \leq \overline{m}$ be fixed, K be a hyperrectangle with sidelengths in $[R, 2R]$. There exists $\eta_0 > 0$ depending only on $d, \underline{m}, \overline{m}$ such that the following holds : let μ be a measurable function on K satisfying $\underline{m} \leq \mu \leq \overline{m}$ and such that $n_{K,\mu} := \int_K \mu$ is an integer, then there exists a family $\Phi^{\mathrm{gen}}(K, \mu)$ of configurations with $n_{K,\mu}$ points in K such that for any $\mathcal{C}^{\mathrm{gen}}$ in $\Phi^{\mathrm{gen}}(K, \mu)$, the following holds :*

1. *The distance between two points of $\mathcal{C}^{\mathrm{gen}}$ and between a point of $\mathcal{C}^{\mathrm{gen}}$ and ∂K is bounded below by η_0.*
2. *There exists E^{gen} satisfying*

$$\begin{cases} \mathrm{div}\,(|y|^\gamma E^{\mathrm{gen}}) = c_{d,s}\,(\mathcal{C}^{\mathrm{gen}} - \mu\delta_{\mathbb{R}^d}) & \text{in } K \times \mathbb{R}^k \\ E^{\mathrm{gen}} \cdot \vec{\nu} = 0 & \text{on } \partial K \times \mathbb{R}^k \end{cases} \tag{2.5.3}$$

and for any $\eta < \eta_0$,

$$\begin{aligned} &\int_{K\times\mathbb{R}^k} |y|^\gamma |E^{\mathrm{gen}}_\eta|^2 - c_{d,s} n_{K,\mu} g(\eta) \\ &\quad \leq C n_{K,\mu} + C R^{d+1-\gamma} \|\mu - m\|^2_{L^\infty(K)} + C\,(n_{K,\mu} g(\eta))^{\frac{1}{2}}\, R^{\frac{d+1-\gamma}{2}} \|\mu - m\|_{L^\infty(K)} \end{aligned} \tag{2.5.4}$$

with a constant C depending only on $d, \overline{m}, \underline{m}$.

3. *The volume of $\Phi^{\mathrm{gen}}(K, \mu)$ is bounded below by*

$$\mathbf{Leb}^{\otimes n_{K,\mu}}\,(\Phi^{\mathrm{gen}}(K, \mu)) \geq (n_{K,\mu}!) C^{n_{K,\mu}} \tag{2.5.5}$$

with a constant C depending only on $d, \overline{m}, \underline{m}$.

We postpone the proof of Lemma 2.5.1 to Section 2.8.

b. The screening result

We now state the screening result from [PS15], in a version rephrased for our needs. As mentioned above this result serves to modify a given electric vector field and the underlying point configuration, in such a way as to satisfy (2.5.1). We call this "screening" because the configuration is modified in such a way that the field generated by the cell can be taken to be zero outside of the cell, i.e. the configuration has no influence outside the cell.

The configuration and the field will only be modified in a thin layer near the boundary of a hyperrectangle K, and remain unchanged in an interior set denoted Old. To be accurate, we do not really need the original configuration to be defined in the whole K, but only in a subcube $C_R \subset K$, the configuration is then completed by hand until the whole K is covered with points. We also need the positions of the points added "by hand" in the layer near the boundary, which will be denoted New, to be flexible enough to have a nonzero volume of associated configurations in phase space. This is accomplished by letting the points move in small balls around their basic positions, which does not alter the estimates.

Proposition 2.5.2 (Screening)**.** *Let $\underline{m}, \overline{m} > 0$ be fixed.*

There exists $R_0 > 0$ universal, $\eta_0 > 0$ depending only on d and $\overline{m}$, there exists a constant C depending on $d, s, \underline{m}, \overline{m}$ such that the following holds.

Let $0 < \varepsilon < \frac{1}{2}$ and $0 < \eta < \eta_0$ be fixed. Let C_R be a hypercube of sidelength R for some $R > 0$ and let K be a hyperrectangle such that $C_R \subset K$. Let μ be a measurable function on K satisfying $\underline{m} \leq \mu \leq \overline{m}$ and such that $\int_K \mu$ is an integer. Let $m = \fint_K \mu$. Let $\mathcal{C}$ be a point configuration in C_R.

Assume that E is a vector field defined in $C_R \times \mathbb{R}^k$ and satisfies

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C} - \mu\delta_{\mathbb{R}^d}) \ \text{ in } C_R.$$

Letting E_η be associated to E as in (2.2.19)*, we define*

$$M_{R,\eta} := \frac{1}{R^d} \int_{C_R \times [-R,R]^k} |y|^\gamma |E_\eta|^2,$$

and in the case $k = 1$

$$e_{\varepsilon,R} := \frac{1}{\varepsilon^4 R^d} \int_{C_R \times \left(\mathbb{R} \setminus (-\frac{1}{2}\varepsilon^2 R, \frac{1}{2}\varepsilon^2 R)\right)} |y|^\gamma |E|^2,$$

and we assume the following inequalities are satisfied

$$R > \max\left(\frac{R_0}{\varepsilon^2}, \frac{CR_0 M_{R,\eta}}{\varepsilon^3}\right), \quad R > \begin{cases} \frac{CR_0 M_{R,\eta}^{1/2}}{\varepsilon^{d+3/2}} & \text{if } k = 0 \\ \max(CR_0 M_{R,\eta}^{1/(1-\gamma)} \varepsilon^{\frac{-1-2d+\gamma}{1-\gamma}}, R_0 \varepsilon^{\frac{2\gamma}{1-\gamma}} e_{\varepsilon,R}^{1/(1-\gamma)}) & \text{if } k = 1 \end{cases}. \tag{2.5.6}$$

Then there exists a (measurable) family $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C}, \mu)$ of point configurations in K and a partition of K as Old $\sqcup$ New *with*

$$\mathrm{Int}_\varepsilon := \{x \in C_R, \mathrm{dist}(x, \partial C_R)\} \geq 2\varepsilon R\} \subset \mathrm{Old} \tag{2.5.7}$$

such that for any $\mathcal{C}^{\mathrm{scr}}$ in $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C}, \mu)$ we have

1. *The configurations $\mathcal{C}$ and $\mathcal{C}^{\mathrm{scr}}$ coincide on* Old*.*
2. *For $\eta < \eta_0$, it holds that*

$$\sum_{x_i \neq x_j \in \mathcal{C}^{\mathrm{scr}}, |x_i - x_j| \leq 2\eta} g(x_i - x_j) = \sum_{x_i \neq x_j \in \mathcal{C}, |x_i - x_j| \leq 2\eta} g(x_i - x_j),$$

i.e. the contribution to the energy due to pairs of points which are 2η-close is left unchanged. Moreover we have

$$\min_{x\in\mathcal{C}^{\rm scr}} \mathrm{dist}(x,\partial K) \geq \eta_0 \tag{2.5.8}$$

$$\min_{x\in\mathcal{C}^{\rm scr}\cap \mathrm{New},\ y\in\mathcal{C}^{\rm scr}} |x-y| \geq \eta_0. \tag{2.5.9}$$

3. *There exists a vector field $E^{\rm scr}\in L^p_{\rm loc}(\mathbb{R}^{d+k},\mathbb{R}^{d+k})$ satisfying*

(a)

$$\begin{cases} -\mathrm{div}\ (|y|^\gamma E^{\rm scr}) = c_{d,s}\,(E^{\rm scr}-\mu\delta_{\mathbb{R}^d}) & \text{in } K\times\mathbb{R}^k \\ E^{\rm scr}\cdot\vec{\nu}=0 & \text{on } \partial K\times\mathbb{R}^k, \end{cases} \tag{2.5.10}$$

In particular the configuration $\mathcal{C}^{\rm scr}$ has exactly $\int_K \mu$ points in K.

(b) *Letting $E^{\rm scr}_\eta$ be associated to $E^{\rm scr}$ as in* (2.2.19) *we have:*

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E^{\rm scr}_\eta|^2 \leq I+II+III \tag{2.5.11}$$

with

$$\begin{aligned} I &= \left(\int_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2\right)(1+C\varepsilon) + Cg(\eta)\left((1+M_{R,\eta})\varepsilon R^d + |K| - |C_R|\right) + Ce_{\varepsilon,R}\varepsilon R^d \\ II &= CR^{d+1-\gamma}\,\|\mu-m\|^2_{L^\infty(K)} \\ III &= \sqrt{I\cdot II} \end{aligned}$$

for some constant C depending only on $s,d,\underline{m},\overline{m}$.

Moreover the number of points of $\mathcal{C}^{\rm scr}$ in New *is a constant $n_{\rm New}$ on $\Phi^{\rm scr}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ and we have*

$$\mathbf{Leb}^{\otimes n_{\rm New}}\left(\Phi^{\rm scr,New}_{\varepsilon,\eta,R}(\mathcal{C},\mu)\right) \geq (n_{\rm New})!c^{n_{\rm New}} \tag{2.5.12}$$

for a certain constant $c>0$ depending only on $d,\overline{m}$, where $\Phi^{\rm scr,New}_{\varepsilon,\eta,R}$ denotes the restriction of the configurations to the subset New $\subset K$.

Proof. The statement is based on a re-writing of [PS15, Proposition 6.1.] provided by a careful examination of its proof. First, let us assume that $\mu\equiv 1$. In that case we may apply directly [PS15, Proposition 6.1]. For the reader's convenience let us sketch that proof.

The first step is to find by a mean-value argument a good boundary, that is the boundary of a hypercube Old included in C_R and containing Int_ε, on which $\int |y|^\gamma |E_\eta|^2$ is not too large, more precisely controlled in terms of $M_{R,\eta}$. In the case where $k=1$, i.e. the interaction potential is not coulombic and we need to use the extension representation, cf. Section 2.2.3, then we need to do the same "vertically" i.e. find by mean value a good height z such that $\int_{\mathrm{Old}\times\{-z,z\}} |y|^\gamma |E_\eta|^2$ is controlled in terms of $e_{\varepsilon,R}$.

The configuration $\mathcal{C}$ and the field E are kept unchanged inside Old. We then tile New := $K\backslash$Old by small hypercubes of sidelength $O(1)$ (and uniformly bounded below) and place one point near the center of each of these hypercubes (they may be chosen freely in a small ball near the center), see Figure b.. This way the new points are well separated by construction, and the distances between two points (of the configurations) in New or between a point (of the configuration) in New and a point (of the configuration) in Old is bounded below by $2\eta_0$. In particular if $\eta<\eta_0$ no new 2η-close pair has been created and property 2) holds.

We then construct a global electric field on New as described at the beginning of the section by pasting together vector fields defined on each hypercube. For the global vector field to satisfy

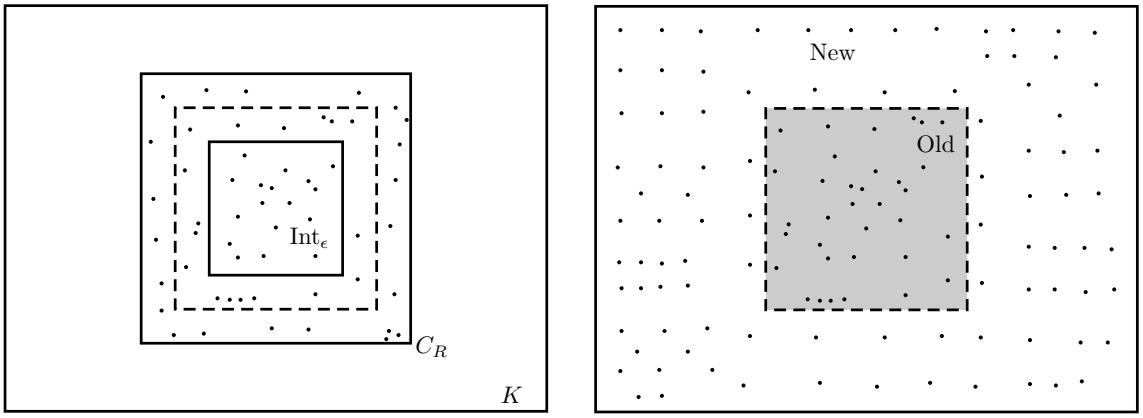

Figure 2.1 – The original configuration (on the left) and the screened configuration (on the right). The dashed line corresponds to the good boundary. Proportions are distorted and Int_ε really contains most of the set K.

the relation (2.5.3), we need the normal components to be continuous (and not necessarily 0) across the interfaces. These normal components are chosen to agree with that of E on ∂Old and to be 0 on ∂K. As explained at the beginning of this section, the energy of the new electric field E^{scr} is then bounded by the energy of E in Old, plus the energy of the new E in all the added hypercubes of $K\backslash$Old. In [PS15], the construction is made for ∂K and ∂C_R at distance proportional to εR from each other. We may apply the result of [PS15] for such a K', and if ∂K is further away from C_R, then it suffices to tile $K\backslash K'$ and paste vector fields constructed exactly as in the proof of Lemma 2.5.1. In the end we obtain a global vector field E^{scr} satisfying item 3 (a). The energy due to the part $K\backslash K'$ is controlled just as in Lemma 2.5.1 by a constant times the number of points added there, i.e. $C(|K|-|K'|) \leq C(|K|-|C_R|)$. More precisely, we may write

$$\int_{(K\backslash K')\times\mathbb{R}^k} (|y|^\gamma |E_\eta^{\mathrm{scr}}|^2 - c_{d,s}g(\eta)) \leq C(|K|-|C_R|).$$

The energy in K' is proven in [PS15] to be controlled in terms of $M_{\eta,R}$ and $e_{\varepsilon,R}$ as follows (taking $\eta=\eta'$ in [PS15, Proposition 6.1])

$$\int_{K'\times\mathbb{R}^k} |y|^\gamma |E_\eta^{\mathrm{scr}}|^2 \leq \left(\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2\right)(1+C\varepsilon) + Cg(\eta)(1+M_{R,\eta})\varepsilon R^d + Ce_{\varepsilon,R}\varepsilon R^d.$$

Combining the two relations, we obtain

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E_\eta^{\mathrm{scr}}|^2 \leq \left(\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2\right)(1+C\varepsilon) + Cg(\eta)(1+M_{R,\eta})\varepsilon R^d + Ce_{\varepsilon,R}\varepsilon R^d + Cg(\eta)(|K|-|C_R|), \quad (2.5.13)$$

where C depends only on d, s.

For each of the hypercubes in New, we may move the point placed therein by a small distance $C = \frac{\eta_0}{4}$ without affecting the conclusions, as done in the proof of Lemma 2.5.1 or as explained in [PS15, Remark 6.7]. This way we obtain a set of configurations (and associated electric fields) satisfying all the requirements and whose n_{New}-dimensional volume is bounded below as in (2.5.12).

To conclude the proof, there remains to handle the fact that μ, in our setting, is not equal to 1 but may vary between $\underline{m}$ and $\overline{m}$. We start by treating the case where $\mu \equiv m$ is a constant function. We may then apply the scaling map σ_m as defined in (2.2.24) to the electric field E. By a change of variables, letting $E' := \sigma_m E$, we get

$$\int_{m^{1/d}(K_R\times[-R,R]^k)} |y|^\gamma |E'_{m^{1/d}\eta}|^2 = m^{-s/d} \int_{K_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2 = R^d M_{R,\eta}$$

and

$$\int_{m^{1/d}\left(K_R\times(\mathbb{R}\setminus(-\frac{1}{2}\varepsilon^2,\frac{1}{2}\varepsilon^2))\right)} |y|^\gamma |E'|^2 = m^{-s/d}\varepsilon^4 R^d e_{\varepsilon,R},$$

i.e. with obvious notation $M_{R,\eta}(E) = m^{1+s/d} M_{m^{1/d}R,m^{1/d}\eta}(E')$ and $e_{\varepsilon,R}(E) = m^{1+s/d} e_{\varepsilon,m^{1/d}R}(E')$. If the constant C is chosen large enough (depending on $\underline{m}, \overline{m}, d, s$) we see that inequalities (2.5.6) imply that the assumptions (6.2) of [PS15, Proposition 6.1] are satisfied by E' with R replaced by $m^{1/d}R$, η replaced by $m^{1/d}\eta$. We may then apply the result of [PS15, Proposition 6.1], i.e. what we have just outlined and get a family of configurations such that the desired conclusions are satisfied, up to a global scaling of all sets and distances by a factor $m^{1/d}$. In particular in view of (2.5.13) we control the energy by

$$\int_{m^{1/d}K\times\mathbb{R}^k} |y|^\gamma |E^{\mathrm{scr}'}_{m^{1/d}\eta}|^2 \le \left(\int_{C_{m^{1/d}R}\times[-m^{1/d}R,m^{1/d}R]^k} |y|^\gamma |E'_{m^{1/d}\eta}|^2\right)(1+C\varepsilon)$$
$$+ Cg(m^{1/d}\eta)(1+m^{-1-s/d}M_{R,\eta}(E))\varepsilon m R^d + Cm^{-1-s/d}e_{\varepsilon,R}mR^d + Cmg(m^{1/d}\eta)(|K|-|C_R|)$$

We then apply the inverse map $\sigma_{m^{-1}}$ to this family of configurations and associated electric fields, and we obtain a family of configurations satisfying all the results of the proposition with μ replaced by m and for the energy bound

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E^{\mathrm{scr}}_\eta|^2 \le \left(\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2\right)(1+C\varepsilon)$$
$$+ Cg(m^{1/d}\eta)(m^{s/d} + m^{-1}M_{R,\eta}(E))\varepsilon m R^d + Ce_{\varepsilon,R}R^d + Cm^{1+s/d}g(m^{1/d}\eta)(|K|-|C_R|)$$

thus in view of the exact form of g, (2.1.3) or (2.1.1)–(2.1.2), we obtain in all cases

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E^{\mathrm{scr}}_\eta|^2 \le \left(\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2\right)(1+C\varepsilon)$$
$$+ Cg(\eta)(1+M_{R,\eta}(E))\varepsilon R^d + Ce_{\varepsilon,R}R^d + Cg(\eta)(|K|-|C_R|)$$

with a constant C which may now depend on $\underline{m}, \overline{m}$. To get from a constant background μ to a variable μ we proceed as in the proof of Lemma 2.5.1 using Lemma 2.8.1. We obtain a family of configurations satisfying the desired conclusions. □

Let us now estimate how this procedure changes the volume of a set of configurations in phase-space.

Lemma 2.5.3. *Let* Int_ε *be as in* (2.5.7) *and* $\mathrm{Ext}_\varepsilon := C_R\setminus\mathrm{Int}_\varepsilon$. *Assume* A *is a (measurable) set of point configurations in* $\mathcal{X}(C_R)$ *such that each configuration of* A *has* n *points in* C_R *and* n_{int} *points in* Int_ε, *with* n_{int} *satisfying*

$$n_{K,\mu} - n_{\mathrm{int}} \le \frac{|\mathrm{Ext}_\varepsilon|}{2c}, \tag{2.5.14}$$

where c is the constant in (2.5.12). Let us also assume that $(\mathcal{C},\mu)$ satisfies the conditions of Proposition 2.5.2 for all $\mathcal{C}$ in A. Then we have

$$\log \mathbf{Leb}^{\otimes n_{K,\mu}}\left(\bigcup_{\mathcal{C}\in A}\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)\right)\geq \log \mathbf{Leb}^{\otimes n}(A) + \log\left((n_{K,\mu}-n_{int})!\left(\frac{c}{|\mathrm{Ext}_\varepsilon|}\right)^{n_{K,\mu}-n_{\mathrm{int}}}\right)+(n_{K,\mu}-n)\log|\mathrm{Ext}_\varepsilon|. \quad (2.5.15)$$

Proof. Using the terminology of Proposition 2.5.2, we may partition A into

$$A=\cup_{n_{\mathrm{New}}=0}^{n_{K,\mu}}(A|n_{\mathrm{New}})$$

according to the number of points n_{New} that are created in $K\backslash C_R$ and in a thin layer (of width $\approx \varepsilon R$) close to ∂C_R. We denote by $A|n_{\mathrm{New}}$ the subset of A consisting of configurations for which n_{New} points are created.

We note that following the construction of Proposition 2.5.2, the number of points in Old (points which remain unchanged) is given by $n_{K,\mu}-n_{\mathrm{New}}$ by definition of n_{New}. By construction again, we have $\mathrm{Int}_\varepsilon\subset$ Old which yields

$$n_{\mathrm{int}}\leq n_{K,\mu}-n_{\mathrm{New}}.$$

Thus for each configuration in $A|n_{\mathrm{New}}$, when applying the construction, a number $n_{K,\mu}-n_{\mathrm{New}}$ of points are left untouched while the other ones i.e. $n-(n_{K,\mu}-n_{\mathrm{New}})$ points (all belonging to Ext_ε), are deleted and replaced by n_{New} points (up to permutation of indices) which live in some small balls in K. We may thus write, using (2.5.12)

$$\mathbf{Leb}^{\otimes n_{K,\mu}}\left(\bigcup_{\mathcal{C}\in(A|n_{\mathrm{New}})}\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)\right)\geq\frac{\mathbf{Leb}^{\otimes n}(A|n_{\mathrm{New}})(n_{\mathrm{New}})!c^{n_{\mathrm{New}}}}{|\mathrm{Ext}_\varepsilon|^{n-(n_{K,\mu}-n_{\mathrm{New}})}}$$

But we have seen that $n_{\mathrm{New}}\leq n_{K,\mu}-n_{\mathrm{int}}$, while one may check that $x\mapsto x!\left(\frac{c}{|\mathrm{Ext}_\varepsilon|}\right)^x$ is decreasing as long as $x\leq\frac{|\mathrm{Ext}_\varepsilon|}{2c}$, so we may write

$$\mathbf{Leb}^{\otimes n_{K,\mu}}\left(\bigcup_{\mathcal{C}\in A|n_{\mathrm{New}}}\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)\right) \geq \mathbf{Leb}^{\otimes n}(A|n_{\mathrm{New}})|\mathrm{Ext}_\varepsilon|^{n_{K,\mu}-n}\left((n_{K,\mu}-n_{\mathrm{int}})!\frac{c}{|\mathrm{Ext}_\varepsilon|}\right)^{n_{K,\mu}-n_{\mathrm{int}}}.$$

Summing over n_{New} and taking the log yields the result. □

c. Screenability

The conditions (2.5.6) borrowed from [PS15] and which are sufficient for the screening result Proposition 2.5.2 to hold, provide (up to a condition on the number of points) the definition of "screenability", whose meaning we explained at the beginning of the section. Our main concern is then to prove the upper semi-continuity of the screening procedure, which forces us to go into its topological details.

Let $0<\underline{m},\overline{m}<+\infty$ be fixed, let η_0 be as in Lemma 2.5.2 (it depends only on d and $\overline{m}$). For any $R,M,e,\varepsilon>0$ such that the following inequalities are satisfied

$$R>\max\left(\frac{R_0}{\varepsilon^2},\frac{CR_0M}{\varepsilon^3}\right),\quad R>\begin{cases}\frac{CR_0M^{1/2}}{\varepsilon^{d+3/2}} & \text{if } k=0\\ \max(CR_0M^{1/(1-\gamma)}\varepsilon^{\frac{-1-2d+\gamma}{1-\gamma}},R_0\varepsilon^{\frac{2\gamma}{1-\gamma}}e^{1/(1-\gamma)}) & \text{if } k=1\end{cases}, \quad (2.5.16)$$

with the constants C, R_0 as in (2.5.6), and for any $0 < \eta < \eta_0$, for any configuration of points $\mathcal{C}$ in C_R and any bounded function μ on C_R we define $\mathcal{O}^{M,e,\varepsilon}_{R,\eta,+}(\mathcal{C},\mu)$ as the set of vector fields E such that

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C} - \mu\delta_{\mathbb{R}^d}) \text{ in } C_R$$

and such that

$$\frac{1}{R^d}\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2 \le M \tag{2.5.17}$$

and in addition, in the case $k = 1$,

$$\frac{1}{\varepsilon^4 R^d}\int_{C_R\times\left(\mathbb{R}\setminus(-\frac{1}{2}\varepsilon^2 R,\frac{1}{2}\varepsilon^2 R)\right)} |y|^\gamma |E|^2 \le e. \tag{2.5.18}$$

The set $\mathcal{O}^{M,e,\varepsilon}_{R,\eta,-}(\mathcal{C},\mu)$ is defined in the same way except that the inequalities (2.5.17), (2.5.18) are taken to be strict.

We will denote by $\mathcal{O}_R(\mathcal{C},\mu)$ the set of vector fields E in C_R such that

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C} - \mu\delta_{\mathbb{R}^d}) \text{ in } C_R$$

without any condition on the energy.

Definition 2.5.4 (Screenability)**.** *We denote by $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ (resp. $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}$) the set of screenable couples $(\mathcal{C},\mu)$ i.e. such that*

1. *$\mathcal{O}^{M,e,\varepsilon}_{R,\eta,+}(\mathcal{C},\mu)$ (resp. $\mathcal{O}^{M,e,\varepsilon}_{R,\eta,-}(\mathcal{C},\mu)$) is not empty.*
2. *The number of points of $\mathcal{C}$ in C_R is bounded above by*

$$\mathcal{N}(0,R) \le MR^d \text{ resp. } \mathcal{N}(0,R) < MR^d. \tag{2.5.19}$$

In the following we see $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ as embedded into the product space $\mathcal{X}(C_R)\times L^\infty(C_R)$ endowed with the natural topology.

Remark 2.5.5. *The condition* (2.5.19) *on the number of points is closed (resp. open for the second one) because $\mathcal{C}\mapsto\mathcal{N}(0,R)(\mathcal{C})$ is continuous on $\mathcal{X}(C_R)$.*

It is clear by definition that we have

$$\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-} \subset \mathcal{S}^{M,e,\varepsilon}_{R,\eta,+} \subset \mathcal{S}^{2M,2e,\varepsilon}_{R,\eta,-}. \tag{2.5.20}$$

Definition 2.5.6. *For any $(\mathcal{C},\mu)\in\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ we define $F^{M,e,\varepsilon}_{R,\eta}$ to be the "best screenable energy"*

$$F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu) = \inf\left\{\frac{1}{R^d}\int_{C_R\times[-R,R]^k}|y|^\gamma|E_\eta|^2,\quad E\in\mathcal{O}^{2M,2e,\varepsilon}_{R,\eta,-}(\mathcal{C},\mu)\right\}.$$

We extend the function $F^{M,e,\varepsilon}_{R,\eta}$ by zero on the complement of $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ so that $F^{M,e,\varepsilon}_{R,\eta} = F^{M,e,\varepsilon}_{R,\eta}\mathbf{1}_{\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}}$.

Remark 2.5.7. *It is easy to see that for any $(\mathcal{C},\mu)$ and any E in $\mathcal{O}_R(\mathcal{C},\mu)$,*

1. *If* (2.5.18) *holds then*

$$F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu) \le \min\left(\frac{1}{R^d}\int_{C_R\times[-R,R]^k}|y|^\gamma|E_\eta|^2, M\right).$$

2. *If* (2.5.18) *fails to hold then*

$$F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu) \le M.$$

Indeed, we note that we always have $F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu) \le M$: either $(\mathcal{C},\mu) \in \mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$, in which case any element of $\mathcal{O}^{M,e,\varepsilon}_{R,\eta,+}(\mathcal{C},\mu)$ gives a test vector-field for $F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu)$ whose energy is less than M, or $(\mathcal{C},\mu) \notin \mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ in which case $F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu)$ is defined to be 0. To prove the statements of the remark, it thus suffices to verify that if $\frac{1}{R^d}\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2 \le M$ then $F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu) \le \frac{1}{R^d}\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2$. But this is clear since in that case the configuration is in $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ and E is a test vector field for $F^{M,e,\varepsilon}_{R,\eta}$.

Lemma 2.5.8. *The set $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ (resp. $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}$) is closed (resp. open) in $\mathcal{X}(C_R)\times L^\infty(C_R)$, and the function $F^{M,e,\varepsilon}_{R,\eta}$ is upper semi-continuous on $\mathcal{X}(C_R)\times L^\infty(C_R)$.*

For that we need a lemma which proves the continuity of the energy, say the local one, with respect to the background density μ and with respect to the points.

Lemma 2.5.9. *Let $R>0$ and let $\mathcal{C},\mathcal{C}'$ be two configurations and μ,μ' be two bounded measurable functions on C_R as above. Let $\tilde{E}$ be the electric field generated by the algebraic difference of $(\mathcal{C},\mu)$ and $(\mathcal{C}',\mu')$ i.e.*

$$\tilde{E} := c_{d,s}\nabla g * (\mathcal{C}-\mathcal{C}' - (\mu-\mu')\delta_{\mathbb{R}^d}) . \tag{2.5.21}$$

Then for any $\eta>0$ the energy $\int_{C_R\times[-R,R]^k} |y|^\gamma |\tilde{E}_\eta|^2$ tends to 0 when $(\mathcal{C}',\mu')$ converges to $(\mathcal{C},\mu)$ in $\mathcal{X}(C_R)\times L^\infty(C_R)$.

Proof. We recall that by Remark 2.2.1, letting $g_\eta(x) = \min(g(x), g(\eta))$ we have

$$\tilde{E}_\eta = \nabla g_\eta * (\mathcal{C}-\mathcal{C}') - \nabla g * ((\mu-\mu')\delta_{\mathbb{R}^d}).$$

To prove the result it suffices to prove that letting $H_1 := g_\eta * (\mathcal{C}-\mathcal{C}')$ and $H_2 := g * ((\mu-\mu')\mathbf{1}_{C_R}\delta_{\mathbb{R}^d})$, both $\int_{C_R\times[-R,R]^k} |y|^\gamma |\nabla H_1|^2$ and $\int_{C_R\times[-R,R]^k} |y|^\gamma |\nabla H_2|^2$ tend to 0 as $(\mathcal{C}',\mu')$ converges to $(\mathcal{C},\mu)$ in $\mathcal{X}(C_R)\times L^\infty(C_R)$. But the number of points in C_R is locally constant for the topology on $\mathcal{X}(C_R)$, so we may assume that the distribution $\mathcal{C}-\mathcal{C}'$ is compactly supported and with total mass 0. Therefore H_1 (resp. ∇H_1) decays like $|x|^{-s-1}$ (resp. like $|x|^{-s-2}$) as $|x|\to\infty$ as noticed in the proof of Lemma 2.3.13. Integrating by parts we may thus write

$$\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H_1|^2 = \iint g_\eta(x-y)(\mathcal{C}-\mathcal{C}')(x)(\mathcal{C}-\mathcal{C}')(y)$$

and the desired result for H_1 follows by continuity of g_η. For H_2, we first notice that by integrability of g we have

$$|H_2| \le C\|\mu-\mu'\|_{L^\infty(C_R)} \tag{2.5.22}$$

where the constant C depends on R, and that

$$-\mathrm{div}\,(|y|^\gamma \nabla H_2) = c_{d,s}(\mu-\mu')\mathbf{1}_{C_R}\delta_{\mathbb{R}^d}$$

in view of (2.2.11). Let then χ be a smooth compactly supported positive function equal to 1 in $C_R\times[-R,R]$, and such that $|\nabla\chi|\le 1$. Integrating by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^{d+k}} \chi^2 |y|^\gamma |\nabla H_2|^2 &= -\int_{\mathbb{R}^{d+k}} \chi^2 \mathrm{div}\,(|y|^\gamma \nabla H_2) H_2 - 2\int_{\mathbb{R}^{d+k}} \chi\nabla\chi\cdot\nabla H_2 |y|^\gamma H_2 \\ &\le c_{d,s}\left|\int_{\mathbb{R}^{d+k}} \chi^2 H_2(\mu-\mu')\delta_{\mathbb{R}^d}\right| + \int_{\mathbb{R}^{d+k}} \chi|\nabla\chi||y|^\gamma |H_2||\nabla H_2|.\end{aligned}$$

From(2.5.22) we may thus write, using the Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}^{d+k}} \chi^2|y|^\gamma|\nabla H_2|^2 \le C\|\mu-\mu'\|_{L^\infty}^2+C\|\mu-\mu'\|_{L^\infty}\left(\int_{\mathbb{R}^{d+k}} \chi^2|y|^\gamma|\nabla H_2|^2\right)^{\frac12}\left(\int_{\mathbb{R}^{d+k}} |\nabla\chi|^2|y|^\gamma\right)^{\frac12}$$

therefore

$$\int_{C_R\times[-R,R]^k} |y|^\gamma|\nabla H_2|^2 \le \int_{\mathbb{R}^{d+k}} \chi^2|y|^\gamma|\nabla H_2|^2 \le C(\|\mu-\mu'\|_{L^\infty}^2+\|\mu-\mu'\|_{L^\infty}^4),$$

with C depending on R. This completes the proof. □

Proof of Lemma 2.5.8. Let us first prove the upper semi-continuity of $F_{R,\eta}^{M,e,\varepsilon}$. Let $(\mathcal{C},\mu)$ in $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$ be fixed and for any $(\mathcal{C}',\mu')$ in $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$ such that $\mathcal{C}'(\overline{C_R})=\mathcal{C}(\overline{C_R})$ (i.e. $\mathcal{C}'$ and $\mathcal{C}$ have the same number of points in C_R), let $\tilde{E}$ be the vector field defined in (2.5.21). By definition of $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$ and $F_{R,\eta}^{M,e,\varepsilon}$, for any $\delta>0$ we may find an electric field E in $\mathcal{O}_{R,\eta,-}^{2M,2e,\varepsilon}(\mathcal{C},\mu)$ such that

$$\frac{1}{R^d}\int_{C_R\times[-R,R]^k} |y|^\gamma|E_\eta|^2 \le F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C},\mu)+\delta.$$

Then $E':=E^\delta+\tilde{E}$ satisfies

$$-\mathrm{div}\,(|y|^\gamma E') = c_{d,s}\,(\mathcal{C}'-\mu'\delta_{\mathbb{R}^d}) \text{ in } C_R.$$

In view of Lemma 2.5.9, we easily deduce that if $(\mathcal{C}',\mu')$ is sufficiently close to $(\mathcal{C},\mu)$ in $\mathcal{X}(\overline{C_R})\times L^\infty(C_R)$, then

$$\frac{1}{R^d}\int_{C_R\times[-R,R]^k} |y|^\gamma|E'_\eta|^2 \le F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C},\mu)+2\delta, \tag{2.5.23}$$

and E' is in $\mathcal{O}_{R,\eta,-}^{2M,2e,\varepsilon}(\mathcal{C}',\mu')$ provided $\delta>0$ was small enough. In particular we have found a test vector-field E' which is in $\mathcal{O}_{R,\eta,-}^{2M,2e,\varepsilon}(\mathcal{C}',\mu')$ and satisfies (2.5.23) hence by definition of $F_{R,\eta}^{M,e,\varepsilon}$ we have

$$F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C}',\mu') \le F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C},\mu)+2\delta$$

if $(\mathcal{C}',\mu')$ is close enough to $(\mathcal{C},\mu)$. Taking δ arbitrarily small this ensures that $F_{R,\eta}^{M,e,\varepsilon}$ is upper semi-continuous at any point $(\mathcal{C},\mu)$ in $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$.

Following the same line of reasoning, together with Remark 2.5.5 we obtain that $\mathcal{S}_{R,\eta,-}^{M,e,\varepsilon}$ is open in $\mathcal{X}(\overline{C_R})\times L^\infty(C_R)$. On the other hand the fact that $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$ is closed is a consequence of Lemma 2.3.6 together with Remark 2.5.5. This in turn ensures that $F_{R,\eta}^{M,e,\varepsilon}=F_{R,\eta}^{M,e,\varepsilon}\mathbf{1}_{\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}}$ is upper semi-continuous at any point. □

The next lemma shows that tagged point process $\bar{P}$ of finite energy have good properties: most configurations under $\bar{P}$ are "screenable" in the sense of Section c., their energies are controlled by that of $\bar{P}$ and the truncation errors due to close pairs of points are small. These controls are then extended to point processes in small balls $B(\bar{P},\nu)$ around $\bar{P}$. In the following when a couple $(x,\mathcal{C})$ is fixed the implicit background measure is $\mu_V(x)$ i.e.

$$(x,\mathcal{C})\in\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}/\mathcal{S}_{R,\eta,-}^{M,e,\varepsilon} \iff (\mathcal{C},\mu_V(x))\in\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}/\mathcal{S}_{R,\eta,-}^{M,e,\varepsilon}.$$

Lemma 2.5.10. *Let $\bar{P}$ be a tagged point process in $\mathcal{P}_s(\Sigma\times\mathcal{X})$ such that $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$ is finite. Then we have*

1. *For $\eta > 0$ small enough and any $e, \varepsilon > 0$,*

$$\lim_{M,R\to\infty} \lim_{\nu\to 0} \inf_{\bar{Q}\in B(\bar{P},\nu)} \bar{Q}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}) = 1 \text{ and } \lim_{M,R\to\infty} \lim_{\nu\to 0} \inf_{\bar{Q}\in B(\bar{P},\nu)} \bar{Q}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}) = 1$$

where $M, R \to \infty$ in such a way that the conditions (2.5.16) *are satisfied.*

2. *For any $\eta, e, \varepsilon > 0$,*

$$\limsup_{M,R\to\infty} \limsup_{\nu\to 0} \sup_{\bar{Q}\in B(\bar{P},\nu)} \int \left(F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C}, \mu_V(x)) - c_{d,s}\mu_V(x)g(\eta)\right) d\bar{Q}(x,\mathcal{C}) \leq \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + C\eta^{\frac{d-s}{2}}. \quad (2.5.24)$$

3. *For any η, τ, with $0 < \tau < \eta^2/2 < 1$, any $x \in \mathbb{R}^d$, any $R > 0$*

$$\limsup_{\eta\to 0} \limsup_{\tau\to 0} \left(\frac{g(2\tau)}{\tau^d} \mathbf{E}_{\bar{P}}[(\mathcal{N}(x,\tau)^2 - 1)_+] + \frac{1}{R^d}\mathbf{E}_{\bar{P}}\left[\left(\sum_{p\neq q\in\mathcal{C}\cap C_R, \tau\leq|p-q|\leq\eta^2/2} g(|p-q|)\right)\right]\right) = 0. \quad (2.5.25)$$

Note that we cannot directly extend (2.5.25) to a small ball around $\bar{P}$ because functions like $\mathcal{C} \mapsto (\mathcal{N}(x,\tau)^2 - 1)_+(\mathcal{C})$ are not bounded.

Proof. As a consequence of Lemma 2.2.12 and (2.2.33) we know that since $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$ is finite we may find a tagged random electric field $\bar{P}^{\text{elec}}$ in $\mathcal{P}_s\left(\Sigma \times L^p_{\text{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})\right)$ such that

$$\forall x \in \Sigma,\ \text{Conf}_{\mu_V(x)}\#\bar{P}^{\text{elec},x} = \bar{P}^x, \qquad \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\text{elec}}) \leq \overline{\mathbb{W}}_{\mu_V}(\bar{P}),$$

where $\text{Conf}_{\mu_V(x)}\#\bar{P}^{\text{elec},x}$ denotes the push-forward of $\bar{P}^{\text{elec},x}$ by $\text{Conf}_{\mu_V(x)}$. By stationarity, in view of Lemma 2.2.8 we have for all $R > 0$

$$\int \left(\frac{1}{R^d}\int_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2 - c_{d,s}\mu_V(x)g(\eta)\right) d\bar{P}^{\text{elec}}(x,E) = \fint_\Sigma \widetilde{\mathcal{W}}_\eta(\bar{P}^{\text{elec},x})dx$$

and by Markov's inequality we see that for any $M, R > 0$ we have for η small enough

$$\bar{P}^{\text{elec}}\left(\frac{1}{R^d}\int_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2 \geq M\right) \leq \frac{\overline{\mathbb{W}}_{\mu_V}(\bar{P}) + c_{d,s}g(\eta)}{M}. \quad (2.5.26)$$

On the other hand we have $\bar{P}^{\text{elec}}$-almost surely

$$\lim_{R\to\infty} \int_{C_1\times(\mathbb{R}\setminus(-\varepsilon^2 R,\varepsilon^2 R))^k} |y|^\gamma |E_\eta|^2 = 0$$

which in turn implies, by stationarity (Lemma 2.2.8 again) that for any $e > 0$

$$\lim_{R\to\infty} \bar{P}^{\text{elec}}\left(\frac{1}{R^d}\int_{C_R\times(\mathbb{R}\setminus(-\varepsilon^2 R,\varepsilon^2 R))^k} |y|^\gamma |E_\eta|^2 \geq e\right) = 0. \quad (2.5.27)$$

Finally from Lemma 2.3.10 we see that $\mathbf{E}_{\bar{P}}\left[\mathcal{N}(0,R)^2\right] \leq CR^{2d}$ with a constant C depending only on $\bar{P}$ hence by Markov's inequality we have

$$\bar{P}\left(\mathcal{N}(0,R) \geq MR^d\right) \leq \frac{C}{M^2} \quad (2.5.28)$$

uniformly on R.

Combining (2.5.26), (2.5.27) and (2.5.28) yields that $\lim_{M,R\to\infty} \bar{P}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}) = 1$ and also, in view of (2.5.20), $\lim_{M,R\to\infty} \bar{P}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}) = 1$. Let us emphasize that although we do not need to satisfy the conditions (2.5.16), we may require them to be satisfied. Since $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}$ is open, $\mathbf{1}_{\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}}$ is lower semi-continuous, hence

$$\lim_{\nu\to 0} \inf_{\bar{Q}\in B(\bar{P},\nu)} \bar{Q}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}) \geq \bar{P}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-})$$

and the first item of the lemma follows using again (2.5.20) to handle $\bar{Q}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+})$.

To prove the second point, according to Lemma 2.2.12 and (2.2.33) we may consider for any $\delta > 0$ a random tagged electric field $\bar{P}^{\mathrm{elec},\delta}$ in $\mathcal{P}_s\left(\Sigma \times L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})\right)$ such that

$$\forall x \in \Sigma,\ \mathrm{Conf}_{\mu_V(x)} \# \bar{P}^{\mathrm{elec},\delta,x} = \bar{P}^x, \qquad \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec},\delta}) \leq \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \delta.$$

For $\eta > 0$ small enough, we get with Lemma 2.3.4

$$\fint_\Sigma \widetilde{\mathcal{W}}_\eta(\bar{P}^{\mathrm{elec},\delta,x})dx \leq \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec},\delta}) + C\eta^{\frac{d-s}{2}}. \tag{2.5.29}$$

We still have for any $R > 0$, by stationarity,

$$\int \left(\frac{1}{R^d}\int_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2 - c_{d,s}\mu_V(x)g(\eta)\right) d\bar{P}^{\mathrm{elec},\delta}(x,E) = \fint_\Sigma \widetilde{\mathcal{W}}_\eta(\bar{P}^{\mathrm{elec},\delta,x})dx$$
$$\leq \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec},\delta}) + C\eta^{\frac{d-s}{2}} \leq \overline{\mathcal{W}}_{\mu_V}(\bar{P}) + \delta + C\eta^{\frac{d-s}{2}}$$

where we have used Lemma 2.3.4. Using Remark 2.5.7 we see that

$$\int F^{M,e,\varepsilon}_{R,\eta} d\bar{P} \leq \int \left(\frac{1}{R^d}\int_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2\right) d\bar{P}^{\mathrm{elec},\delta}(x,E)$$
$$+ M\bar{P}^{\mathrm{elec},\delta}\left(\int_{\mathbb{R}^d\times(\mathbb{R}^k\setminus(-\varepsilon^2 R,\varepsilon^2 R))} |y|^\gamma |E_\eta|^2 \geq e\right).$$

Together with (the analogue of) (2.5.27) and the upper semi-continuity of $F^{M,e,\varepsilon}_{R,\eta}$, the last two relations yield the second item of the lemma, taking $\delta \to 0$, and $R \to \infty$ in (2.5.29).

We turn to the third item: for $\eta > 0$ and $\tau < \eta^2/2$ we have by Lemma 2.3.12 that

$$\fint_\Sigma \widetilde{\mathcal{W}}_\eta(\bar{P}^{\mathrm{elec},\delta,x})dx + C\left(\frac{g(2\tau)}{\tau^d}\mathbf{E}_{\bar{P}}[(\mathcal{N}(0,\tau)^2 - 1)_+] + \mathbf{E}_{\bar{P}}\left[\sum_{p\neq q\in\mathcal{C}\cap C_1, \tau\leq|p-q|\leq\eta^2/2} g(|p-q|)\right]\right)$$
$$\leq \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec},\delta}) + C\eta^{\frac{d-s}{2}}$$

and $\mathcal{N}(0,\tau)$ can be replaced by $\mathcal{N}(x,\tau)$ for any $x \in \mathbb{R}^d$ and C_1 by an average over C_R, by stationarity of $\bar{P}$ (cf. Lemma 2.2.8). Letting $\eta \to 0$ gives the result. □

2.5.2 Regularization of point configurations

The singularity of the interaction kernel g has been dealt with via the truncation procedure at the level η, which renormalizes the energy by truncating the short distance interactions. As $\eta \to 0$ the truncated energy converges for any configuration to the renormalized energy w_N (cf

(2.2.22)). However to prove the Large Deviation Principle we need to have a *uniform* control on the error made by truncating. This is what allows us to obtain a conclusion of the type of Varadhan's Integral Lemma (e.g. [DZ10, Theorem 4.3.1.]). Note that this difficulty already appears for example in [BG99] at the leading order of the LDP. In view of Lemma 2.3.3 to control the truncation, we need to control pairs of η-close points. We will do this in two steps: by separating points by a minimum distance τ with $\tau < \eta^2/2$ and then by estimating the interaction of pairs of points whose distance is between τ and η via Lemma 2.3.12. Let us note that since the screening procedure already erases "bad" (non-screenable) configurations and replace them by configurations for which there is no pair of points at distance $\leq \eta$ we only need to apply this regularization to screenable configurations.

a. The regularization procedure

When $l > 0$ is fixed, for any $\vec{i} \in l\mathbb{Z}^d$ by "the hypercube of center $\vec{i}$" we mean the closed hypercube of sidelength l of center $\vec{i}$ and whose edges are parallel to the axes of $\mathbb{Z}^d$, and we identify such a hypercube with its center. If $\vec{i} \in l\mathbb{Z}^d$ and $r > 0$ we let again $\mathcal{N}(\vec{i}, r)$ denote the number of points in the hypercube of sidelength $r > 0$ centered at $\vec{i}$.

The purpose of the following lemma is to "regularize" a point configuration by spacing out the points that are too close to each other, while remaining close to the original configuration. This operation generates a certain volume of configurations $\mathcal{C}^{\mathrm{reg}}$ ("reg" as "regularized") for which we control the contribution of the energy due to pair of close points.

Lemma 2.5.11. *For any $\tau \in (0,1)$ and any hyperrectangle K whose sidelengths are in $[R, 2R]$ there exists a measurable multivalued function $\Phi^{\mathrm{reg}}_{\tau,R}$ mapping $\mathcal{X}(K)$ into the set of subsets of $\mathcal{X}(K)$ such that any configuration $\mathcal{C}^{\mathrm{reg}}$ in $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ has the same number of points as $\mathcal{C}$ and satisfies*

1. *The distance to the original configuration goes to zero when $\tau \to 0$ (uniformly on $\mathcal{C}^{\mathrm{reg}}$)*

$$\sup\{d_{\mathcal{X}}(\mathcal{C}, \mathcal{C}^{\mathrm{reg}}) \mid \mathcal{C}^{\mathrm{reg}} \in \Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})\} \underset{\tau\to 0}{\longrightarrow} 0.$$

2. *For any finite configuration $\mathcal{C}$ and any $\mathcal{C}^{\mathrm{reg}} \in \Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ we have for any $\eta \geq 8\tau$*

$$\sum_{x_i \neq x_j \in \mathcal{C}^{\mathrm{reg}}, |x_i - x_j| \leq \eta} g(x_i - x_j) \leq Cg(\tau)\Big(\sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \Big(\mathcal{N}(\vec{i}, 12\tau)^2(\mathcal{C}) - 1\Big)_+ + \sum_{x_i \neq x_j \in \mathcal{C}, \tau \leq |x_i - x_j| \leq 2\eta} g(x_i - x_j)\Big)$$

   where C is a universal constant (depending only on d).

3. *For any integer n_K and any set A of configurations with n_K points, we have:*

$$\log \mathbf{Leb}^{\otimes n_K}\Big(\bigcup_{\mathcal{C} \in A} \Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})\Big) \geq \log \mathbf{Leb}^{\otimes n_K}(A) - C \fint_{\mathcal{C} \in A} \sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \mathcal{N}(\vec{i}, 6\tau)(\mathcal{C}) \log \mathcal{N}(\vec{i}, 6\tau)(\mathcal{C}) \tag{2.5.30}$$

   where C is a universal constant (depending only on d).

Proof. **Definition of the regularization procedure.**
For any $\tau > 0$ and $\mathcal{C} \in \mathcal{X}(K)$ we consider two categories of hypercubes in $6\tau\mathbb{Z}^d$:

— $S_\tau(\mathcal{C})$ is the set of hypercubes $\vec{i} \in 6\tau\mathbb{Z}^d$ such that $\mathcal{C}$ has at most one point in $\vec{i}$ and no point in the adjacent hypercubes.

— $T_\tau(\mathcal{C})$ is the set of the hypercubes that are not in $S_\tau(\mathcal{C})$ and that contain at least one point of $\mathcal{C}$.

We define $\varphi_\tau(\mathcal{C})$ to be the following configuration: the points of $\mathcal{C}$ that belong to some hypercube of $S_\tau(\mathcal{C})$ are left unchanged, whereas for any $\vec{i} \in T_\tau(\mathcal{C})$ we replace the configuration $\mathcal{C} \cap \vec{i}$ by a well-separated configuration in a smaller hypercube. More precisely we consider the lattice $3\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau\mathbb{Z}^d$ translated so that the origin coincides with the point $\vec{i} \in 6\tau\mathbb{Z}^d$ and place $\mathcal{N}(\vec{i}, 6\tau)$ points on this lattice in such a way that they are all contained in the hypercube of sidelength 3τ and center $\vec{i}$ (a simple argument of volume shows that this is indeed possible, the precise way of disposing points is not important - it is easy to see that one may do it in a measurable fashion). This defines a measurable function $\varphi_{\tau,R} : \mathcal{X}(K) \to \mathcal{X}(K)$ (see Figure 2.2).

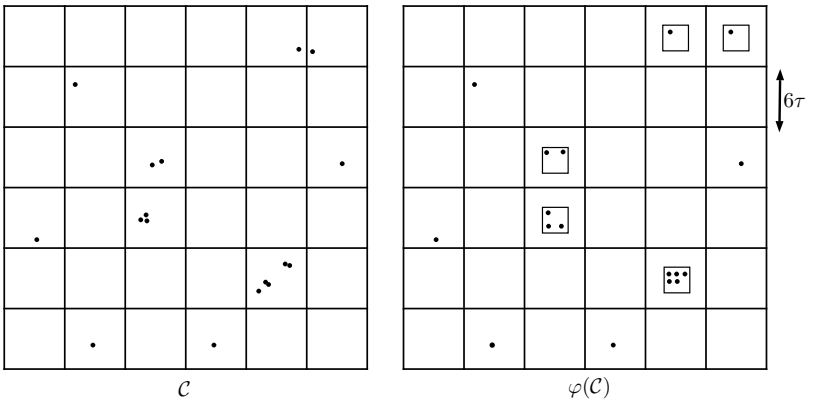

Figure 2.2 – Effect of the regularization. On the right are shown the smaller hypercubes in which the new configurations are created for $\vec{i} \in T_\tau(\mathcal{C})$.

We then define $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ to be the set of configurations that are obtained from $\mathcal{C}$ the following way: the points of $\mathcal{C}$ that belong to some hypercube of $S_\tau(\mathcal{C})$ are left unchanged and for any $\vec{i} \in T_\tau(\mathcal{C})$ we allow the points of $\varphi_{\tau,R}(\mathcal{C}) \cap \vec{i}$ to move arbitrarily (and independently) within a radius $\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau$. We claim that $\Phi^{\mathrm{reg}}_{\tau,R}$ has the three desired properties.

A. Distance estimate.

The first claim of the lemma is easy to check: since for any $\mathcal{C}^{\mathrm{reg}} \in \Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ the configurations $\mathcal{C}$ and $\mathcal{C}^{\mathrm{reg}}$ have the same number of points in every hypercube of $6\tau\mathbb{Z}^d$ it implies that every point of $\mathcal{C}$ is moved by a distance at most $O(\tau)$ (with a constant depending only on the dimension) which in view of the definition (2.2.29) of $d_{\mathcal{X}}$ yields $d_{\mathcal{X}}(\mathcal{C}, \mathcal{C}^{\mathrm{reg}}) = O(\tau)$ uniformly for $\mathcal{C}^{\mathrm{reg}} \in \Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ (it really depends only on the number of points of $\mathcal{C}$ in K).

B. Truncation estimate.

To prove the second point let us distinguish three types of pairs of points $x_i, x_j \in \mathcal{C}^{\mathrm{reg}}$ which might satisfy $|x_i - x_j| \leq \eta$:

1. The pairs of points x_i, x_j belonging to some hypercube of $T_\tau(\mathcal{C})$.
2. The pairs of points x_i, x_j belonging to two adjacent hypercubes of $T_\tau(\mathcal{C})$.
3. The pairs of points x_i, x_j such that $|x_i - x_j| \leq \eta$ but neither of the two previous cases holds.

To bound the contributions of the first type of pairs, let us observe that in any hypercube $\vec{i} \in T_\tau(\mathcal{C})$ the sum of pairwise interactions is bounded above by

$$\sum_{x_i \neq x_j \in \mathcal{C}^{\mathrm{reg}} \cap \vec{i}} g(x_i - x_j) \leq C g(\tau)(\mathcal{N}(\vec{i}, 6\tau)^2(\mathcal{C}) - 1)_+. \tag{2.5.31}$$

Indeed by construction the point configuration $\mathcal{C}^{\mathrm{reg}}$ in a hypercube $\vec{i} \in T_\tau(\mathcal{C})$ consists in a subset of the lattice $3\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau\mathbb{Z}^d$ where each point has been allowed to move within a ball of radius $\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau$. The minimal distance between points is hence at least $2\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau$ and moreover a simple combinatorial argument shows that for each point of $\mathcal{C}^{\mathrm{reg}}$ in this hypercube $\vec{i}$ there is $O(r^{d-1})$ other points at distance $r\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau$ hence we have

$$\sum_{x_i \neq x_j \in \mathcal{C}^{\mathrm{reg}} \cap \vec{i}} g(x_i - x_j) = \mathcal{N}(\vec{i}, 6\tau)\ O\left(\int_2^{O\left(\mathcal{N}(\vec{i},6\tau)^{1/d}\right)} g(r\mathcal{N}(\vec{i}, 6\tau) - 1/d\tau) r^{d-1} dr \right),$$

which in turn yields (2.5.31) [2], since the sum is obviously zero when $\mathcal{N}(\vec{i}, 6\tau) = 1$.

For the second type of pairs, let us denote by $Z_{\tau,0}$ the set of $\vec{i} \in \mathbb{Z}^d$ whose coordinates belong to $\{0, 1\}$ and by $n_{\vec{i},2\tau,\vec{i_0}}$ the number of points in $\vec{i} + \vec{i_0}$ where $\vec{i} \in 12\tau\mathbb{Z}^d$. We notice that the points x_i and x_j belong to the same hypercube in either $12\tau\mathbb{Z}^d$ or one of the translates $12\tau\mathbb{Z}^d + 6\tau\vec{i_0}$ for some $\vec{i_0} \in Z_{\tau,0}$. Observing that the distance between the points in $\mathcal{C}^{\mathrm{reg}}$ which belong to different hypercubes is bounded below by 6τ, we find that the total contribution of the second type of pairs is bounded by

$$Cg(\tau) \sum_{\vec{i_0} \in Z_{\tau,0}} \sum_{\vec{i} \in 12\tau\mathbb{Z}^d} (n^2_{\vec{i},2\tau,\vec{i_0}} - 1)_+.$$

We may simplify the previous expression by extending the size of the hypercube in which we count the points and bound the total contribution of the second type of pairs by

$$Cg(\tau) \sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \left(\mathcal{N}(\vec{i}, 12\tau)^2(\mathcal{C}) - 1 \right)_+.$$

Finally the contribution of the third type of pairs is easily bounded by

$$\sum_{x_i, x_j \in \mathcal{C}, \tau \leq |x_i - x_j| \leq \eta + 8\tau} Cg(x_i - x_j),$$

indeed any such two points live in non-adjacent hypercubes hence were at distance $|x_i - x_j| \geq 12\tau$ in $\mathcal{C}$ and their distance is at worst reduced by 8τ during the regularization (then one discusses according to whether g is logarithmic or satisfies (2.1.3)).

C. Volume loss estimate.

Finally we turn to the volume consideration. The fibers of $\Phi^{\mathrm{reg}}_{\tau,R}$ have a simple description: we have $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C}) = \Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C}')$ only if $\mathcal{C}' \cap \vec{i} = \mathcal{C} \cap \vec{i}$ for $\vec{i} \in S_\tau(\mathcal{C})$ and $\mathcal{N}(\vec{i}, 6\tau)(\mathcal{C}') = \mathcal{N}(\vec{i}, 6\tau)(\mathcal{C})$ for $\vec{i} \in T_\tau(\mathcal{C})$ (these conditions are sufficient once symmetrized with respect to the roles of $\mathcal{C}$ and $\mathcal{C}'$). For a given configuration $\mathcal{C}$ in K with N points this describes a submanifold of $(\mathbb{R}^d)^N$ of co-dimension $\#S_\tau(\mathcal{C})$. The volume of a fiber is bounded by

$$\left(\sum_{\vec{i} \in T_\tau(\mathcal{C})} \mathcal{N}(\vec{i}, 6\tau) \right)! \left(\tau^d\right)^{\sum_{\vec{i} \in T_\tau(\mathcal{C})} \mathcal{N}(\vec{i}, 6\tau)}$$

whereas the volume of $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ is given by

$$\left(\sum_{\vec{i} \in T_\tau(\mathcal{C})} \mathcal{N}(\vec{i}, 6\tau) \right)! \prod_{\vec{i} \in T_\tau(\mathcal{C})} \left(\frac{\tau}{\mathcal{N}(\vec{i}, 6\tau)^{1/d}} \right)^{\mathcal{N}(\vec{i}, 6\tau) d}$$

which after taking the logarithm yields the volume comparison of equation (2.5.30). □

2. Another way of seeing (2.5.31) is to recall that according to the "first-order" results (2.1.5) the minimal energy of N points in a fixed compact set grows as N^2 and (2.5.31) follows by scaling.

b. Effect on the energy

We argue that the regularization procedure at scale τ has a negligible influence on the screened energy e.g. for configurations obtained by the screening procedure of Proposition 2.5.2. Let K and μ, $\mathcal{C}$ satisfying the assumptions in Proposition 2.5.2 and let $\Phi^{\rm scr}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ be the set of configurations generated by the screening procedure of Proposition 2.5.2. For any $\mathcal{C}^{\rm scr}$ in $\Phi^{\rm scr}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ let $E^{\rm scr}$ be the corresponding screened vector field (2.5.10). Let $\Phi^{\rm reg}_{\tau,R}(\mathcal{C}^{\rm scr})$ be the set of configurations generated by the regularization procedure applied to $\mathcal{C}^{\rm scr}$. The following holds

Lemma 2.5.12. *For any $\mathcal{C}^{\rm reg}$ in $\Phi^{\rm reg}_{\tau,R}(\mathcal{C}^{\rm scr})$ there exists a vector field $E^{\rm reg}\in L^p_{\rm loc}(\mathbb{R}^{d+k},\mathbb{R}^{d+k})$ satisfying*

1. $$\begin{cases} -\text{div }(|y|^\gamma E^{\rm reg}) = c_{d,s}\,(\mathcal{C}^{\rm reg}-\mu\delta_{\mathbb{R}^d}) & \text{in } K\times\mathbb{R}^k \\ E^{\rm reg}\cdot\vec{\nu}=0 & \text{on } \partial K\times\mathbb{R}^k, \end{cases} \tag{2.5.32}$$

2. *Letting $E^{\rm reg}_\eta$ be associated to $E^{\rm reg}$ as in* (2.2.19) *we have*

$$\int_{K\times\mathbb{R}^k}|y|^\gamma|E^{\rm reg}_\eta|^2 \le \left(\int_{K\times\mathbb{R}^k}|y|^\gamma|E^{\rm scr}_\eta|^2\right)(1+o_\tau(1)) \tag{2.5.33}$$

where the error term $o_\tau(1)$ goes to zero as $\tau\to 0$, depending only on η and R.

Proof. Let $g^{\rm Neu}$ be the unique solution with mean zero to

$$\begin{cases} -\text{div}\,(|y|^\gamma\nabla g^{\rm Neu}) = c_{d,s}\left(\delta_0-\frac{1}{|K|}\delta_{\mathbb{R}^d}\right) & \text{in } K\times\mathbb{R}^k \\ \nabla g^{\rm Neu}\cdot\vec{\nu}=0 & \text{on } \partial K\times\mathbb{R}^k, \end{cases}$$

and let $g^{\rm Neu}_\eta$ be the truncated kernel at scale η as above. For any $\mathcal{C}^{\rm scr}$ in $\Phi^{\rm scr}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ and any $\mathcal{C}^{\rm reg}$ in $\Phi^{\rm reg}_{\tau,R}(\mathcal{C}^{\rm scr})$ let us consider the vector field $\tilde{E}$ generated by the difference $\mathcal{C}^{\rm reg}-\mathcal{C}^{\rm scr}$ with Neumann boundary conditions on ∂K

$$\tilde{E}(x) := \int \nabla g^{\rm Neu}(x-p)(\mathcal{C}^{\rm reg}-\mathcal{C}^{\rm scr})(p).$$

Since the regularization procedure preserves the number of points, it is clear that $E^{\rm reg}:=E^{\rm scr}+\tilde{E}$ satisfies (2.5.32). To bound its energy we proceed as in the proof of Lemma 2.5.9 and it is enough to bound $\int_{\mathbb{R}^{k+d}}|y|^\gamma|\tilde{E}_\eta|^2$ by a $o_\tau(1)$. Integrating by parts we are left to bound

$$\iint g^{\rm Neu}_\eta(x-y)(\mathcal{C}^{\rm reg}-\mathcal{C}^{\rm scr})(x)(\mathcal{C}^{\rm reg}-\mathcal{C}^{\rm scr})(y).$$

By construction there is no point of $\mathcal{C}^{\rm scr}$ or $\mathcal{C}^{\rm reg}$ closer than some constant $\eta_0>0$ to ∂K, and $g^{\rm Neu}_\eta$ is uniformly continuous at distance $\ge\eta_0$ from ∂K. Moreover there is the same number of points in $\mathcal{C}^{\rm reg}$ and $\mathcal{C}^{\rm scr}$, this number is at most $C||\mu||_\infty R^d$, and the minimal connection distance between the points of $\mathcal{C}^{\rm scr}$ and $\mathcal{C}^{\rm reg}$ is then bounded by $C||\mu||_\infty R^d\tau$ because each point of $\mathcal{C}^{\rm scr}$ has been moved by a distance at most $C\tau$ during the regularization (see Item 1 of Lemma 2.5.11). We may then bound

$$\iint g^{\rm Neu}_\eta(x-y)(\mathcal{C}^{\rm reg}-\mathcal{C}^{\rm scr})(x)(\mathcal{C}^{\rm reg}-\mathcal{C}^{\rm scr})(y) = O(\tau)$$

with a $O(\tau)$ depending only on $R,d,||\mu||_\infty$, but independent of $\mathcal{C}^{\rm reg}$ and $\mathcal{C}^{\rm scr}$. □

2.5.3 Conclusion

We may now combine the previous ingredients to accomplish the program stated at the beginning of the section.

For any point configuration $\mathcal{C}$ in $\mathcal{X}(C_R)$, any hyperrectangle K containing C_R and any bounded measure μ on K such that $n_{K,\mu} := \int_K \mu$ is an integer we define a family $\Phi^{\mathrm{mod}}(\mathcal{C}, \mu, K)$ (depending on the other parameters $\eta, \varepsilon, M, R, \tau$) of point configurations which are contained in K and have $n_{K,\mu}$ points the following way :

1. If $(\mathcal{C}, \mu)$ **is** *screenable* i.e. is in $\mathcal{S}^{M,1,\varepsilon}_{R,\eta,+}$ then we let $\Phi^{\mathrm{mod}}(\mathcal{C}, \mu)$ be the image by $\Phi^{\mathrm{reg}}_{\tau,R}$ of the family $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C}, \mu)$ of point configurations in K obtained by applying Proposition 2.5.2 to any electric field $E \in \mathcal{O}^{2M,2,\varepsilon}_{R,\eta,-}(\mathcal{C}, \mu)$ such that

$$\frac{1}{R^d} \int_{C_R \times [-R,R]^k} |y|^\gamma |E_\eta|^2 \leq F^{M,1,\varepsilon}_{R,\eta}(\mathcal{C}, \mu) + \varepsilon.$$

   By Proposition 2.5.2 together with Lemma 2.5.12, to any of these point configurations is associated a *screened* and *regularized* electric field whose energy is bounded in terms of $\frac{1}{R^d} \int_{C_R \times \mathbb{R}^k} |y|^\gamma |E_\eta|^2$, hence in terms of $F^{M,1,\varepsilon}_{R,\eta}(\mathcal{C}, \mu) + \varepsilon$ as in (2.5.11) and (2.5.33).

2. If $(\mathcal{C}, \mu)$ is **not** in $\mathcal{S}^{M,1,\varepsilon}_{R,\eta,+}$ then we let $\Phi^{\mathrm{mod}}(\mathcal{C}, \mu, K)$ be the family of configurations $\Phi^{\mathrm{gen}}(K, \mu)$ defined in Lemma 2.5.1. By the conclusions of Lemma 2.5.1, to any of these point configurations is associated an electric field whose energy is bounded as in (2.5.4) and which vanishes outside K.

Let us evaluate the effect of this operation on the volume of configurations i.e. we compare the volume of a certain set of configurations in C_R with the volume of the resulting configurations after applying Φ^{mod}. We distinguish between the cases of a set of *screenable* configurations and a set of *non-screenable* configurations.

Lemma 2.5.13. *Let R, K, μ be as above. Assume A is a (measurable) set of point configurations in $\mathcal{X}(C_R)$ such that each configuration of A has n points in C_R and n_{int} points in Int_ε.*

1. If $(\mathcal{C}, \mu)$ is in $\mathcal{S}^{M,1,\varepsilon}_{R,\eta,+}$ for all $\mathcal{C} \in A$ and (2.5.14) holds then

$$\begin{aligned}\log \mathbf{Leb}^{\otimes n_{K,\mu}} \left(\bigcup_{\mathcal{C} \in A} \Phi^{\mathrm{mod}}(\mathcal{C}, \mu) \right) &\geq \log \mathbf{Leb}^{\otimes n}(A) \\ &+ \log \left((n_{K,\mu} - n_{int})! \left(\frac{c}{|\mathrm{Ext}_\varepsilon|} \right)^{n_{K,\mu} - n_{\mathrm{int}}} \right) + (n_{K,\mu} - n) \log |\mathrm{Ext}_\varepsilon| \\ &- C \fint_{\mathcal{C} \in A} \sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \mathcal{N}(\vec{i}, 12\tau) \log \mathcal{N}(\vec{i}, 12\tau). \quad (2.5.34)\end{aligned}$$

2. If $(\mathcal{C}, \mu)$ is not in $\mathcal{S}^{M,1,\varepsilon}_{R,\eta,+}$ for all $\mathcal{C} \in A$ then

$$\log \mathbf{Leb}^{\otimes n_{K,\mu}} \left(\bigcup_{\mathcal{C} \in A} \Phi^{\mathrm{mod}}(\mathcal{C}, \mu, K) \right) \geq \log \mathbf{Leb}^{\otimes n}(A) + \log \left((n_{K,\mu}! C^{n_{K,\mu}} |R^d|^{-n} \right). \quad (2.5.35)$$

Proof. The bound (2.5.34) follows from combining (2.5.15) with (2.5.30) whereas (2.5.35) follows directly from (2.5.5). □

2.6 Construction of configurations

This section is devoted to the proof of Proposition 2.1.7 by expliciting a set of compatible configurations with a large enough asymptotic (logarithmic) volume. To do so, we follow the strategy initiated in the previous section, i.e. first partition (some subset of) $\mathbb{R}^d$ into hyperrectangles K such that $\int_K \mu_V'$ is an integer (here and in the following we mostly deal with quantities defined at the blown-up scale $N^{1/d}$). Each hyperrectangle K will contain a hypercube translate of C_R such that $|K| - |C_R|$ is small and each hypercube will contain a point configuration. We want the global configurations (when considering all hyperrectangles together) to approximate (after averaging over translations) a given tagged point process $\bar{P}$. To do so, we will draw the point configurations in each hypercube jointly at random according to a (slightly modified) Poisson point process, and standard large deviations results will allow us to show that the correct ones end up occupying enough volume in phase space i.e. that sufficiently many of the (averaged) point configurations ressemble $\bar{P}$.

Then these configurations drawn "abstractly" at random using Sanov's theorem are modified as described in the previous section by screening-then-regularizing the parts for which it is possible to do so, and replacing the "bad" parts by "standard" configurations constructed by hand. This will allow to eventually obtain a global configuration with N points whose energy can be computed additively with respect to the hyperrectangles. At each step we need to check that the transformations imposed to the configurations do not alter much their phase-space volume, their energy, and keep them close to the given tagged process $\bar{P}$.

One of the additional technical difficulties is that the density of the equilibrium measure μ_V is in general not bounded from below near the boundary $\partial\Sigma$ and that its support Σ cannot be exactly tiled by hyperrectangles. To deal with this, we follow the construction made in [PS15] which consists in removing a thin layer near the boundary, and in placing in that layer some "frozen configuration" constructed by hand where the points are only free to move within small balls. We will later have to show again that the contributions to the energy and to the volume of this thin layer are negligible.

In the following we always assume that $\bar{P}$ is a stationary tagged point processes with finite energy $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$ (otherwise Proposition 2.1.7 reduces to Proposition 2.1.6).

2.6.1 Subdividing the domain

We start the construction as in [PS15, Section 7] : we divide the domain between a neighborhood of the boundary, where the density is not bounded below and which must be treated "by hand", and a large interior. We recall that Σ is the support of the equilibrium measure μ_V. We let $\Sigma' = N^{1/d}\Sigma$ (which depends on N) be its blown-up and $\mu_V'(x') = \mu_V(N^{-1/d}x')$ the blow-up of the equilibrium measure, and recall that its density is bounded above by $\overline{m}$.

For convenience we recall the construction of [PS15, Section 7]. For $t > 0$ we define the tubular neighborhood of $\partial\Sigma'$ and its boundary to be

$$\Sigma_t' = \{x \in \Sigma', \mathrm{dist}(x, \partial\Sigma') > t\} \qquad \Gamma_t = \{x \in \Sigma', \mathrm{dist}(x, \partial\Sigma') = t\}.$$

Since (2.2.5) holds, Γ_t is C^1 for $t < t_c$ small enough.

Pick $1 > \underline{m} > 0$ a small number. By assumption (2.2.7), if $\alpha > 0$ in that assumption, rescaling by $N^{1/d}$, if $\mathrm{dist}(x, \partial\Sigma') \geq \frac{N^{1/d}}{c_1^{1/\alpha}}\underline{m}^{1/\alpha}$ where c_1 is the constant in (2.2.7), then $\mu_V'(x) \geq \underline{m}$. Thus we may find

$$T = T(N) \in \left[\frac{N^{1/d}}{c_1^{1/\alpha}}\underline{m}^{1/\alpha}, \frac{N^{1/d}}{c_1^{1/\alpha}}\underline{m}^{1/\alpha} + c\right], \tag{2.6.1}$$

for a constant c depending on $\underline{m}$, such that : $\mathcal{N}(T) := \mu'_V(\Sigma'_T) \in \mathbb{N}$, and $\mu'_V \geq \underline{m}$ in Σ'_T. We note that we may have taken $\underline{m}$ small enough so that $T < t_c$ and

$$\mathcal{H}^{d-1}(\partial\Sigma'_t) \leq 2\mathcal{H}^{d-1}(\partial\Sigma') = O(N^{\frac{d-1}{d}}) \quad \text{for all } t \leq T. \tag{2.6.2}$$

If $\alpha = 0$ in assumption (2.2.7) then μ'_V is bounded below by a positive constant on its support and we simply take $T = 0$ (of course (2.6.2) holds also in this case). By (2.6.1) the quantity $N^{-1/d}T$ tends to $\frac{\underline{m}^{1/\alpha}}{c_1^{1/\alpha}}$ as $N \to \infty$. In the following we let $r_{\underline{m}} := \frac{\underline{m}^{1/\alpha}}{c_1^{1/\alpha}}$, $\Sigma_{\underline{m}} := \{x \in \Sigma, \mathrm{dist}(x, \partial\Sigma) \geq r_{\underline{m}}\}$ and $\Sigma'_{\underline{m}} := \{x \in \Sigma', \mathrm{dist}(x, \partial\Sigma') \geq N^{1/d} r_{\underline{m}}\}$.

In the region Σ'_T we have the lower bound $\mu'_V \geq \underline{m}$ and there is no degeneracy. We now tile Σ'_T by hypercubes whose size is large but independent of N. The next lemma is a straightforward modification of [SS15b, Lemma 6.5].

Lemma 2.6.1 (Tiling the interior of the domain)**.** *There exists a constant $C_0 > 0$ depending on $\underline{m}, \overline{m}$ such that, given any $R > 1$, there exists for any $N \in \mathbb{N}^*$ a collection $\mathcal{K}_N$ of closed hyperrectangles in Σ'_T with disjoint interiors, whose sidelengths are between R and $R + C_0/R$, and which are such that*

$$\{x \in \Sigma'_T : d(x, \partial\Sigma'_T) \geq C_0 R\} \subset \bigcup_{K \in \mathcal{K}_N} K := \Sigma'_{\mathrm{int}}, \tag{2.6.3}$$

$$\forall K \in \mathcal{K}_N, \quad \int_K \mu'_V \in \mathbb{N}.$$

Moreover, an inspection of the proof allows us to observe that the hyperrectangles have their axes parallel to those of $\mathbb{R}^d$.

Let us enumerate the elements of $\mathcal{K}_N$ as $K_1, \ldots, K_{m_{N,R}}$ where $m_{N,R}$ is the number of hyperrectangles in $\mathcal{K}_N$. For any hyperrectangle K_i in $\mathcal{K}_N$ we denote by x_i the center of K_i and by $\overline{C}_i$ the closed hypercube of sidelength R contained in K_i whose center is x_i and whose axes are parallel to those of K_i, we also let $N_i := \int_{K_i} \mu'_V$ (which is an integer by construction). Since μ'_V is bounded above by $\overline{m}$ and below by $\underline{m}$ on Σ'_{int} and since K_i has its sidelengths in $[R, R + C_0/R]$ we have

$$C_1 R^d \leq N_i \leq C_2 R^d \tag{2.6.4}$$

with constants $C_1, C_2 > 0$ depending only on $\underline{m}, \overline{m}$. For any hypercube $\overline{C}_i$ we denote

$$\mathrm{Int}_{\varepsilon,i} := \{x \in \overline{C}_i \mid \mathrm{dist}(x, \partial\overline{C}_i) \geq 2\varepsilon R^d\}$$

as defined in (2.5.7) and by $\mathrm{Ext}_{\varepsilon,i}$ its complement in $\overline{C}_i$. We denote by $\mathcal{N}_i$ (resp. $\mathcal{N}_i^{\mathrm{int}}$) the function "number of points of a configuration in $\overline{C}_i$" (resp. in $\mathrm{Int}_{\varepsilon,i}$).

In the following lemma we collect some useful estimates about the quantities related to the tiling.

Lemma 2.6.2. *We have for any $R > 0$:*

1.

$$\lim_{R\to\infty} \lim_{N\to\infty} \frac{R^d}{N} m_{N,R} = |\Sigma_{\underline{m}}|. \tag{2.6.5}$$

More precisely we have

$$\frac{R^d}{N} m_{N,R} = |\Sigma_{\underline{m}}|(1 + o_{N\to\infty}(1))(1 + O_{R\to\infty}(R^{-2})). \tag{2.6.6}$$

2.

$$|K_i| = R^d + O(R^{d-2}) \tag{2.6.7}$$

$$N_i = R^d \mu'_V(x_i) + o_{N\to\infty}(1)R^d + O(R^{d-2}) \tag{2.6.8}$$

where $O(R^{d-2})$ depends only on $\underline{m}, \overline{m}$.

3.

$$|\mathrm{Ext}_{\varepsilon,i}| = R^d(1-(1-2\varepsilon)^d) = 2d\varepsilon R^d + O(\varepsilon^2)R^d \tag{2.6.9}$$

where the $O(\varepsilon^2)$ depends only on ε.

Proof. From (2.6.3) we see that $|\Sigma'_T - \Sigma'_{\mathrm{int}}|$ is bounded by $|\{x \in \Sigma'_T, \mathrm{dist}(x, \partial\Sigma'_T) \le C_0 R\}|$. From (2.6.2) we see that

$$|\{x \in \Sigma'_T, \mathrm{dist}(x, \partial\Sigma'_T) \le C_0 R\}| = O(RN^{\frac{d-1}{d}}) = o(|\Sigma'_T|)$$

because $|\Sigma'_T|$ is of order N. This implies

$$|\Sigma'_{\mathrm{int}}| \sim |\Sigma'_T| \text{ when } N \to \infty. \tag{2.6.10}$$

By construction the $m_{N,R}$ hyperrectangles partition Σ'_{int} and have sidelengths in $[R, R + C_0/R]$ hence the following holds

$$m_{N,R}R^d \le |\Sigma'_{\mathrm{int}}| \le m_{N,R}\left(R + \frac{C_0}{R}\right)^d, \tag{2.6.11}$$

in particular

$$m_{N,R}R^d = |\Sigma'_{\mathrm{int}}|\left(1 + O_{R\to\infty}\left(R^{-2}\right)\right).$$

Moreover we have from (2.6.1) and by definition

$$\lim_{N\to\infty} \frac{|\Sigma'_T|}{N} = \lim_{N\to\infty} \frac{|\Sigma'_{\underline{m}}|}{N} = |\Sigma_{\underline{m}}|. \tag{2.6.12}$$

The three relations (2.6.10), (2.6.11), (2.6.12) easily yield (2.6.5) and (2.6.6).

The bound (2.6.7) holds because by construction the sidelengths of K_i are between R and $R + C_0/R$ with a constant C_0 depending only on $\underline{m}$). To get (2.6.8) we use the fact that $|K_i| = R^d + O(R^{d-2})$ and from the Hölder condition (2.2.6) we see that $\|\mu'_V(x) - \mu'_V(x_i)\|_{L^\infty(K_i)}$ tends to 0 as $N \to \infty$ (depending only on the size of K_i, hence on R and $\underline{m}$).

The bound (2.6.9) follows immediately from the definitions.

□

From now on, until Section 2.6.4 we work only in Σ'_{int} defined in (2.6.3), which we recall is a disjoint union of hyperrectangles (see Figure 2.3, where the region in grey corresponds to Σ'_{int}).

2.6.2 Generating approximating microstates

This step is devoted to presenting an argument in the spirit of Sanov's theorem in order to generate "abstractly" a whole family of point configurations in Σ'_{int} whose continuous and discrete averages over translations are close to some fixed tagged point process. The proof of Lemma 2.6.3 follows the same line as the proof of Proposition 2.1.6 and is given in Section 2.7.

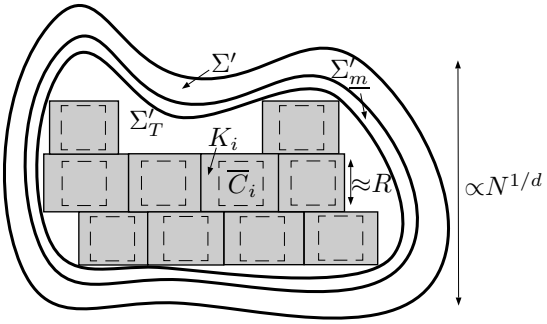

Figure 2.3 – The tiling of Σ'.

For any $\bar{P}$ in $\mathcal{P}_s(\Sigma \times \mathcal{X})$ we let $\bar{P}_{\underline{m}}$ be the tagged point process induced by restricting the "tag" coordinates to $\Sigma_{\underline{m}} \subset \Sigma$ i.e.

$$\bar{P}_{\underline{m}} := \frac{1}{|\Sigma_{\underline{m}}|} \int_{\Sigma_{\underline{m}}} \bar{P}^x dx.$$

Since $|\Sigma - \Sigma_{\underline{m}}| \to 0$ as $\underline{m} \to 0$, if F is a measurable function on $\Sigma \times \mathcal{X}$ such that F is $L^1(\bar{P})$, then

$$\int F d\bar{P}_{\underline{m}} \to \int F d\bar{P} \quad \text{as } \underline{m} \to 0$$

and this convergence is uniform for $F \in \mathrm{Lip}_1(\Sigma \times \mathcal{X})$. This also implies that for any $r > 0$ we have

$$B(\bar{P}_{\underline{m}}, r/2) \subset B(\bar{P}, r) \subset B(\bar{P}_{\underline{m}}, 2r)$$

for $\underline{m}$ small enough (depending on r).

The following lemma says that the discrete space average as well as the continuum space average of randomly chosen configurations occupy a volume in $B(\bar{P}_m, \varepsilon)$ which is given by the entropy of $\bar{P}$. We will prove it in Section 2.7 together with Proposition 2.1.6.

Lemma 2.6.3. *Let $(\mathbf{C}_1, \ldots, \mathbf{C}_{m_{N,R}})$ be $m_{N,R}$ independent Poisson point processes of intensity 1 on each hypercube $\overline{C}_i$ conditioned so that the total number of points is equal to*

$$N_{\mathrm{int}} := \mu'_V(\Sigma'_{\mathrm{int}}) = \sum_{i=1}^{m_{N,R}} N_i. \tag{2.6.13}$$

We define $\dot{\mathfrak{M}}_{N,R}$ as the law of the following random variable in $\Sigma'_{\mathrm{int}} \times \mathcal{X}$:

$$\frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/d} x_i, \theta_{x_i} \cdot \mathbf{C}_i)}.$$

Moreover let $\mathbf{C}$ be the point process obtained as the union of the point processes $\mathbf{C}_i$ i.e.

$$\mathbf{C} := \sum_{i=1}^{m_{N,R}} \mathbf{C}_i$$

as a sum of random measures, and let us define $\widehat{\mathfrak{M}}_{N,R}$ as the law of the random variable in $\Sigma'_{\underline{m}} \times \mathcal{X}$

$$\frac{1}{N|\Sigma_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d} x, \theta_x \cdot \mathbf{C})} dx.$$

Let us denote by $\bar{P}_{\underline{m},R}$ the law induced by $\bar{P}_{\underline{m}}$ in the hypercube C_R, i.e. the push-forward of $\bar{P}_{\underline{m}}$ by the map $(x,\mathcal{C}) \mapsto (x, \mathcal{C} \cap C_R)$. Finally let

$$r_{\mu_V,\underline{m}} := -\frac{\mu_V(\Sigma_{\underline{m}})}{|\Sigma_{\underline{m}}|} \log \frac{\mu_V(\Sigma_{\underline{m}})}{|\Sigma_{\underline{m}}|} + \frac{\mu_V(\Sigma_{\underline{m}})}{|\Sigma_{\underline{m}}|} - 1.$$

Then for any $\bar{P} \in \mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$ the following inequality holds :

$$\liminf_{R\to\infty} \frac{1}{R^d} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log \dot{\mathfrak{M}}_{N,R}(B(\bar{P}_{\underline{m},R}, \nu)) \geq -\fint_{\Sigma_{\underline{m}}} \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx - r_{\mu_V,\underline{m}}, \quad (2.6.14)$$

moreover, for any $\delta > 0$ we have

$$\liminf_{R\to\infty} \frac{1}{R^d} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log(\dot{\mathfrak{M}}_{N,R}, \widehat{\mathfrak{M}}_{N,R}) \left(B(\bar{P}_{\underline{m},R}, \nu) \times B(\bar{P}_{\underline{m}}, \delta)\right) \geq -\fint_{\Sigma_{\underline{m}}} \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx - r_{\mu_V,\underline{m}} \quad (2.6.15)$$

where by $(\dot{\mathfrak{M}}_{N,R}, \widehat{\mathfrak{M}}_{N,R})$ we denote the joint law of $\dot{\mathfrak{M}}_N$ and $\widehat{\mathfrak{M}}_N$ (with the natural coupling).

2.6.3 Regularizing and screening microstates

In this subsection we take the approximating microstates introduced in Lemma 2.6.3 and apply to them the screening-then-regularization procedure described in Section 2.5.3.

We obtain the following (recall N_{int} is defined in (2.6.13)) :

Lemma 2.6.4. *Let $\bar{P} \in \mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$. Given $\delta_1, \eta, \varepsilon, M, R, \tau, \nu, N$ positive with (2.5.6) satisfied, there exists a set (depending on the parameters) A^{mod} of point configurations in Σ'_{int} which are of the form $\mathcal{C}^{\text{mod}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\text{mod}}$ where $\mathcal{C}_i^{\text{mod}}$ is a configuration in K_i and such that the following holds :*

1. *For any $\mathcal{C}^{\text{mod}}$ in A^{mod}, if η is small enough, ε small enough, R, M large enough satisfying (2.5.16), τ, ν small enough and N large enough then*

$$\frac{1}{|\Sigma'_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d}x, \theta_x \cdot \mathcal{C}^{\text{mod}})} dx \in B(\bar{P}_{\underline{m}}, \frac{3\delta_1}{4}). \quad (2.6.16)$$

2. $$\lim_{\eta\to 0} \lim_{M,R\to\infty} \lim_{\tau\to 0} \lim_{\nu\to 0} \lim_{N\to\infty} \sup_{\mathcal{C}^{\text{mod}} \in A^{\text{mod}}} \frac{1}{N} \left(\sum_{x_i \neq x_j \in \mathcal{C}^{\text{mod}}, |x_i - x_j| \leq \eta} g(x_i - x_j) \right) = 0. \quad (2.6.17)$$

3. *For any $\mathcal{C}^{\text{mod}} \in A^{\text{mod}}$ there exists an electric field E^{mod} satisfying*

$$\begin{cases} \operatorname{div}(|y|^\gamma E^{\text{mod}}) = c_{d,s} \left(\mathcal{C}^{\text{mod}} - \mu'_V \delta_{\mathbb{R}^d} \right) & \text{in } \Sigma'_{\text{int}} \\ E^{\text{mod}} \cdot \vec{\nu} = 0 & \text{on } \partial\Sigma'_{\text{int}} \end{cases} \quad (2.6.18)$$

and

$$\limsup_{\varepsilon\to 0, M,R\to\infty, \tau,\nu\to 0, N\to\infty} \left(\frac{1}{|\Sigma'_{\text{int}}|} \int_{\Sigma'_{\text{int}} \times \mathbb{R}^k} |y|^\gamma |E_\eta^{\text{mod}}|^2 - \int F_{R,\eta}^{M,1,\varepsilon} d\bar{P}_{\underline{m}} \right) \leq 0. \quad (2.6.19)$$

4. There is a good volume of such microstates

$$\liminf_{\varepsilon\to 0}\liminf_{R,M\to\infty}\liminf_{\tau,\nu\to 0}\liminf_{N\to\infty}\frac{1}{N_{\rm int}}\log\frac{\mathbf{Leb}^{N_{\rm int}}}{|\Sigma'_{\rm int}|^{N_{\rm int}}}\left(A^{\rm mod}\right)\geq -\int_{\Sigma_{\underline{m}}}\mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]\\ -|\Sigma_{\underline{m}}|r_{\mu_V,\underline{m}}. \quad (2.6.20)$$

Proof. Let $\bar{P}\in\mathcal{P}_{s,1}(\Sigma\times\mathcal{X})$ of finite energy and $\delta_1>0$ be given (as in the statement of Proposition 2.1.7). For any δ and ν, let us write the conditions for a point configuration $\mathcal{C}:=\sum_{i=1}^{m_{N,R}}\mathcal{C}_i$

$$\frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}}\delta_{(N^{-1/d}x,\theta_x\cdot\mathcal{C})}dx\in B(\bar{P}_{\underline{m}},\delta) \quad (2.6.21)$$

and

$$\frac{1}{m_{N,R}}\sum_{i=1}^{m_{N,R}}\delta_{(N^{-1/d}x_i,\theta_{x_i}\cdot\mathcal{C}_i)}\in B(\bar{P}_{\underline{m},R},\nu). \quad (2.6.22)$$

By Lemma 2.6.3 we know that given N,R,δ,ν there exists a set $A^{\rm abs}$ ("abs" as "abstract" because we generate them abstractly - and not by hand - using Sanov theorem as explained in the previous section) of configurations $\mathcal{C}^{\rm abs}=\sum_{i=1}^{m_{N,R}}\mathcal{C}_i^{\rm abs}$ (understood of a sum of measures) with $N_{\rm int}$ points, where $\mathcal{C}_i^{\rm abs}$ is a point configuration in the hypercube $\overline{C}_i$, such that

$$\liminf_{R\to\infty}\liminf_{\nu\to 0}\liminf_{N\to\infty}\frac{1}{m_{N,R}R^d}\log\frac{\mathbf{Leb}^{N_{\rm int}}}{(m_{N,R}R^d)^{N_{\rm int}}}\left(\{\mathcal{C}^{\rm abs}\in A^{\rm abs},\text{(2.6.21) and (2.6.22) hold}\}\right)\\ \geq -\fint_{\Sigma_{\underline{m}}}\mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]-r_{\mu_V,\underline{m}}. \quad (2.6.23)$$

To see how Lemma 2.6.3 yields (2.6.23) it suffices to note that the law of the $N_{\rm int}$-points point process $\mathbf{C}$ of Lemma 2.6.3 coincides with the law of the point process induced by the $N_{\rm int}$-th product of the normalized Lebesgue measure on $\cup_{i=1}^{m_{N,R}}\overline{C}_i$, and then (2.6.15) gives (2.6.23).

We let $A^{\rm mod}$ be the set of configurations obtained after applying the procedure described in Section 2.5. More precisely, for each $\mathcal{C}^{\rm abs}$ in $A^{\rm abs}$ we decompose $\mathcal{C}^{\rm abs}$ as $\sum_{i=1}^{m_{N,R}}\mathcal{C}_i^{\rm abs}$ where $\mathcal{C}_i^{\rm abs}$ is a point configuration in $\overline{C}_i$, and for any $i=1\dots m_{N,R}$ we let $\Phi_i^{\rm mod}(\mathcal{C}^{\rm abs})$ be the set of configurations obtained after screening-then-regularizing $\mathcal{C}_i^{\rm abs}$ by the map $\Phi^{\rm mod}$ (in the following, for good definition, we have to translate back $\overline{C}_i$ and the other quantities by a vector x_i)

$$\Phi_i^{\rm mod}(\mathcal{C}):=\Phi^{\rm mod}(\theta_{x_i}\cdot\mathcal{C}_i^{\rm abs},\mu'_V(x_i+\cdot),\theta_{x_i}\cdot K_i).$$

We then let $\overline{\Phi^{\rm mod}}(\mathcal{C}^{\rm abs})$ be the set of global configurations obtained as the product of the $\Phi_i^{\rm mod}(\mathcal{C}^{\rm abs})$

$$\overline{\Phi^{\rm mod}}(\mathcal{C}^{\rm abs}):=\prod_{i=1}^{m_{N,R}}\Phi_i^{\rm mod}(\mathcal{C}^{\rm abs})$$

and $A^{\rm mod}$ ("mod" as "modified") is finally defined as the image of $A^{\rm abs}$ by $\overline{\Phi^{\rm mod}}$.

Let us now check that $A^{\rm mod}$ satisfies the properties of items 1 to 4.

a. Dealing with the variation of μ_V.

Lemma 2.6.5. *We have for any $N,R,\underline{m}$ and $i\in 1\dots m_{N,R}$*

$$\|\mu'_V(x_i)-\mu'_V\|_{L^\infty(\overline{C}_i)}\leq C\left(\frac{R}{N^{1/d}}\right)^\kappa \quad (2.6.24)$$

for some $\kappa > 0$ with a constant C depending only on d, V and $\underline{m}$. In particular for any $M' > M$ and $e' > e$, for any $R, \varepsilon, \eta > 0$ we have for N large enough (depending on the other parameters but not on the configuration):

$$(\mathcal{C}_i, \mu'_V(x'_i)) \in \mathcal{S}^{M,e,\varepsilon}_{R,\eta,+} \Longrightarrow (\mathcal{C}_i, \mu'_V) \in \mathcal{S}^{M',e',\varepsilon}_{R,\eta,+}, \tag{2.6.25}$$

moreover if $(\mathcal{C}_i, \mu'_V(x'_i))$ and $(\mathcal{C}_i, \mu'_V)$ are in $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ we have

$$F^{M',e',\varepsilon}_{R,\eta}(\mathcal{C}_i, \mu'_V) \leq F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C}_i, \mu'_V(x'_i))(1 + o(1)) + o(1) \tag{2.6.26}$$

where the terms $o(1)$ tend to zero when $N \to \infty$ depending only on R and $\underline{m}$ not on the configuration $\mathcal{C}_i$ nor on $i = 1 \dots m_{N,R}$.

Proof. The bound (2.6.24) follows immediatly from the Hölder assumption (2.2.6) on the density of μ_V and the definition of μ'_V as the blown-up quantity associated to μ_V. The two controls (2.6.25) and (2.6.26) follow then from Lemma 2.5.8, and the fact that they are uniform (independent of the configurations) results from the proofs of Lemmas 2.5.8 and 2.5.9 (the energy of the "difference electric field" can be expressed in terms of $\|\mu'_V(x_i) - \mu'_V\|$ and using the Cauchy-Schwarz inequality together with (2.6.24) is enough to conclude). □

An important consequence is the following : if $\mathcal{C}$ is a finite configuration whose discrete average (over translations) is close to $\bar{P}$, then most of the configurations in the discrete average are screenable. Indeed by construction, configurations in A^{abs} verify (2.6.21) and (2.6.22). In particular, combining Item 1 of Lemma 2.5.10 and Lemma 2.6.5 we see that

$$\lim_{M,R\to\infty} \lim_{\nu\to 0} \lim_{N\to\infty} \inf_{\mathcal{C}^{\text{abs}} \in A^{\text{abs}}} \frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/d}x, \mathcal{C}^{\text{abs}}_i)}(\mathcal{S}^{M,1,\varepsilon}_{R,\eta,+}) = 1. \tag{2.6.27}$$

b. Distance to $\bar{P}_{\underline{m}}$.

To prove the first item of Lemma 2.6.4 we claim that the screening-then-regularizing procedure preserves the closeness of the continuous average to $\bar{P}_{\underline{m}}$ (however in general it does not preserve that of the discrete average). For that purpose we have to distinguish between hyperrectangles where the configuration is screenable (where the configuration is only modified in a thin layer or by moving points by a distance at most τ) and hyperrectangles where it is not (where the configuration is then completely modified).

Let $\mathcal{C}^{\text{mod}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}^{\text{mod}}_i$ be in A^{mod} (where $\mathcal{C}^{\text{mod}}_i$ is the point configuration in the hyperrectangle K_i), we may find $\mathcal{C}^{\text{abs}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}^{\text{abs}}_i$ in A^{abs} such that equation (2.6.21) holds and for any $i = 1 \dots m_{N,R}$

$$\mathcal{C}^{\text{mod}}_i \in \theta_{-x_i} \cdot \Phi^{\text{mod}}_i(\mathcal{C}^{\text{abs}}_i)$$

i.e. $\mathcal{C}^{\text{mod}}$ has been obtained from $\mathcal{C}^{\text{abs}}$ by screening-then-regularizing.

We want to show that the continuous average

$$\frac{1}{|\Sigma'_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d}x, \theta_x \cdot \mathcal{C}^{\text{mod}})} dx \in \mathcal{P}(\Sigma \times \mathcal{X})$$

satisfies (2.6.16).

We claim that we may evaluate the distance between the continuous averages of $\mathcal{C}^{\text{abs}}$ and $\mathcal{C}^{\text{mod}}$ in terms of the distance between the configurations in each hypercube K_i:

$$d_{\mathcal{P}(\Sigma\times\mathcal{X})}\left(\frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}}\delta_{\left(N^{-1/d}x,\theta_x\cdot\mathcal{C}^{\text{abs}}\right)},\frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}}\delta_{\left(N^{-1/d}x,\theta_x\cdot\mathcal{C}^{\text{mod}}\right)}\right)$$
$$\leq C\sum_{i=1}^{m_{N,R}} d_{\mathcal{X}(K_i)}\left(\mathcal{C}_i^{\text{abs}},\mathcal{C}_i^{\text{mod}}\right)+\frac{\delta_1}{3}+o_{R,N\to\infty}(1) \quad (2.6.28)$$

for a certain constant C depending only on δ_1. The proof of (2.6.28) is elementary and we sketch it below.

First, from the approximation property of Lipschitz functions on $\mathcal{X}$ by local functions (see Lemma 2.2.5, Item 3) and the definition (2.2.27) of the distance on $\mathcal{P}(\Sigma\times\mathcal{X})$ we see that there exists $k\geq 0$ large enough such that for any tagged point process $\bar{Q}$ in $\mathcal{P}(\Sigma\times\mathcal{X})$, if $\bar{Q}_k$ denotes the push-forward of $\bar{Q}$ by $(x,\mathcal{C})\mapsto(x,\mathcal{C}\cap C_k)$ (in other words, the point process induced on $\Sigma\times C_k$), we have

$$d_{\mathcal{P}(\Sigma\times\mathcal{X})}(\bar{Q},\bar{Q}_k)\leq\frac{\delta_1}{6}. \quad (2.6.29)$$

This means that when comparing two point processes we can localize the configurations to some hypercube of fixed size up to a small uniform error and in the following we let k be an integer such that (2.6.29) holds. Hence in order to evaluate the distance between the two continuous averages we may reduce ourselves to evaluate the distance between their projection on $\mathcal{P}(\Sigma\times\mathcal{X}(C_k))$ up to an error $\frac{\delta}{6}+\frac{\delta}{6}$ according to (2.6.29). It amounts to testing the averages against Lipschitz functions $F\in\text{Lip}_1(\Sigma\times\mathcal{X})$ such that $F(x,\mathcal{C})=F(x,\mathcal{C}\cap C_K)$ for any $(x,\mathcal{C})$. The continuous average (over translates in $\Sigma_{\underline{m}}$) of such a function F can be compared to its discrete average on the hypercubes K_i up to an error comparable to the fraction of the volume $\Sigma'_{\underline{m}}$ which is at distance less than k of $\Sigma'_{\underline{m}}\backslash\Sigma'_{\text{int}}$. By Lemma 2.6.2 we see that this fraction is $o(1)$ as $R,N\to\infty$.

Now for any $i=1\dots m_{N,R}$ we want to evaluate $d_{\mathcal{X}(K_i)}\left(\mathcal{C}_i^{\text{abs}},\mathcal{C}_i^{\text{mod}}\right)$. We denote by I_1 the set of indices $i=1\dots m_{N,R}$ such that $(\mathcal{C}_i^{\text{abs}},\mu'_V)$ is in $\mathcal{S}^{M,1,\varepsilon}_{R,\eta,+}$ and I_2 the set of indices $i=1\dots m_{N,R}$ such that $(\mathcal{C}_i^{\text{abs}},\mu'_V)$ is **not** in $\mathcal{S}^{M,1,\varepsilon}_{R,\eta,+}$. Let us recall that the distance $d_{\mathcal{X}(K_i)}$ has been defined in (2.2.28) by testing against Lipschitz functions which are bounded by 1 in sup-norm. Consequently if $i\in I_2$ we have

$$d_{\mathcal{X}(K_i)}\left(\mathcal{C}_i^{\text{abs}},\mathcal{C}_i^{\text{mod}}\right)\leq 2|K_i|\leq CR^d \quad (2.6.30)$$

which is the maximal distance between two configurations of $\mathcal{X}(K_i)$. On the other hand if $i\in I_1$ we have

$$d_{\mathcal{X}(K_i)}\left(\mathcal{C}_i^{\text{abs}},\mathcal{C}_i^{\text{mod}}\right)\leq 2(|K_i|-|\overline{C}_i|)+2(|\overline{C}_i|-|\text{Int}^i_\varepsilon|)+CR^d\tau,$$

where we denote $\text{Int}_{\varepsilon,i}:=\{x\in\overline{C}_i\mid\text{dist}(x,\partial\overline{C}_i)\geq 2\varepsilon R^d\}$ and C is a constant. This is because the configurations $\mathcal{C}_i^{\text{abs}}$ and $\mathcal{C}_i^{\text{mod}}$ may differ completely on $K_i\backslash\text{Int}^i_\varepsilon$, but in Int^i_ε the screening procedure has not modified $\mathcal{C}_i^{\text{abs}}$ (according to Item 1 of Proposition 2.5.2) and the only modification is due to the regularization procedure which moves the points by at most $C\tau$ (and there are at most CR^d points in Int^i_ε). Using (2.6.9) and (2.6.7) we get

$$d_{\mathcal{X}(K_i)}\left(\mathcal{C}_i^{\text{abs}},\mathcal{C}_i^{\text{mod}}\right)\leq CR^d(\varepsilon+\tau)+O(R^{d-2}). \quad (2.6.31)$$

Combining (2.6.28), (2.6.30) and (2.6.31) we have

$$d_{\mathcal{P}(\Sigma\times\mathcal{X})}\left(\frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}}\delta_{\left(N^{-1/d}x,\theta_x\cdot\mathcal{C}^{\mathrm{abs}}\right)},\frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}}\delta_{\left(N^{-1/d}x,\theta_x\cdot\mathcal{C}^{\mathrm{mod}}\right)}\right)$$
$$\leq \frac{\#I_1}{m_{N,R}}\Big(CR^d(\varepsilon+\tau)+O(R^{d-2})\Big)+\frac{\#I_2}{m_{N,R}}CR^d. \quad (2.6.32)$$

Using (2.6.27) we see that $\frac{\#I_2}{m_{N,R}}=o(1)$ and $\frac{\#I_1}{m_{N,R}}=1-o(1)$ when $M,R\to\infty,\nu\to 0,N\to\infty$ uniformly in A^{abs}. Combined with (2.6.32) it yields (2.6.16) when the parameters are sent to their limit as described in Item 1 of Lemma 2.6.4.

c. Truncation error.

We turn to the second item of Lemma 2.6.4 which bounds the truncation error of the configuration that we construct.

Let $\mathcal{C}^{\mathrm{mod}}=\sum_{i=1}^{m_{N,R}}\mathcal{C}_i^{\mathrm{mod}}$ be in A^{mod} and let $\mathcal{C}^{\mathrm{abs}}$ such that $\mathcal{C}^{\mathrm{mod}}$ has been obtained from $\mathcal{C}^{\mathrm{abs}}$ by screening-then-regularizing. By construction (see (2.5.8) in Item 2 of Proposition 2.5.2 in the case of a screenable configuration and Item 1 of Lemma 2.5.1 in the case of a non-screenable configuration) we have

$$\min_{i=1\dots m_{N,R}}\min_{x\in\mathcal{C}_i^{\mathrm{mod}}}\mathrm{dist}(x,\partial K_i)\geq\eta_0$$

for some $\eta_0>0$ depending only on $d,\underline{m},\overline{m}$. Therefore if η is small enough (depending only on $d,\underline{m},\overline{m}$) the only pair of points in $\mathcal{C}^{\mathrm{mod}}$ at distance less than η are included in some hyperrectangle K_i. We denote by I_1 the set of indices $i=1\dots m_{N,R}$ such that $(\mathcal{C}_i^{\mathrm{abs}},\mu'_V)$ is in $\mathcal{S}^{M,1,\varepsilon}_{R,\eta,+}$ and I_2 the set of indices $i=1\dots m_{N,R}$ such that $(\mathcal{C}_i^{\mathrm{abs}},\mu'_V)$ is not in $\mathcal{S}^{M,1,\varepsilon}_{R,\eta,+}$.

If $i\in I_2$ the configuration $\mathcal{C}_i^{\mathrm{mod}}$ is by construction (see Lemma 2.5.1) made of points which are well-separated by the same constant η_0 hence there is no pair of points at distance less than η in K_i for $i\in I_2$ and for η small enough (depending only on $d,\underline{m},\overline{m}$).

If $i\in I_1$ we know by construction that the only pair of points at distance less than η are in $\overline{C}_i$ (the points in $K_i\backslash\overline{C}_i$ are well-separated, see (2.5.9)). We may apply Item 2 of Lemma 2.5.11 to bound the truncation error in $\overline{C}_i$ in terms of the points of $\mathcal{C}_i^{\mathrm{abs}}$, for any $\eta\geq 8\tau$ it holds:

$$\sum_{x_i\neq x_j\in\mathcal{C}_i^{\mathrm{mod}},|x_i-x_j|\leq\eta}g(x_i-x_j)$$
$$\leq Cg(\tau)\left(\Big(\sum_{\vec{i}\in 6\tau\mathbb{Z}^d}\mathcal{N}(\vec{i},12\tau)^2(\mathcal{C}_i^{\mathrm{abs}})-1\Big)_+ +\sum_{x_i\neq x_j\in\mathcal{C}_i^{\mathrm{abs}},\tau\leq|x_i-x_j|\leq 2\eta}g(x_i-x_j)\right)$$

where C is a universal constant (depending only on d). Since $i\in I_1$ the condition (2.5.19) in the definition of the screenability implies that $\mathcal{C}_i^{\mathrm{abs}}$ has at most MR^d points hence we may also write the previous equation as

$$\sum_{x_i\neq x_j\in\mathcal{C}_i^{\mathrm{mod}},|x_i-x_j|\leq\eta}g(x_i-x_j)\leq Cg(\tau)\Big(\sum_{\vec{i}\in\tau\mathbb{Z}^d}(\mathcal{N}(\vec{i},12\tau)^2(\mathcal{C}_i^{\mathrm{abs}})-1)_+\wedge M^2R^{2d})$$
$$+\sum_{x_i\neq x_j\in\mathcal{C}_i^{\mathrm{abs}},\tau\leq|x_i-x_j|\leq 2\eta}g(x_i-x_j)\wedge MR^dg(\tau)\Big). \quad (2.6.33)$$

This re-writing is only technical, and meant to replace the functions depending on the number of points by bounded functions, which can now be tested against the convergence of point processes.

In view of Item 3 of Lemma 2.5.10, setting $\bar{Q} = \frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/d}x_i, \mathcal{C}_i^{\rm abs})}$ for any $\mathcal{C}^{\rm abs} \in A^{\rm abs}$ (which is by assumption included in $B(\bar{P}_{\underline{m}}, \nu)$) we have for any η, τ, with $0 < \tau < \eta^2/2 < 1$

$$\limsup_{\eta\to 0} \limsup_{\tau\to 0} \limsup_{\nu\to 0} \sup_{\mathcal{C}^{\rm abs}\in A^{\rm abs}} \Big[\frac{g(2\tau)}{\tau^d} \mathbf{E}_{\bar{Q}}[(\mathcal{N}(x,\tau)^2 - 1)_+ \wedge M^2 R^{2d}] + \frac{1}{R^d} \mathbf{E}_{\bar{Q}}\Big(\Big(\sum_{p\neq q \in \mathcal{C}\cap C_R, \tau \le |p-q| \le \eta^2/2} g(|p-q|) \Big) \wedge M R^d g(\tau) \Big) \Big] = 0.$$

In particular both of the expectations must go to zero. Writing $\bar{Q}$ explicitly, this implies that

$$\lim_{\eta\to 0} \lim_{M,R\to\infty} \lim_{\tau\to 0} \lim_{\nu\to 0} \lim_{N\to\infty} \sup_{\mathcal{C}^{\rm abs}\in A^{\rm abs}} \frac{1}{N} \sum_{i=1}^{m_{N,R}} C g(\tau) \Big(\sum_{\vec{i}\in\tau\mathbb{Z}^d} (\mathcal{N}(\vec{i}, 12\tau)^2(\mathcal{C}_i^{\rm abs}) - 1)_+ \wedge M^2 R^{2d}) + \sum_{x_i \neq x_j \in \mathcal{C}_i^{\rm abs}, \tau \le |x_i - x_j| \le 2\eta} g(x_i - x_j) \wedge M R^d g(\tau) \Big) = 0$$

which when combined with (2.6.33) proves (2.6.17) because the sum on $i = 1, \dots, m_{N,R}$ bounds of course the sum on $i \in I_1$.

d. Energy.

We want to control the energy of electric fields associated to the configurations in $A^{\rm mod}$.

First we associate to any $\mathcal{C}^{\rm mod} \in A^{\rm mod}$ a screened electric field $E^{\rm mod}$ satisfying (2.6.18). As explained in Section 2.5.3 we know by definition that for $\mathcal{C}^{\rm mod} \in A^{\rm mod}$, for any $i = 1 \dots m_{N,R}$ there exists an vector field $E_i^{\rm mod}$ such that

$$\begin{cases} \operatorname{div}(|y|^\gamma E_i^{\rm mod}) = c_{d,s}(\mathcal{C}_i^{\rm mod} - \mu_V' \delta_{\mathbb{R}^d}) & \text{in } K_i \times \mathbb{R}^k \\ E_i^{\rm mod} \cdot \vec{\nu} = 0 & \text{on } \partial K_i \times \mathbb{R}^k \end{cases}$$

Setting $E^{\rm mod} = \sum_i E_i^{\rm mod} \mathbf{1}_{K_i \times \mathbb{R}^k}$ provides the vector field mentioned in Item 3 of Lemma 2.6.4 which satisfies (2.6.18).

We now turn to bound its energy, refering again to Section 2.5.3. Let $\mathcal{C}^{\rm abs} \in A^{\rm abs}$ be such that $\mathcal{C}^{\rm mod}$ is obtained from $\mathcal{C}^{\rm abs}$ after screening/regularizing. We denote again by I_1 the set of indices $i = 1 \dots m_{N,R}$ such that $(\mathcal{C}_i^{\rm abs}, \mu_V')$ is in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$ and I_2 the set of indices $i = 1 \dots m_{N,R}$ such that $(\mathcal{C}_i^{\rm abs}, \mu_V')$ is not in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$.

For $i \in I_1$ the energy is bounded as in (2.5.11) (after screening) and (2.5.33) (after regularization). For $i \in I_2$ it is bounded as in (2.5.4).

At this point, we insert the information on μ_V' provided by (2.2.6) with (2.2.8). This ensures that

$$\Big\| \mu_V' - \fint_{K_i} \mu_V' \Big\|_{L^\infty(K_i)} \le C R^\kappa N^{-\kappa/d}$$

for a constant depending only on μ_V. Moreover, we have $0 \le |K_i| - |\overline{C}| = O(R^{d-2})$ as stated in (2.6.7). Inserting these estimates into the bounds of (2.5.4) and (2.5.11), and combining with (2.5.33) in the case $i \in I_1$ we find

$$\frac{1}{R^d} \int_{\Sigma_{\rm int}' \times \mathbb{R}^k} |y|^\gamma |E_\eta^{\rm mod}|^2 \le \Big(\sum_{i\in I_1} (F_{R,\eta}^{M,1,\varepsilon}(\mathcal{C}_i^{\rm abs}, \mu_V') + \varepsilon)(1 + C\varepsilon) + C g(\eta)((1+M)\varepsilon + o_R(1)) + C e \varepsilon + o_N(1) + \frac{C}{R^d} \sum_{i\in I_2} N_i g(\eta) \Big)(1 + o_\tau(1)) \quad (2.6.34)$$

where the term $o_N(1)$ tends to 0 as $N \to \infty$, keeping the other parameters fixed, $o_R(1)$ tends to 0 as $R \to \infty$ independently of the other parameters and $o_\tau(1)$ is as in (2.5.33).

Using Lemma 2.6.5 together with the upper semi-continuity of $F_{R,\eta}^{M,1,\varepsilon}\mathbf{1}_{\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}}$ we obtain

$$\limsup_{\varepsilon\to 0}\limsup_{M,R\to\infty}\limsup_{\tau,\nu\to 0}\limsup_{N\to\infty}\frac{1}{m_{N,R}}\sum_{i\in I_1}F_{R,\eta}^{M,1,\varepsilon}(\mathcal{C}_i^{\rm abs},\mu_V'(x_i)) \le \int \mathbf{1}_{\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}}F_{R,\eta}^{M,1,\varepsilon}d\bar{P}_{\underline{m}}. \quad (2.6.35)$$

Moreover by (2.6.37) again and $N_i = O(R^d)$, the term $\frac{C}{R^d}\sum_{i\in I_2}N_i g(\eta)$ is $o(m_{N,R})$ when η is fixed. Let us also recall that $m_{N,R} \approx \frac{N}{R^d}$ (cf. (2.6.5)).

Combining (2.6.34) and (2.6.35) we get (2.6.19). □

e. Control on the volume loss.

We now wish to bound the volume loss between the set $A^{\rm abs}$ of microstates generated "abstractly" and the set $A^{\rm mod}$ of configurations obtained after modification by the screening-and-regularizing procedure.

For each configuration $\mathcal{C}^{\rm abs}$ in $A^{\rm abs}$ we keep the distinction between $i \in I_1$ and $i \in I_2$ as above. From Lemma 2.5.13 we see that the difference of volume between $A^{\rm mod}$ and $A^{\rm abs}$ is bounded below as follows

$$\begin{aligned}\log \mathbf{Leb}^{\otimes N_{\rm int}}\left(A^{\rm mod}\right) - \log \mathbf{Leb}^{\otimes N_{\rm int}}\left(A^{\rm abs}\right)\\ \ge \fint_{\mathcal{C}^{\rm abs}\in A^{\rm abs}}\sum_{i\in I_1}\log\left((N_i-\mathcal{N}_i^{\rm int})!\left(\frac{c}{|{\rm Ext}_\varepsilon|}\right)^{N_i-\mathcal{N}_i^{\rm int}}\right) + (N_i-\mathcal{N}_i)\log|{\rm Ext}_\varepsilon|\\ - C\sum_{i\in I_1}\sum_{\vec{i}\in 6\tau\mathbb{Z}^d}\mathcal{N}(\vec{i},6\tau)\log\mathcal{N}(\vec{i},6\tau) + \sum_{i\in I_2}\log\left(N_i!C^{N_i}|\overline{C}_i|^{-\mathcal{N}_i}\right)d\mathcal{C}. \quad (2.6.36)\end{aligned}$$

We note that from (2.6.27), we have

$$\lim_{M,R\to\infty}\lim_{\nu\to 0}\lim_{N\to\infty}\frac{\#I_1}{m_{N,R}} = 1 \qquad \limsup_{M,R\to\infty}\lim_{\nu\to 0}\lim_{N\to\infty}\frac{\#I_2}{m_{N,R}} = 0. \quad (2.6.37)$$

In order to apply Lemma 2.5.13 however we need to check that the condition (2.5.14) holds for $i \in I_1$ i.e. that

$$N_i - \mathcal{N}_i^{\rm int} \le \frac{|{\rm Ext}_\varepsilon|}{2c}, \quad (2.6.38)$$

is satisfied for any $i \in I_1$, which can be achieved by taking c small enough. Indeed we have $N_i \le CR^d$ with a constant C depending only on $\underline{m}, \overline{m}$ as observed in (2.6.4). Hence up to changing c in (2.6.36) into

$$c_1 = \min(c_0\varepsilon, c) \quad (2.6.39)$$

where c_0 depends only on d, we see that we can always satisfy (2.6.38).

The integrand in (2.6.36) may be bounded below using Stirling's estimate as follows

$$\begin{aligned}\sum_{i\in I_1}\log\left((N_i-\mathcal{N}_i^{\rm int})!\left(\frac{c_1}{|{\rm Ext}_\varepsilon|}\right)^{N_i-\mathcal{N}_i^{\rm int}}\right) + (N_i-\mathcal{N}_i)\log|{\rm Ext}_\varepsilon|\\ \ge \sum_{i\in I_1}(N_i-\mathcal{N}_i^{\rm int})\log(N_i-\mathcal{N}_i^{\rm int}) - (N_i-\mathcal{N}_i^{\rm int}) - (N_i-\mathcal{N}_i^{\rm int})\log|{\rm Ext}_\varepsilon|\\ + (N_i-\mathcal{N}_i)\log|{\rm Ext}_\varepsilon| + (N_i-\mathcal{N}_i^{\rm int})\log c_1 - C\sum_{\vec{i}\in 6\tau\mathbb{Z}^d}\mathcal{N}(\vec{i},6\tau)\log\mathcal{N}(\vec{i},6\tau) \quad (2.6.40)\end{aligned}$$

(with c_1 as in (2.6.39)) on the one hand, and on the other hand

$$\sum_{i\in I_2} \log\Big(N_i! C^{N_i} |\overline{C}_i|^{-\mathcal{N}_i}\Big) \geq \sum_{i\in I_2} N_i \log N_i - \mathcal{N}_i \log|\overline{C}_i| - N_i(1-\log C). \tag{2.6.41}$$

We now turn to studying the terms in (2.6.40), (2.6.41) which relies on estimating the quantities $\mathcal{N}_i$ and $\mathcal{N}_i^{\rm int}$.

Let $\mathcal{D}^{\rm int}(x,\mathcal{C})$ be the discrepancy quantity $\mathcal{N}^{\rm int}(\mathcal{C}) - \mu_V(x)|{\rm Int}_\varepsilon|$. If $i \in I_1$ the quantity $\mathcal{D}^{\rm int}(x_i,\mathcal{C}_i) := \mathcal{N}_i^{\rm int} - \mu_V'(x_i)|{\rm Int}_\varepsilon|$ is bounded since the uniform bound (2.5.19) on the number of points holds for $i \in I_1$. We may then pass to the limit $\nu \to 0$ using (2.6.22)

$$\lim_{\nu\to 0} \frac{1}{m_{N,R}} \sum_{i\in I_1} \Big|\mathcal{N}_i^{\rm int} - \mu_V'(x_i)|{\rm Int}_\varepsilon|\Big| \leq \int \Big|\mathcal{D}^{\rm int}(x,\mathcal{C})\Big|\, d\bar{P}_{\underline{m},R}(x,\mathcal{C})$$
$$= \int_{\Sigma_{\underline{m}}} dx \int \Big|\mathcal{D}^{\rm int}(x,\mathcal{C})\Big|\, d\bar{P}_R^x(\mathcal{C}). \tag{2.6.42}$$

The discrepancy estimates of Lemma 2.3.10, more precisely (2.3.8), show that

$$\int_{x\in\Sigma_{\underline{m}}} dx \int \Big|\mathcal{D}^{\rm int}(x,\mathcal{C})\Big|\, d\bar{P}_R^x(\mathcal{C}) = O(R^{\frac{1}{2}(d+s)})$$

as $R \to \infty$ with a bound depending only on $\bar{P}$. Inserting the previous estimate in (2.6.42) we obtain that (since $s < d$)

$$\lim_{\nu\to 0} \frac{1}{m_{N,R}} \sum_{i\in I_1} \Big(\mathcal{N}_i^{\rm int} - \mu_V'(x_i)|{\rm Int}_\varepsilon|\Big) = O(R^{\frac{1}{2}(d+s)}) = o(R^d) \tag{2.6.43}$$

as $R \to \infty$ with a $o(R^d)$ depending only on $\bar{P}$. Arguing similarly we also get

$$\lim_{\nu\to 0} \left(\frac{1}{m_{N,R}} \sum_{i\in I_1} \mu_V'(x_i) R^d - \mathcal{N}_i\right) = O(R^{d-\frac{1}{2}(d-s)}), \tag{2.6.44}$$

The same also holds for I_2 and using the fact that $m_{N,R} \approx \frac{N}{R^d}$ we get

$$\lim_{\nu\to 0} \left(\sum_{i\in I_2} \mu_V'(x_i) R^d - \mathcal{N}_i\right) = O(NR^{-\frac{1}{2}(d-s)}). \tag{2.6.45}$$

Next, from (2.6.8) and the definition of Int, Ext, we may write

$$N_i = \mu_V'(x_i)|{\rm Int}_\varepsilon| + \mu_V'(x_i)|{\rm Ext}_\varepsilon| + o_{N\to\infty}(1)R^d + O(R^{d-2}),$$

hence, in view of (2.6.43) we have in the limit $\nu \to 0$ (depending on R)

$$\frac{1}{m_{N,R}} \sum_{i\in I_1} \frac{(N_i - \mathcal{N}_i^{\rm int})}{|{\rm Ext}_\varepsilon|} = \frac{1}{|{\rm Ext}_\varepsilon|}\Big(o(R^d) + o_{N\to\infty}(1)R^d\Big) + \frac{1}{m_{N,R}} \sum_{i\in I_1} \mu_V'(x_i).$$

Since $\int \mu_V(x) = 1$ by definition and since from (2.6.37) we have $\#I_1 \approx m_{N,R}$ it holds that

$$\frac{1}{m_{N,R}} \sum_{i\in I_1} \mu_V'(x_i) = 1 + o(1)$$

as $\underline{m} \to 0$, $M, R \to \infty$, $\nu \to 0$. Moreover when ε is fixed and $R, N \to \infty$ we have by (2.6.9)

$$\frac{1}{|\mathrm{Ext}_\varepsilon|}\Big(o(R^d) + o_{N\to\infty}(1)R^d)\Big) = o(1).$$

Finally we get that

$$\lim_{\underline{m}\to 0}\lim_{\varepsilon\to 0}\lim_{M,R\to\infty}\lim_{\nu\to 0}\lim_{N\to\infty}\sup_{\mathcal{C}\in A^{\mathrm{abs}}}\frac{1}{\#I_1}\sum_{i\in I_1}\frac{(N_i-\mathcal{N}_i^{\mathrm{int}})}{|\mathrm{Ext}_\varepsilon|} = 1 + o(1). \tag{2.6.46}$$

We may now write using Jensen's inequality

$$\sum_{i\in I_1}(N_i-\mathcal{N}_i^{\mathrm{int}})\log(N_i-\mathcal{N}_i^{\mathrm{int}}) - (N_i-\mathcal{N}_i^{\mathrm{int}})\log|\mathrm{Ext}_\varepsilon|$$
$$\geq \#I_1|\mathrm{Ext}_\varepsilon|\left(\frac{1}{\#I_1}\sum_{i\in I_1}\frac{(N_i-\mathcal{N}_i^{\mathrm{int}})}{|\mathrm{Ext}_\varepsilon|}\right)\log\left(\frac{1}{\#I_1}\sum_{i\in I_1}\frac{(N_i-\mathcal{N}_i^{\mathrm{int}})}{|\mathrm{Ext}_\varepsilon|}\right).$$

and using (2.6.46) together with (2.6.5) and (2.6.9) we get that

$$\sum_{i\in I_1}(N_i-\mathcal{N}_i^{\mathrm{int}})\log(N_i-\mathcal{N}_i^{\mathrm{int}}) - (N_i-\mathcal{N}_i^{\mathrm{int}})\log|\mathrm{Ext}_\varepsilon| \geq -C\varepsilon N$$

for some constant $C > 0$ depending only on the dimension and $\bar{P}$. This settles the first terms of the right-hand side in (2.6.40). For the next one, in view of (2.6.44) we have

$$\lim_{\nu\to 0}\log|\mathrm{Ext}_{\varepsilon,R}|\left(\frac{1}{\#I_1}\sum_{i\in I_1}\mu_V'(x_i)R^d - \mathcal{N}_i\right) = o(R^d)$$

which together with (2.6.8) yields that $\sum_{i\in I_1}(N_i-\mathcal{N}_i)\log|\mathrm{Ext}_\varepsilon|$ is a $o(N)$ when $M, R \to \infty$ and $N \to \infty$.

Similarly we may write that

$$\frac{1}{\#I_1}\sum_{i\in I_1}(N_i-\mathcal{N}_i^{\mathrm{int}}) = \frac{1}{\#I_1}\sum_{i\in I_1}\Big((N_i-\mathcal{N}_i)+\mathcal{N}_i^{\mathrm{ext}}\Big) = o(R^d) + \frac{1}{\#I_1}\sum_{i\in I_1}\mathcal{N}_i^{\mathrm{ext}}$$

and arguing as above we see that

$$\frac{1}{\#I_1}\sum_{i\in I_1}\mathcal{N}_i^{\mathrm{ext}} \leq \overline{m}|\mathrm{Ext}_\varepsilon| + o(R^d),$$

which together with the choice (2.6.39) of c_1 yields that

$$\sum_{i\in I_1}(N_i-\mathcal{N}_i^{\mathrm{int}})(\log c_1 - 1) \geq -CN\varepsilon|\log\varepsilon|$$

with a constant C depending only on $\bar{P}, d, \overline{m}$. Concerning the sum on $i \in I_1$ we are left to control the terms $\sum_{\vec{i}\in 6\tau\mathbb{Z}^d}\mathcal{N}(\vec{i},6\tau)\log\mathcal{N}(\vec{i},6\tau)$, which are treated as in Section c.: they are uniformly bounded because of (2.5.19) which allows us to take the limit $\nu \to 0$ and Item 3 of Lemma 2.5.25 together with the trivial bound $n\log n \leq (n^2-1)_+$ ensures that

$$\frac{1}{m_{N,R}}\sum_{i\in I_1}\sum_{\vec{i}\in 6\tau\mathbb{Z}^d}\mathcal{N}(\vec{i},6\tau)\log\mathcal{N}(\vec{i},6\tau) = o(1)$$

as $M, R \to \infty, \tau \to 0, \nu \to 0, N \to \infty$.

We now treat the terms for $i \in I_2$. In order to control (2.6.41) we follow the same line as above. First we decompose the difference as

$$N_i \log N_i - \mathcal{N}_i \log |\bar{C}_i| = (N_i - \mathcal{N}_i) \log R^d + N_i \log \frac{N_i}{R^d}.$$

The sum of $(N_i - \mathcal{N}_i) \log R^d$ is bounded as above thanks to (2.6.45) and (2.6.8) which yield

$$\sum_{i \in I_2} (N_i - \mathcal{N}_i) \log R^d = o(N) \text{ in the limit } R \to \infty, \tau \to 0, N \to \infty.$$

On the other hand, the second term is bounded using (2.6.4)

$$\sum_{i \in I_2} N_i \log \frac{N_i}{R^d} \geq -C \# I_2 R^d$$

and with (2.6.37) we finally get that

$$\sum_{i \in I_2} N_i \log N_i - \mathcal{N}_i \log |\bar{C}_i| = o(N) \text{ in the limit } M, R \to \infty, \tau \to 0, N \to \infty.$$

The last term $\sum_{i \in I_2} N_i (1 - \log C)$ is easily bounded because $N_i \leq C R^d$ and $\# I_2 = o(m_{N,R})$ hence

$$\sum_{i \in I_2} N_i (1 - \log C) = o(N).$$

Inserting all these controls into (2.6.40), (2.6.41), using (2.6.5) and the fact that $|\Sigma_{\underline{m}}| \to |\Sigma|$ as $\underline{m} \to 0$, we get from (2.6.36) that

$$\liminf_{\underline{m} \to 0} \liminf_{\varepsilon \to 0} \liminf_{M,R \to \infty} \liminf_{\tau \to 0} \liminf_{\nu \to 0} \liminf_{N \to \infty} \frac{1}{N} \left(\log \mathbf{Leb}^{N_{\rm int}} \left(A^{\rm mod} \right) \right) - \log \mathbf{Leb}^{N_{\rm int}} \left(A^{\rm abs} \right) \Big) \geq 0. \quad (2.6.47)$$

Combining (2.6.47) with (2.6.23) and (2.6.5) we conclude that (2.6.20) holds.

2.6.4 Completing the construction and conclusion

Once the construction has been made in $\Sigma'_{\rm int}$, there remains to complete it in the thin layer $\Sigma' \backslash \Sigma'_{\rm int}$ by placing "frozen" points there. That precise construction was already made in [PS15, Proposition 7.3, Step 3], where the following is shown :

Lemma 2.6.6. *There exists a family $A^{\rm ext}$ depending on N, R, η, of point configurations with $N - N_{\rm int}$ points in $\Sigma' \backslash \Sigma'_{\rm int}$ and which satisfy the following.*

1. *For any $\mathcal{C}^{\rm ext}$ in $A^{\rm ext}$, the distance between two points of $\mathcal{C}^{\rm ext}$ or between a point of $\mathcal{C}^{\rm ext}$ and the $\partial(\Sigma' \backslash \Sigma'_{\rm int})$ is bounded below by $\eta_0 > 0$ depending only on d and $\overline{m}$.*
2. *For any $\mathcal{C}^{\rm ext}$ in $A^{\rm ext}$, there exists a vector field $E^{\rm ext} \in L^p_{\rm loc}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ such that*

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma E^{\rm ext}) = c_{d,s} \left(\mathcal{C}^{\rm ext} - \mu'_V \right) & \text{in } \Sigma' \backslash \Sigma'_{\rm int} \\ E^{\rm ext} \cdot \vec{\nu} = 0 & \text{on } \partial(\Sigma' \backslash \Sigma'_{\rm int}) \end{cases}.$$

3. *The vector field $E^{\rm ext}$ described above satisfies*

$$\int_{(\Sigma' \backslash \Sigma'_{\rm int}) \times \mathbb{R}^k} |y|^\gamma |E^{\rm ext}_\eta|^2 \leq C(|\Sigma'| - |\Sigma'_{\rm int}|). \quad (2.6.48)$$

4. The volume of $A^{\rm ext}$ is bounded below the following way:

$$\mathbf{Leb}^{\otimes(N-N_{\rm int})}\left(A^{\rm ext}\right) \geq C^{(N-N_{\rm int})}(N-N_{\rm int})! \tag{2.6.49}$$

for some C depending only on d, s and $\overline{m}$.

We may now finish the proof of Proposition 2.1.7. Let $\bar{P}, \delta_1, \delta_2$ be given as in the statement of the proposition, and let N, η, R, ε be given. Let us consider any configuration $\mathcal{C}^{\rm mod}$ in $\Sigma'_{\rm int}$ and electric field $E^{\rm mod}$ provided by Lemma 2.6.4 for these parameters, and any configuration $\mathcal{C}^{\rm ext}$ and associated electric field $E^{\rm ext}$ provided by Lemma 2.6.6. We may then consider the total configuration $\mathcal{C}^{\rm tot} := \mathcal{C}^{\rm mod} + \mathcal{C}^{\rm ext}$ (the sum is again in the sense of measures) and the total vector field $E^{\rm tot} := E^{\rm mod}\mathbf{1}_{\Sigma'_{\rm int}} + E^{\rm ext}\mathbf{1}_{\Sigma'\setminus\Sigma'_{\rm int}}$.

A. Distance to $\bar{P}$.

In view of (2.6.16) together with the fact that $|\Sigma'| - |\Sigma'_{\underline{m}}| = o(N)$ as $\underline{m} \to 0$ it is easy to conclude that

$$\boxed{\frac{1}{|\Sigma|N}\int_{\Sigma'} \delta_{(N^{-1/d}x, \theta_x\cdot\mathcal{C}^{\rm tot})}dx \in B(\bar{P}, \delta_1),} \tag{2.6.50}$$

for $\underline{m}$ small enough, ε small enough, M, R large enough, ν small enough and N large enough. To see it we may choose k large enough so that (2.6.29) holds with $\frac{\delta_1}{10}$ and argue as in the proof of item 1 : the configurations $(\theta_x\cdot\mathcal{C}^{\rm tot})\cap C_k$ and $(\theta_x\cdot\mathcal{C}^{\rm mod})\cap C_k$ coincide on

$$\left\{x \in \Sigma_{\rm int}, d\left(x, \partial\Sigma_{\rm int}\right) \geq Ck^{1/d}\right\}$$

which represents a fraction $1-o(1)$ (when $R, N \to \infty$) of $\Sigma_{\rm int}$, hence of $\Sigma'_{\underline{m}}$ so that

$$d\left(\frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d}x, \theta_x\cdot\mathcal{C}^{\rm tot})}, \bar{P}\right) \leq \frac{\delta_1}{10} + \frac{3\delta_1}{4} + o(1).$$

To conclude it suffices to observe (see Remark 2.7.1 for a precise statement) that since the difference of volume $|\Sigma'| - |\Sigma'_{\underline{m}}| = o(N)$ as $\underline{m} \to 0$, the continuous averages of a given configuration over both domains lie at distance $o(1)$ uniformly as $\underline{m} \to 0$.

B. Energy.

We have

$$\begin{cases} -{\rm div}\left(|y|^\gamma E^{\rm tot}\right) = c_{d,s}\left(\mathcal{C}^{\rm tot} - \mu'_V\delta_{\mathbb{R}^d}\right) & \text{in } \mathbb{R}^{d+k} \\ E^{\rm tot} = 0 & \text{in } \mathbb{R}^{d+k}\backslash(\Sigma'\times\mathbb{R}^k). \end{cases}$$

Moreover from (2.6.19) and (2.6.48), since $|\Sigma'| - |\Sigma'_{\rm int}| = o(N)$ as $\underline{m} \to 0$, we see that the energy of $E^{\rm tot}$ satisfies for any η small enough

$$\limsup_{\underline{m}\to 0}\limsup_{\varepsilon\to 0}\limsup_{M,R\to\infty}\limsup_{\tau,\nu\to 0}\limsup_{N\to\infty}\left(\frac{1}{N|\Sigma|}\int_{\mathbb{R}^{d+k}}|y|^\gamma|E^{\rm tot}|^2 - \int F^{M,1,\varepsilon}_{R,\eta}d\bar{P}\right) \leq 0. \tag{2.6.51}$$

Moreover since the points added in $\Sigma\backslash\Sigma'_{\rm int}$ are well-separated from item 1 of Lemma 2.6.6, we keep (2.6.17).

Combining Lemma 2.3.13 and (2.6.51) we see that for every point configuration obtained this way, the associated electric field as in (2.2.21) satisfies

$$\limsup_{\eta\to 0,\underline{m}\to 0,\varepsilon\to 0,M,R\to\infty,\tau,\nu\to 0 N\to\infty}\left[\left(\frac{1}{N}\int_{\mathbb{R}^{d+k}}|y|^\gamma|\nabla H'_{N,\eta}|^2 - c_{d,s}g(\eta)\right)\right.$$
$$\left. - \int\left(F^{M,1,\varepsilon}_{R,\eta} - c_{d,s}g(\eta)\mu_V(x)\right)d\bar{P}\right] \leq 0.$$

Using the definition (2.2.22), (2.3.2) from Lemma 2.3.3, the fact that (2.6.17) holds, and (2.5.24) from Lemma 2.5.10 we get that given $\delta_1, \delta_2 > 0$ we may obtain (2.6.50) and

$$\boxed{w_N(\mathcal{C}^{\rm tot}) \leq \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \delta_2} \tag{2.6.52}$$

by choosing η small enough, $\underline{m}$ small enough, ε small enough, M, R large enough, ν, τ small enough and N large enough.

C. Volume.

We are left to bound below the volume of configurations $A^{\rm tot} := \{\mathcal{C}^{\rm tot}\}$ that we have constructed, and connect this volume with a large enough probability for $\bar{\Omega}_{N,\beta}$ as in (2.1.25). We may bound the volume of $A^{\rm tot}$ as follows

$$\frac{\mathbf{Leb}^{\otimes N}\left(A^{\rm tot}\right)}{|\Sigma'|^N} \geq \binom{N}{N^{\rm int}} \frac{\mathbf{Leb}^{\otimes N^{\rm int}}\left(A^{\rm mod}\right)}{|\Sigma'|^{N^{\rm int}}} \frac{\mathbf{Leb}^{\otimes (N-N^{\rm int})}\left(A^{\rm ext}\right)}{|\Sigma'|^{N-N^{\rm int}}}$$

which yields by taking the log and using (2.6.20) and (2.6.49)

$$\begin{aligned}\frac{1}{N}\log\frac{\mathbf{Leb}^{\otimes N}\left(A^{\rm tot}\right)}{|\Sigma'|^N} &\geq \frac{1}{N}\log\binom{N}{N^{\rm int}} + \frac{N^{\rm int}}{N}\log\frac{|\Sigma'_{\rm int}|}{|\Sigma'|} - \int_{\Sigma_{\underline{m}}} \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx \\ &- (\log|\Sigma| - |\Sigma| + 1) - \frac{1}{N}(N-N^{\rm int})\log|\Sigma'| + \frac{1}{N}(N-N^{\rm int})\log C + \frac{1}{N}\log(N-N^{\rm int})! - r\end{aligned}$$

with an error r going to zero as $\eta \to 0, \underline{m} \to 0, \varepsilon \to 0, M, R \to \infty, \tau, \nu \to 0, N \to \infty$. The terms of combinatorial nature may be re-arranged and bounded below by Stirling's formula

$$\frac{1}{N}\log\binom{N}{N^{\rm int}} + \frac{1}{N}\log(N-N^{\rm int})! \geq \log N + \frac{1}{N}(N^{\rm int}-N) - \frac{N^{\rm int}}{N}\log N^{\rm int}.$$

Moreover we have $(N-N^{\rm int})\log|\Sigma'| = (N-N^{\rm int})\log N + (N-N^{\rm int})\log|\Sigma|$. Let us also observe that $\frac{1}{N}(N^{\rm int}-N) = o(1)$ and $\frac{|\Sigma'_{\rm int}|}{|\Sigma'|} \to 1$ when $\underline{m} \to 0, R \to \infty, N \to \infty$. We thus have

$$\begin{aligned}\frac{1}{N}\log\frac{\mathbf{Leb}^{\otimes N}\left(A^{\rm tot}\right)}{|\Sigma'|^N} \geq &- \int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx - (\log|\Sigma| - |\Sigma| + 1) \\ &+ \log N - \frac{N^{\rm int}}{N}\log N^{\rm int} - (N-N^{\rm int})\log N + o(1).\end{aligned}$$

Since $N^{\rm int} \leq N$ we finally get

$$\frac{1}{N}\log\frac{\mathbf{Leb}^{\otimes N}\left(A^{\rm tot}\right)}{|\Sigma'|^N} \geq -\int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx - (\log|\Sigma| - |\Sigma| + 1) + o(1).$$

Conclusion. To complete the proof of Proposition 2.1.7 we need to link the preceding construction with a large enough volume of "good" events for $\bar{\Omega}_{N,\beta}$. This is done by conditioning $\mathbb{Q}_{N,\beta}$ into having all N points in Σ, the resulting conditional expectation (after scaling) is equal in law to $\frac{\mathbf{Leb}^{\otimes N}}{|\Sigma'|^N}$. The probability of this event is bounded below by

$$\liminf_{N\to\infty} \frac{1}{N}\log\mathbb{Q}_{N,\beta}\left(N \text{ points in } \Sigma\right) \geq \log\frac{|\Sigma|}{|\omega|} \tag{2.6.53}$$

because $\mathbb{Q}_{N,\beta}$ is essentially the (N times product of the) normalized Lebesgue measure on ω. More precisely $\mathbb{Q}_{N,\beta}$ is the N times product of the measure $\frac{e^{-N\beta\zeta(x)}dx}{\int e^{-N\beta\zeta(x)}dx}$, but we know that ζ vanishes on ω and is positive outside ω and moreover from (2.4.1) we know that $\int e^{-N\beta\zeta(x)}dx \to_{N\to\infty} |\omega|$ (see equation (2.7.15) and the proof of the lower bound after it for more details).

In view of (2.6.50) and (2.6.52) and using (2.6.53) we have (with the notations of Proposition 2.1.7):

$$\begin{aligned}\liminf_{N\to\infty}\frac{1}{N}\log\bar{\mathfrak{Q}}_{N,\beta}\left(B(\bar{P},\delta_1)\cap T_{N,\delta_2}(\bar{P})\right) &\geq -\int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx-(\log|\Sigma|-|\Sigma|+1)\\ &\quad+\liminf_{N\to\infty}\frac{1}{N}\log\mathbb{Q}_{N,\beta}\left(N\text{ points in }\Sigma\right)\\ &\geq -\int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx-(\log|\omega|-|\Sigma|+1),\end{aligned}$$

which, in view of (2.1.23), concludes the proof of Proposition 2.1.7.

2.7 Proof of the LDPs for the reference measure

In this section, we prove Proposition 2.1.6 and Lemma 2.6.3. Proposition 2.1.6 is a "process-level" (or type 3) LDP, whereas Lemma 2.6.3 is closer to a (type 2) Sanov-like large deviation result.

In order to prove Proposition 2.1.6 we rely on a similar result, Proposition 2.7.5 below proved in [GZ93] with the Poisson process $\mathbf{\Pi}^1$ as reference measure instead of $\bar{\mathfrak{Q}}_{N,\beta}$. What we have to do is then to show that the result remains true when perturbing away from the Poisson case, which will take several steps. The proof of Lemma 2.6.3, on the other hand, relies on the classical Sanov's theorem whose adaptation to our setting is very similar to the previous one. We believe that some (if not all) of these variations around classical results belong to folklore knowledge but we provide a proof for the sake of completeness.

2.7.1 Two comparison lemmas

We start by introducing some notions that will allow to replace point processes by equivalent ones.

Definition 2.7.1. *Let (X,d_X) be a metric space and let $\{R_N\}_N$ and $\{R'_N\}_N$ be two coupled sequences of random variables with value in X, defined on some probability spaces $\{(\Omega_N,\mathcal{B}_N,\pi_N)\}_N$. For any $\delta>0$ we say that $\{R_N\}_N$ and $\{R'_N\}_N$ are eventually almost surely (e.a.s.) δ-close when for N large enough we have*

$$\pi_N\left(d_X(R_N,R'_N)\geq\delta\right)=0.$$

It two sequences are e.a.s. δ-close for any $\delta>0$ we say that they are eventually almost surely equivalent (e.a.s.e.).

Let us emphasize that being eventually almost surely equivalent is strictly stronger than the usual convergence in probability. It is also easily seen to be stronger than the classical notion of "exponential equivalence" (see [DZ10, Section 4.2.2]) and thanks to that, large deviation principles may be transfered from one sequence to the other.

Lemma 2.7.2. *If the sequences $\{R_N\}_N$ and $\{R'_N\}_N$ are eventually almost surely equivalent and an LDP with good rate function holds for $\{R_N\}_N$, then the same LDP holds for $\{R'_N\}_N$.*

Proof. This is a straightforward consequence of [DZ10, Theorem 4.2.13]. □

A first example is given by the averages of a configuration over (translations in) two close sequences of sets.

Remark 2.7.3. *Let $\{V_N\}_N, \{W_N\}_N$ be two sequences of Borel sets in $\mathbb{R}^d$ of bounded Lebesgue measure and let f be a bounded measurable function on $\mathcal{X}$. Then for any configuration $\mathcal{C} \in \mathcal{X}$ we have*

$$\frac{1}{|W_N|}\left|\int_{W_N} f(\theta_x \cdot \mathcal{C})dx - \int_{V_N} f(\theta_x \cdot \mathcal{C})dx\right| \leq \frac{|W_N \triangle V_N|}{|W_N|}\|f\|_\infty, \tag{2.7.1}$$

where $\triangle$ denotes the symmetric difference between sets. In particular if P is a point process and $\lim_{N\to\infty} \frac{|W_N \triangle V_N|}{|W_N|} = 0$, the random variables obtained as the push-forward of P by the maps

$$\mathcal{C} \mapsto \frac{1}{|W_N|}\int_{W_N} \delta_{\theta_x \cdot \mathcal{C}} dx \text{ and } \mathcal{C} \mapsto \frac{1}{|V_N|}\int_{V_N} \delta_{\theta_x \cdot \mathcal{C}} dx$$

are eventually almost surely equivalent.

The point (2.7.1) is straightforward. To get e.a.s. δ-closeness it suffices to recall that the distance between point processes is defined in (2.2.27) by testing against functions in $\mathrm{Lip}_1(\mathcal{X})$ (which are in particular bounded in sup-norm). In a similar spirit we have

Lemma 2.7.4. *Let P be a point process in $\mathbb{R}^d$ and $\{\Lambda_N\}_N$ be a sequence of Borel sets of $\mathbb{R}^d$ of finite Lebesgue measure, such that*

$$\forall k \in \mathbb{N}, \quad |\{x \in \Lambda_N, d(x, \partial\Lambda_N) \geq k\}| = o(|\Lambda_N|).$$

In particular the assumption holds when $\Lambda_N = N^{1/d}\Lambda$ where Λ is a compact set with Lipschitz boundary.

Let us denote by R_N, resp. R'_N the push-forward of P by the map $\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x \cdot \mathcal{C}} dx$, resp. $\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x \cdot (\mathcal{C} \cap \Lambda_N)} dx$. Then the sequences $\{R_N\}_N$ and $\{R'_N\}_N$ are e.a.s.e.

Proof. Let us observe that the operation of taking the intersection with Λ_N affects only a small portion of the translates, indeed we have for any $k \geq 1$

$$(\theta_x \cdot \mathcal{C}) \cap C_k = (\theta_x \cdot (\mathcal{C} \cap \Lambda_N))) \cap C_k$$

for all x such that $d(x, \partial\Lambda_N) \geq k^{1/d}$. Thus, combining the uniform approximation of functions in $\mathrm{Lip}_1(\mathcal{X})$ by bounded local functions as in Lemma 2.2.5 and the definition (2.2.27) of $d_{\mathcal{P}(\mathcal{X})}$ as testing against functions in $\mathrm{Lip}_1(\mathcal{X})$ we get that for any $\delta > 0$ there exists $k \geq 1$ such that

$$d_{\mathcal{P}(\mathcal{X})}(R_N, R'_N) \leq \delta + \frac{|x \in \Lambda, d(x, \partial\Lambda_N) \geq k|}{|\Lambda_N|}, \ P\text{-almost surely.}$$

By assumption the second term in the right-hand side is $o(1)$ when $N \to \infty$ hence R_N, R'_N are e.a.s. 2δ-close and this holds for any $\delta > 0$. □

2.7.2 Continuous average, proof of Proposition 2.1.6

We now turn to the proof of the large deviation result for $\bar{\mathfrak{Q}}_{N,\beta}$ stated in Proposition 2.1.6. We start by recalling the following fundamental large deviation principle for empirical fields.

Proposition 2.7.5 (Georgii-Zessin)**.** *Let $\{\Lambda_N\}_N$ be a fixed sequence of cubes increasing to $\mathbb{R}^d$ and let R_N be the push-forward of $\mathbf{\Pi}^1$ by the map*

$$\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x \cdot \mathcal{C}} dx.$$

Then $\{R_N\}_N$ satisfies a large deviation principle at speed $|\Lambda_N|$ with rate function $\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$.

This is a consequence of [GZ93, Theorem 3.1] together with [GZ93, Remark 2.4] to get rid of the periodization used in their definition of R_N (see also [FO88]). One could also adapt the method of [RAS09, Chapter 6] from the discrete case (point processes on $\mathbb{Z}^d$) to the case of point processes on $\mathbb{R}^d$, where the Gärtner-Ellis theorem (see [DZ10, Section 4.5.3]) is used by establishing the existence of a pressure and studying its Legendre-Fenchel transform. We now need to extend the result to our setting.

a. Extension to Lipschitz boundaries

In this first step we extend the LDP of Proposition 2.7.5 to more general shapes of $\{\Lambda_N\}_N$.

Lemma 2.7.6. *Let Λ be a compact set of $\mathbb{R}^d$ with a non-empty interior and a Lipschitz boundary, and let $\Lambda_N := N^{1/d}\Lambda$. Let R_N be the push-forward of $\mathbf{\Pi}^1$ by the map*

$$\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x \cdot \mathcal{C}} dx.$$

Then $\{R_N\}_N$ satisfies a large deviation principle at speed $N|\Lambda|$ with rate function $\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$.

Proof. In the following every hypercube is such that its edges are parallel to the axes of $\mathbb{R}^d$. Let N be given. Let us consider the hypercubes centered at the points of $\Lambda \cap \frac{1}{n}\mathbb{Z}^d$ and of sidelength $\frac{1}{n}$, and remove those that are centered at points in

$$A_n := \left\{ x \in \Lambda \cap \frac{1}{n}\mathbb{Z}^d, d(x, \partial\Lambda) \leq 2c_d n^{-1/d} \right\},$$

where c_d is the distance between the center of the unit hypercube in dimension d and any vertex of this hypercube. Since the boundary of Λ is Lipschitz, we have $\lim_{n\to\infty} |A_n| = 0$, so that the total volume lost when removing the boundary hypercubes is less than $2^{-N}|\Lambda|$ for n large enough. In other words, we have found a family of $m = m(N)$ hypercubes $\{\Lambda^{(i,N)}\}_{i=1}^{m(N)}$ included in Λ and such that $|\Lambda| - |\cup_{i=1}^m \Lambda^{(i,N)}| \leq 2^{-N}|\Lambda|$.

For any N we may then define $\tilde{\Lambda}^{(N)}$ as the hypercube of center 0 and such that

$$|\tilde{\Lambda}^{(N)}| = \sum_{i=1}^m |\Lambda^{(i,N)}| = m|\Lambda^{(1,N)}| \geq |\Lambda| - 2^{-N}|\Lambda|. \tag{2.7.2}$$

There exists a measurable bijection $\Phi_N : \cup_{i=1}^m \Lambda^{(i,N)} \to \tilde{\Lambda}^{(N)}$ which is a translation on each hypercube $\Lambda^{(i,N)}$ $(i = 1, \ldots, m)$.

Next, we let R_N be as before the push-forward of $\mathbf{\Pi}^1$ by the map $\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x\cdot\mathcal{C}} dx$ and R'_N be the push-forward of $\mathbf{\Pi}^1$ by the map

$$\mathcal{C} \mapsto \frac{1}{Nm|\Lambda^{(1,N)}|}\int_{\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}} \delta_{\theta_x\cdot\mathcal{C}} dx.$$

Finally, from any configuration of points $\mathcal{C}$ on $\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}$ we get by applying $x \mapsto N^{1/d}\Phi_N(N^{-1/d}(x))$ a configuration in $N^{1/d}\tilde{\Lambda}^{(N)}$, which by abusing notation we denote again by $\Phi_N(\mathcal{C})$. We denote by R''_N the push-forward of $\mathbf{\Pi}^1_{|\Lambda_N}$ by:

$$\mathcal{C} \mapsto \frac{1}{N|\tilde{\Lambda}^{(N)}|}\int_{\Lambda_N} \delta_{\theta_{\Phi_N(x)}\cdot\Phi_N(\mathcal{C})} dx.$$

We impose that the random variables R_N, R'_N, R''_N are coupled together the natural way.

It is easily seen that the push-forward of $\mathbf{\Pi}^1_{|\Lambda_N}$ – or more precisely of the process induced on the subset $\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}$ – by the map $\mathcal{C} \mapsto \Phi_N(\mathcal{C})$ is equal in law to $\mathbf{\Pi}^1_{N^{1/d}\tilde{\Lambda}^{(N)}}$. The sequence of hypercubes $\{N^{1/d}\tilde{\Lambda}^{(N)}\}_N$ satisfies the hypothesis of Proposition 2.7.5 hence a Large Deviation Principle holds at speed $|\Lambda|N$ for the sequence $\{R''_N\}_N$ (the fact that we consider the push-forward of $\mathbf{\Pi}^1_{|\Lambda_N}$ instead of that of $\mathbf{\Pi}^1$ is irrelevant thanks to Lemma 2.7.4). To show that the same principle holds for $\{R_N\}_N$ it is enough to show that the two sequences are e.a.s. equivalent in the sense of Definition 2.7.1.

Let us first observe that the sequences $\{R_N\}_N$ and $\{R'_N\}$ are e.a.s.e. because as a consequence of (2.7.2) the tiling of Λ_N by the hypercubes $\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}$ only misses a $o(1)$ fraction of the volume of Λ_N and e.a.s. equivalence is then a consequence of Lemma 2.7.3.

As for the pair of sequences $\{R'_N\}_N$ and $\{R''_N\}_N$, let us observe that for any $k \geq 1$ we have

$$(\theta_x \cdot \mathcal{C}) \cap C_k = \left(\theta_{\Phi_N(x)} \cdot \Phi_N(\mathcal{C})\right) \cap C_k$$

for any x in one of the tiling hypercubes $\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}$ except for the points that are near the boundary of their hypercube - those such that

$$d\left(x, \cup_{i=1}^m \partial N^{1/d}\Lambda^{(i,N)}\right) \leq |C_k|^{1/d}.$$

For any k the fraction of volume of points in the hypercube that are close to the boundary in the previous sense is negligible as $N \to \infty$. Arguing as in the proof of Lemma 2.7.4 gives the result. □

b. Tagged point processes

We now recast the result of Lemma 2.7.6 in the context of tagged point processes (as defined in Section 2.2.4) which necessitates to replace the specific relative entropy ent by its analogue with tags.

Lemma 2.7.7. *Let Λ be a compact set of $\mathbb{R}^d$ with C^1 boundary and non-empty interior and let $\bar{R}_N$ be the push-forward of $\mathbf{\Pi}^1$ by the map*

$$\mathcal{C} \mapsto \frac{1}{|\Lambda|}\int_\Lambda \delta_{(x,\theta_{N^{1/d}x}\cdot\mathcal{C})}dx.$$

Then $\{\bar{R}_N\}_N$ satisfies a large deviation principle at speed N with rate function

$$\bar{P} \mapsto \int_\Lambda \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx.$$

Proof. **Upper bound.** Let $\bar{P}$ be a stationary tagged point process. We claim that

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log \mathbf{\Pi}^1(\bar{R}_N \in B(\bar{P},\varepsilon)) \leq -\int_\Lambda \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx. \tag{2.7.3}$$

Let us observe that the "forgetful" map $\varphi : \mathcal{P}(\Lambda\times\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ obtained by pushing forward the map $(x,\mathcal{C}) \mapsto \mathcal{C}$ is continuous. This yields

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log \mathbf{\Pi}^1(\bar{R}_N \in B(\bar{P},\varepsilon)) \leq \limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log \mathbf{\Pi}^1\left(\varphi(\bar{R}_N) \in B(\varphi(\bar{P}),\varepsilon)\right).$$

The definition of φ implies that $R_N := \varphi(\bar{R}_N)$ is the push-forward of $\mathbf{\Pi}^1$ by the map $\mathcal{C} \mapsto \frac{1}{N|\Lambda|}\int_{\Lambda_N}\delta_{\theta_x\cdot\mathcal{C}}dx$. From Lemma 2.7.6 we know that an LDP holds for R_N at speed $|\Lambda|N$ with

rate function $\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$ (or equivalently at speed N with rate function $|\Lambda|\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$) hence the right-hand side is bounded by $|\Lambda|\mathsf{ent}[\varphi(\bar{P})|\mathbf{\Pi}^1]$. Now let us note that $\varphi(\bar{P}) = \frac{1}{|\Lambda|}\int_\Lambda \bar{P}^x dx$ (where $\bar{P}^x$ is the disintegration of $\bar{P}$ with respect to the first coordinate of $\Sigma\times\mathcal{X}$ and where the integral is understood in the Gelfand-Pettis sense), but the relative specific entropy is affine hence we have

$$\mathsf{ent}[\varphi(\bar{P})|\mathbf{\Pi}^1] = \frac{1}{|\Lambda|}\int_\Lambda \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx$$

and this shows the upper bound of the lemma.

Lower bound. Let $\bar{P}$ be a tagged point process. We want to prove that

$$\liminf_{\varepsilon\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\mathbf{\Pi}^1(\bar{R}_N\in B(\bar{P},\varepsilon)) \geq -\int_\Lambda \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx. \tag{2.7.4}$$

For any $\varepsilon>0$ we claim that there exists a covering of Λ by compact sets $A_1,\dots,A_M\subset\Lambda$ of pairwise disjoint interiors such that each set A_i has a Lipschitz boundary and such that, denoting by $R_N^{(i)}$ the push-forward of $\mathbf{\Pi}^1$ by $\mathcal{C}\mapsto\frac{1}{N|A_i|}\int_{N^{1/d}A_i}\delta_{\theta_x\cdot(\mathcal{C}\cap N^{1/d}A_i)}dx$ (we impose that the random variables $\bar{R}_N$ and $R_N^{(1)},\dots,R_N^{(M)}$ are coupled together in the natural way), the following holds for any δ small enough and N large enough:

$$\bigcap_{i=1}^M\left\{R_N^{(i)}\in B\left(⨍_{A_i}\bar{P}^x dx,\delta\right)\right\}\subset\left\{\bar{R}_N\in B(\bar{P},\varepsilon)\right\}. \tag{2.7.5}$$

This is shown by the following successive approximations.

1. By definition of the topology of weak convergence, the ball $B(\bar{P},\varepsilon)$ contains a certain open set of the type
$$\bigcap_{i\in I}\left\{\left|\int F_i(x,\mathcal{C})(d\bar{R}_N-d\bar{P})\right|\leq\delta_1\right\}$$
for a finite family of continuous functions $F_i\in C^0(\Lambda\times\mathcal{X})$.
2. A standard application of the Stone-Weierstrass theorem implies that each function F_i can be approximated in sup-norm by a finite sum $\sum_j f_{i,j}g_{i,j}$ where $f_{i,j}$ are continuous functions on Λ and $g_{i,j}$ are continuous functions on $\mathcal{X}$.
3. We may then approximate each $f_{i,j}$ by step functions on Λ with a common partition $\{A_1,\dots,A_M\}$ for all functions $f_{i,j}$. Each set in the partition can be chosen to be either a hypercube or the intersection of a hypercube with Λ so that they all have a Lipschitz boundary. At this point (2.7.5) is seen to hold for some $\delta>0$ small enough, only with the random variable $R_N'^{(i)}$ instead of $R_N^{(i)}$, where $R_N'^{(i)}$ is the push-forward of $\mathbf{\Pi}^1$ by the map $\mathcal{C}\mapsto⨍_{N^{1/d}A_i}\delta_{\theta_x\cdot\mathcal{C}}dx$.
4. We may also approximate each function $g_{i,j}$ by a bounded local function in $\mathrm{Loc}_k(\mathcal{X})$ (as in Lemma 2.2.5) with the same k for all functions $g_{i,j}$. This allows us to argue as in Lemma 2.7.4 to neglect the points that are close to the boundary between two elements of the partition, hence passing from $R_N'^{(i)}$ to $R_N^{(i)}$.

Since the $A_1,\dots,A_M$ are pairwise disjoint (up to a boundary of zero Lebesgue measure), the events $\left\{R_N^{(i)}\in B\left(⨍_{A_i}\bar{P}^x dx,\delta\right)\right\}$ are globally independent so that (2.7.5) yields

$$\frac{1}{N}\log\mathbf{\Pi}^1\left(\left\{\bar{R}_N\in B(\bar{P},\varepsilon)\right\}\right)\geq\frac{1}{N}\sum_{i=1}^M\log\mathbf{\Pi}^1\left(\left\{R_N^{(i)}\in B\left(⨍_{A_i}\bar{P}^x dx,\delta\right)\right\}\right).$$

Moreover, for any $i = 1 \dots M$, we have

$$\liminf_{\delta\to 0}\liminf_{N\to\infty}\frac{1}{|A_i|N}\log\mathbf{\Pi}^1\left(\left\{R_N^{(i)}\in B\left(\fint_{A_i}\bar{P}^x dx,\delta\right)\right\}\right) = -\mathsf{ent}\left[\fint_{A_i}\bar{P}^x dx\Big|\mathbf{\Pi}^1\right] = -\fint_{A_i}\mathsf{ent}\left[\bar{P}^x|\mathbf{\Pi}^1\right]dx$$

by the large deviation principle of Lemma 2.7.6 and by the fact that $\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$ is affine. This finally implies that

$$\liminf_{N\to\infty}\frac{1}{N}\log\mathbf{\Pi}^1\left(\left\{\bar{R}_N\in B(\bar{P},\varepsilon)\right\}\right)\geq\sum_{i=1}^M\int_{A_i}\mathsf{ent}\left[\bar{P}^x|\mathbf{\Pi}^1\right]dx = -\int_\Lambda\mathsf{ent}\left[\bar{P}^x|\mathbf{\Pi}^1\right]dx$$

for any $\varepsilon > 0$, which implies (2.7.4).

Conclusion. From (2.7.3) and (2.7.4) we get a weak LDP for the sequence $\{\bar{R}_N\}_N$. The full LDP is obtained by observing that $\{\bar{R}_N\}_N$ is exponentially tight, a fact for which we only sketch the (elementary) proof : for any integer M we may find an integer $T(M)$ large enough such that a point process has less than $T(M)$ points in C_M expect for a fraction $\leq \frac{1}{M}$ of the configurations, with $\bar{R}_N$-probability bounded below (when $N\to\infty$) by $1-e^{-NM}$. The union (on $N\geq N_0$ large enough) of such events has a large $\bar{R}_N$-probability (bounded below by $1-e^{-NM}$ when $N\to\infty$) and is easily seen to be compact. □

c. From Poisson to Bernoulli

Modification in the lower bound. From Lemma 2.7.4 we know that the large deviation principle of Lemma 2.7.7 is still true when restricting the Poisson point process to $N^{1/d}\Lambda$. If we consider an N-point Bernoulli point process on $N^{1/d}\Lambda$ instead of the restriction of a Poisson point process as the reference measure i.e. if we constrain $\mathbf{\Pi}^1$ into having a fixed number of points in $N^{1/d}\Lambda$ then the LDP is modified. The large deviation upper bound holds but the large deviation lower bound ceases to be true in general, for the limit point processes might have large excesses of points with non-negligible probability e.g. in the case of the Poisson point process itself. Let us recall that we denote by $\mathcal{P}_{s,1}(\Lambda\times\mathcal{X})$ the set of stationary tagged point processes (with space coordinate taken in Λ) such that the integral on $x\in\Lambda$ of the intensity of the disintegration measure $\bar{P}^x$ (which is by assumption a stationary point process) is 1.

In what follows, when a set Λ is fixed if M, N are integers we denote by $\mathbf{B}_{M,N}$ the Bernoulli point process with M points in $N^{1/d}\Lambda$ and we let $\mathbf{B}_N := \mathbf{B}_{N,N}$ for any integer N. We want to prove

Lemma 2.7.8. *Let Λ be a compact set of $\mathbb{R}^d$ with C^1 boundary and non-empty interior and let $\bar{S}_N$ be the push-forward of $\mathbf{B}_N$ by the map*

$$\mathcal{C}\mapsto\frac{1}{N|\Lambda|}\int_{N^{1/d}\Lambda}\delta_{(N^{-1/d}x,\theta_x\cdot\mathcal{C})}dx.$$

Then for any $A\subset\mathcal{P}_s(\Lambda\times\mathcal{X})$ we have:

$$\left(-\inf_{\bar{P}\in\mathring{A}\cap\mathcal{P}_{s,1}(\Lambda\times\mathcal{X})}\int_\Lambda\mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx\right)-(\log|\Lambda|-|\Lambda|+1)\leq\liminf_{N\to\infty}\frac{1}{N}\log\bar{S}_N(A)$$
$$\leq\limsup_{N\to\infty}\frac{1}{N}\log\bar{S}_N(A)\leq\left(-\inf_{\bar{P}\in\bar{A}}\int_\Lambda\mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx\right)-(\log|\Lambda|-|\Lambda|+1). \quad (2.7.6)$$

Let us emphasize that in the lower bound of (2.7.6) the *infimum* is taken on the restriction $\mathring{A} \cap \mathcal{P}_{s,1}(\Lambda \times \mathcal{X})$.

Variations of the domain and the number of points. For M, N integers we denote by $\bar{S}_{M,N}$ the push-forward of $\mathbf{B}_{M,N}$ by the map

$$\mathcal{C} \mapsto \frac{1}{|\Lambda|} \int_\Lambda \delta_{(x, \theta_{N^{1/d}x} \cdot \mathcal{C})} dx.$$

Let us observe that $\bar{S}_{N,N} = \bar{S}_N$ as defined in Lemma 2.7.8. The following lemma allows us to handle the variations of the number of points.

Lemma 2.7.9. *Let $L = L_N$ and $M = M_N$ be two sequences depending on N with $L \geq \max(M, N)$ and let $l := \limsup_{N\to\infty} \max(|\frac{N}{L} - 1|, |\left(\frac{N}{L}\right)^{1/d} - 1|, |\frac{M}{N} - 1|, |\frac{N}{M} - 1|)$.*

Let $\bar{P}$ be a stationary tagged point process. The following holds:

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \bar{S}_{M,N} \left(B(\bar{P}, \delta) \right) \leq \limsup_{N\to\infty} \frac{1}{N} \log \bar{S}_L \left(B(\bar{P}, O(l)) \right) + O(l).$$

Proof. The probability of the point process $\mathbf{B}_L$ having exactly M points in $N^{1/d}\Lambda$ is given by $\left(\frac{N}{L}\right)^M \left(1 - \frac{N}{L}\right)^{L-M} \binom{L}{M}$ and conditionally to this event $\mathbf{B}_L$ induces a point process on $N^{1/d}\Lambda$ which is equal in law to $\mathbf{B}_{M,N}$. Moreover it easy to see from the definitions that for any $\mathcal{C}$ in $\mathcal{X}(N^{1/d}\Lambda)$ we have

$$d_{\mathcal{P}(L^{1/d}\Lambda \times \mathcal{X})} \left(\frac{1}{L|\Lambda|} \int_{L^{1/d}\Lambda} \delta_{(L^{-1/d}x, \theta_x \cdot \mathcal{C})} dx, \frac{1}{N|\Lambda|} \int_{N^{1/d}\Lambda} \delta_{(N^{-1/d}x, \theta_x \cdot \mathcal{C})} dx \right) = O\left(\left| (N/L)^{1/d} - 1 \right| + |N/L - 1| \right) = O(l) \quad (2.7.7)$$

Indeed for any $F \in \mathrm{Lip}_1(\Lambda \times \mathcal{X})$ we may write

$$\begin{aligned} &\frac{1}{L|\Lambda|} \int_{L^{1/d}\Lambda} F(L^{-1/d}x, \theta_x \cdot \mathcal{C}) dx - \frac{1}{N|\Lambda|} \int_{N^{1/d}\Lambda} F(N^{-1/d}x, \theta_x \cdot \mathcal{C}) dx \\ &\qquad = \frac{1}{L|\Lambda|} \int_{L^{1/d}\Lambda} \left(F(L^{-1/d}x, \theta_x \cdot \mathcal{C}) - F(N^{-1/d}x, \theta_x \cdot \mathcal{C}) \right) dx \\ &\qquad\qquad + \left(\frac{1}{N|\Lambda|} - \frac{1}{L|\Lambda|} \right) \int_{N^{1/d}\Lambda} F(N^{-1/d}x, \theta_x \cdot \mathcal{C}) dx. \quad (2.7.8) \end{aligned}$$

We have, for any $x \in N^{1/d}\Lambda$

$$|F(L^{-1/d}x, \theta_x \cdot \mathcal{C}) - F(N^{-1/d}x, \theta_x \cdot \mathcal{C})| \leq C|L^{-1/d} - N^{-1/d}|N^{1/d} = C\left((N/L)^{1/d} - 1 \right)$$

and on the other hand F is bounded by 1, which together with (2.7.8) and the fact that $N \leq L$ yields (2.7.7).

Conditioning $\mathbf{B}_L$ to have exactly M points in $N^{1/d}\Lambda$ we get

$$\begin{aligned} \log \bar{S}_{M,N} \left(B(\bar{P}, \delta) \right) &\leq \log \bar{S}_L \left(B(\bar{P}, \delta + O(l)) \right) \\ &\quad - \left(M \log(\frac{N}{L}) + (L - M) \log(1 - \frac{N}{L}) \right). \quad (2.7.9) \end{aligned}$$

By definition of l we have $\liminf_{N\to\infty} \frac{1}{N} \left(M \log(\frac{N}{L}) + (L - M) \log(1 - \frac{N}{L}) \right) = O(l) + O(l^2) = O(l)$, hence taking the limit $N \to \infty, \delta \to 0$ in (2.7.9) yields the lemma. □

We now turn to the proof of Lemma 2.7.8.

Proof. In what follows $\bar{R}_N$ will denote the push-forward of $\mathbf{\Pi}^1_{|N^{1/d}\Lambda}$ by the map

$$\mathcal{C} \mapsto \frac{1}{|\Lambda|}\int_\Lambda \delta_{(x,\theta_{N^{1/d}x}\cdot\mathcal{C})}dx.$$

To establish the upper bound of (2.1.24) it is enough to condition $\mathbf{\Pi}^1_{|N^{1/d}\Lambda}$ into having exactly N points. The conditional expectation is then equal in law to $\mathbf{B}_N$ so that

$$\begin{aligned}\frac{1}{N}\log \bar{R}_N(A) &\geq \frac{1}{N}\log \bar{S}_N(A) + \frac{1}{N}\log \mathbf{\Pi}^1_{|N^{1/d}\Lambda}(N \text{ points})\\ &= \frac{1}{N}\log \bar{S}_N(A) + \frac{1}{N}\log e^{-N|\Lambda|}\frac{1}{N!}(N|\Lambda|)^N\\ &= \frac{1}{N}\log \bar{S}_N(A) + N(\log|\Lambda| - |\Lambda| + 1) + o_{N\to\infty}(1)\end{aligned}$$

hence the upper bound of (2.1.24) follows from the LDP upper bound of Lemma 2.7.7.

We now turn to the lower bound in (2.1.24). Let us denote by $\#\Lambda_N$ the number of points of a configuration in $N^{1/d}\Lambda$ and by $\#\partial\Lambda_N$ the number of points in a 2-tubular neighborhood of $\partial\Lambda_N$. Let χ be a non-negative smooth function compactly supported in the unit ball of $\mathbb{R}^d$ such that $\int\chi = 1$ and let us denote by $\tilde{\chi}$ the continuous function on $\mathcal{X}$ obtained by testing χ against the point configurations (seen as Radon measures). If $\mathcal{C}$ is a point configuration in $N^{1/d}\Lambda$ we have

$$\frac{1}{N|\Lambda|}\int_{N^{1/d}\Lambda}\tilde{\chi}(\mathcal{C})\delta_{(N^{-1/d}x,\theta_x\cdot\mathcal{C})}dx = \frac{\#\Lambda_N}{N|\Lambda|} + O\left(\frac{\#\partial\Lambda_N}{N|\Lambda|}\right). \tag{2.7.10}$$

Moreover for all $\bar{P}\in\mathcal{P}_{s,1}(\Lambda\times\mathcal{X})$ we have by definition of the intensity $\int\tilde{\chi}(\mathcal{C})d\bar{P}(x,\mathcal{C}) = \frac{1}{|\Lambda|}$. It implies that for all $\varepsilon>0$

$$\begin{aligned}&\lim_{\delta\to0}\lim_{N\to\infty}\frac{1}{N}\log\bar{R}_N\left(B(\bar{P},\delta)\right)\\ &\qquad\leq \lim_{\delta\to0}\lim_{N\to\infty}\frac{1}{N}\log\bar{R}_N\left(\left\{\bar{Q}\in B(\bar{P},\delta)\mid \left|\int\tilde{\chi}(\mathcal{C})d\bar{Q}(x,\mathcal{C}) - \frac{1}{|\Lambda|}\right|\leq\varepsilon\right\}\right).\end{aligned} \tag{2.7.11}$$

We now observe that under a Poisson point process $\mathbf{\Pi}^1$ there are at most $\frac{N}{\log\log N}$ points near the boundary $\partial\Lambda_N$ with overwhelming probability :

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbf{\Pi}^1\left(\left\{\frac{\#\partial\Lambda_N}{N|\Lambda|}\geq\frac{1}{\log\log N}\right\}\right) = -\infty.$$

It means in particular that in the right-hand side of (2.7.11) we may neglect the (intersection with the) event $\{\frac{\#\partial\Lambda_N}{N|\Lambda|}\geq\frac{1}{\log\log N}\}$ since this event has a logarithmically negligible probability. We may then neglect the $O(\frac{\#\partial\Lambda_N}{N|\Lambda|})$ error term in (2.7.10) and replace the right-hand side of (2.7.11) by

$$\lim_{N\to\infty}\frac{1}{N}\log\bar{R}_N\left(B(\bar{P},\delta)\cap\left|\frac{\#\Lambda_N}{|\Lambda_N|} - \frac{1}{|\Lambda|}\right|\leq\varepsilon\right).$$

In the previous equation and in the rest of the proof we make a slight abuse of notation since $\bar{R}_N$ is the push-forward of $\mathbf{\Pi}^1$ by a certain map.

Next, up to replacing ε by $|\Lambda|\varepsilon$ let us write this term as

$$\lim_{N\to\infty}\frac{1}{N}\log\bar{R}_N\left(B(\bar{P},\delta)\cap\left|\frac{\#\Lambda_N}{N}-1\right|\leq\varepsilon\right)$$
$$=\lim_{N\to\infty}\frac{1}{N}\log\sum_{\frac{M}{N}=1-\varepsilon}^{\frac{M}{N}=\frac{1}{1+\varepsilon}}\bar{S}_{M,N}\left(B(\bar{P},\delta)\right)\mathbf{\Pi}^1_{\Lambda_N}(M\text{ points in }\Lambda_N).$$

The previous expression is obtained by applying the law of the total probability with respect to the possible values of $\#\Lambda_N$, and observing that the conditional expectation of $\bar{R}_N$ knowing the event $\{\#\Lambda_N=M\}$ is equal in law to $\bar{S}_{M,N}$.

We bound the $O(\varepsilon N)$ terms in the sum by their maximum to get

$$\limsup_{N\to\infty}\frac{1}{N}\log\sum_{\frac{M}{N}=1-\varepsilon}^{\frac{M}{N}=1+\varepsilon}\bar{S}_{M,N}\left(B(\bar{P},\delta)\right)\leq\limsup_{N\to\infty}\frac{1}{N}\log\max_{M=N(1-\varepsilon),\dots,N(1+\varepsilon)}\log\bar{S}_{M,N}\left(B(\bar{P},\delta)\right)$$
$$+\limsup_{N\to\infty}\frac{1}{N}\max_{M=N(1-\varepsilon),\dots,N(1+\varepsilon)}\log\mathbf{\Pi}^1_{\Lambda_N}(M\text{ points in }\Lambda_N).\quad(2.7.12)$$

Applying lemma 2.7.9 with $|\frac{M}{N}-1|\leq\varepsilon$ and $L=N(1+\varepsilon)$ we get

$$\lim_{\delta\to0}\limsup_{N\to\infty}\frac{1}{N}\log\bar{S}_{M,N}\left(B(\bar{P},\delta)\right)\leq\limsup_{N\to\infty}\frac{1}{N}\log\bar{S}_L\left(B(\bar{P},O(1))\right)+o(1),\quad(2.7.13)$$

where $O(1),o(1)$ hold when $\varepsilon\to0$, whereas an elementary computation yields for $\mathbf{\Pi}^1_{\Lambda_N}$

$$\limsup_{N\to\infty}\frac{1}{N}\max_{M=N(1-\varepsilon),\dots,N(1+\varepsilon)}\log\mathbf{\Pi}^1_{\Lambda_N}(M\text{ points in }\Lambda_N)\leq(\log|\Lambda|-|\Lambda|+1)+O(\varepsilon).\quad(2.7.14)$$

Combining (2.7.12), (2.7.13), (2.7.14), using that $L=N(1+\varepsilon)$ and letting $\varepsilon\to0$ we obtain

$$\lim_{\delta\to0}\lim_{N\to\infty}\frac{1}{N}\log\bar{R}_N\left(B(\bar{P},\delta)\right)\leq\liminf_{\delta\to0}\liminf_{N\to\infty}\frac{1}{N}\bar{S}_N\left(B(\bar{P},\delta)\right)+(\log|\Lambda|-|\Lambda|+1).$$

The lower-bound for $\bar{S}_N$ is now a consequence of the LDP lower bound obtained for $\bar{R}_N$ in Lemma 2.7.7. This completes the proof of Lemma 2.7.8. □

d. From Bernoulli to $\bar{\Omega}_{N,\beta}$

We now wish to extend the large deviation principle to the case of the point process $\bar{\Omega}_{N,\beta}$, defined as the push-forward of $\mathbb{Q}_{N,\beta}$ by i_N, cf. (2.1.9) and (2.1.21). Let us observe that the probability measure $\mathbb{Q}_{N,\beta}$ has a constant density on ω^N since by definition ζ vanishes on ω, and that its (marginal) density tends to zero like $\exp(-\beta N\zeta(x))$ outside ω. Hence we expect $\bar{\Omega}_{N,\beta}$ to behave like a Bernoulli point process with roughly N points on $N^{1/d}\omega$ (which would correspond to the case where $\zeta=+\infty$ outside ω). We may now turn to proving Proposition 2.1.6.

Proof. **Lower bound.** The lower bound of (2.1.24) is obtained by conditioning the points to all fall inside $N^{1/d}\Sigma$. Denote by $\#\Sigma_N$ the number of points in $N^{1/d}\Sigma$, by definition of $\mathbb{Q}_{N,\beta}$ we have

$$\mathbb{Q}_{N,\beta}\left(\{\#\Sigma_N=N\}\right)=\left(\frac{|\Sigma|}{\int_{\mathbb{R}^d}e^{-\beta N\zeta(x)}dx}\right)^N.$$

It is easy to deduce using (2.4.1) that

$$\frac{1}{N}\log \mathbb{Q}_{N,\beta}\left(\{\#\Sigma_N = N\}\right) = \log\frac{|\Sigma|}{|\omega|} + o(1). \tag{2.7.15}$$

Conditionally to $\#\Sigma_N = N$, the point process generated by $\mathbb{Q}_{N,\beta}$ is equal in law to an N-point Bernoulli process in $N^{1/d}\Sigma$ hence we have, with the notation $\bar{S}_N$ of the previous paragraph (the reference set is now $\Lambda = \Sigma$)

$$\frac{1}{N}\log \bar{\mathfrak{Q}}_{N,\beta}(A) \geq \frac{1}{N}\log \bar{S}_N(A) - \frac{1}{N}\log \mathbb{Q}_{N,\beta}\left(\{\#\Sigma_N = N\}\right).$$

Using the LDP lower bound for $\bar{S}_N$ proven in Lemma 2.7.8 together with (2.7.15) we get the lower bound for $\bar{\mathfrak{Q}}_{N,\beta}$.

Upper bound. The law of total probabilities yields (abusing notation as in the proof of the lower bound of Lemma 2.7.8)

$$\limsup_{N\to\infty}\frac{1}{N}\log \bar{\mathfrak{Q}}_{N,\beta}(A) \leq \limsup_{N\to\infty}\frac{1}{N}\log\sum_{k=0}^{N}\bar{\mathfrak{Q}}_{N,\beta}(A\cap\#\Sigma_N = k)\mathbb{Q}_{N,\beta}(\#\Sigma_N = k)$$

Conditionally to $\#\Sigma_N = k$ the point process generated by $\mathbb{Q}_{N,\beta}$ is equal in law to a Bernoulli point process with $k \leq N$ points in $N^{1/d}\Sigma$ and the LDP upper bound of Lemma 2.7.8 allows us to bound each term, so that the upper bound follows from that of Lemma 2.7.8. More precisely it is easy to see that with overwhelming probability the number of points $\#\Sigma_N$ tends to infinity as e.g. $\sqrt{N}$ so that we may bound

$$\frac{1}{N}\log\sum_{k=0}^{N}\bar{\mathfrak{Q}}_{N,\beta}(A\cap\#\Sigma_N = k)\mathbb{Q}_{N,\beta}(\#\Sigma_N = k)$$
$$\leq \frac{1}{N}\log\sum_{k=\sqrt{N}}^{N}\bar{\mathfrak{Q}}_{N,\beta}(A\cap\#\Sigma_N = k)\mathbb{Q}_{N,\beta}(\#\Sigma_N = k).$$

Bounding $\mathbb{Q}_{N,\beta}(\#\Sigma_N = k)$ by 1 and the terms $\frac{1}{N}\log\bar{\mathfrak{Q}}_{N,\beta}(A\cap\#\Sigma_N = k)$ by $\frac{1}{k}\log\bar{\mathfrak{Q}}_{N,\beta}(A\cap\#\Sigma_N = k)$ and using Lemma 2.7.8 we get the result. □

2.7.3 Discrete average, proof of Lemma 2.6.3

In this section we give the proof of Lemma 2.6.3. The line of reasoning is analogous to the continuous case and we will only sketch the argument. Let us first forget about the condition on the total number of points (i.e. we consider independent Poisson point processes) and about the tags (i.e. the coordinate in Σ'_{int}), then there holds for any fixed R a Large Deviation Principle for $\dot{\mathfrak{M}}_{N,R}$ at speed $m_{N,R}$ with rate function $\text{Ent}[\cdot|\mathbf{\Pi}^1_{|C_R}]$. This is a consequence of the classical Sanov theorem (see [DZ10, Section 6.2]) since in this case the random variables $\theta_{x_i}\cdot\mathbf{C}_i$ are independent and identically distributed Poisson point processes on each hypercube. Taking the limit $R\to\infty$ yields, in view of the asymptotics (2.6.5) on $m_{N,R}$ and the definition (2.1.12) of the specific relative entropy,

$$\lim_{R\to\infty}\liminf_{\nu\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\dot{\mathfrak{M}}_{N,R}(B(P_{\underline{m}|C_R},\varepsilon)) \geq -\mathsf{ent}\left[P_{\underline{m}}|\mathbf{\Pi}^1\right].$$

We may then extend this LDP to the context of tagged point processes by following essentially the same argument as in the proof of the continuous case.

We then argue as before in the proof of LDP lower bound for the continuous average in the Bernoulli case. To condition the point process into having $N_{\text{int}} \approx N\mu_V(\Sigma_{\underline{m}})$ points in $\Sigma'_{\text{int}} \approx \Sigma'_{\underline{m}}$ modifies the LDP lower bound obtained from Sanov's theorem by a quantity which is an adaptation of (2.7.14) in this setting

$$\limsup_{N\to\infty} \frac{1}{N|\Sigma_{\underline{m}}|} \log\left(e^{-N|\Sigma_{\underline{m}}|} \frac{(N|\Sigma_{\underline{m}}|)^{N\mu_V(\Sigma_{\underline{m}})}}{(N\mu_V(\Sigma_{\underline{m}}))!} \right),$$

hence the constant $r_{\mu_V,\underline{m}}$ in (2.6.14). This settles the first point of Lemma 2.6.3.

The second point follows from the first one by elementary manipulations. The main argument is that if one knows that a discrete average of large hypercubes is very close to some point process P, then the continuous average of much smaller hypercubes is also close to P since it can be re-written using the discrete average up to a small error. More precisely for any fixed $\delta > 0$ establishing that a point process is in $B(P, \delta)$ can be done by testing against local functions in Loc_k for some k large enough (because of the topology on $\mathcal{X}$ and the approximation Lemma 2.2.5). For R, N large enough an overwhelming majority of all translates of C_k by a point in $\Sigma'_{\underline{m}}$ is included in one of the hypercubes $\overline{C}_i$ ($i = 1 \dots m_{N,R}$) (this follows from the definitions and (2.6.5)).

For any such local function $f \in \text{Loc}_k$ we have

$$\frac{1}{|\Sigma'_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} f(\theta_x \cdot \mathcal{C}) \approx \frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \frac{1}{R^d} \int_{\bar{C}_i} f(\theta_x \cdot \mathcal{C}) dx, \tag{2.7.16}$$

which allows us to pass from the assumption that the discrete average (in the right-hand side of (2.7.16)) of a configuration is close to P to the fact that the continuous average (in the left-hand side of (2.7.16)) is close to P. These considerations are easily adapted to the situation of tagged point processes.

2.8 Additional proofs

We collect here the proofs of various lemmas used in the course of the paper.

2.8.1 Proof of Lemma 2.2.5

The first point ($\mathcal{X}$ is a Polish space) is well-known, see e.g. [DVJ08, Proposition 9.1.IV]). It is easy to see that $d_{\mathcal{X}}$ is a well-defined distance (the only point to check is the separation property). It is also clear that any sequence converging for $d_{\mathcal{X}}$ converges for the topology on $\mathcal{X}$. Conversely, let $\{\mu_n\}_n$ be a sequence in $\mathcal{X}$ which converges vaguely to μ and let $\varepsilon > 0$. There exists an integer K such that $\sum_{k\geq K} \frac{1}{2^k} \leq \frac{\varepsilon}{2}$ so we might restrict ourselves to the first K terms in the series defining $d_{\mathcal{X}}(\mu_n, \mu)$. For each $k = 1, \dots, K$ and for any n, let $\mu_{n,k}$ and μ_k be the restriction to the hypercube C_k of each term (and of the limit). For any $k = 1, \dots, K$ the sequence of masses $(\mu_{n,k}(C_k))_{k\geq 1}$ is an integer sequence and up to passing to a common subsequence by a standard diagonal argument, we may assume that for each k the sequence $\{\mu_{n,k}(C_k)\}_{k\geq 1}$ is either constant or diverging to $+\infty$. We may then restrict ourselves to the terms k for which the sequence is constant $\equiv N_k$. By compactness we may then assume that the N_k points of the configuration converge to some N_k-uple $x_1 \dots x_{N_k}$ of points in C_k. It is easy to see that N_k must be equal to $\mu_k(C_k)$ and that the points $x_1 \dots x_{N_k}$ must correspond to the points of the configuration μ_k. This implies the convergence in the sense of $d_{\mathcal{X}}$. From any sequence $\{\mu_n\}_n$ which converges weakly to μ we may extract a subsequence which converges to

μ in the sense of $d_{\mathcal{X}}$ (and the converse is true), which ensures that $d_{\mathcal{X}}$ is compatible with the topology on $\mathcal{X}$.

We now prove the approximation property stated in the third point of Lemma 2.2.5. By density it is enough to prove the second part of the statement i.e. the (uniform) approximation of Lipschitz functions by local functions. Let F be in $\mathrm{Lip}_1(\mathcal{X})$ and $\delta > 0$. From the definition (2.2.29) of $d_{\mathcal{X}}$ we see that there exists k such that if two configurations $\mathcal{C}, \mathcal{C}'$ coincide on C_k then $d_{\mathcal{X}}(\mathcal{C}, \mathcal{C}') \leq \delta$. We let $f_k := F(\mathcal{C} \cap C_k)$. By definition f is a local function in Loc_k, we have chosen k such that $d_{\mathcal{X}}(\mathcal{C}, \mathcal{C} \cap C_k) \leq \delta$ for any configuration $\mathcal{C}$ and since by assumption F is 1-Lipschitz we have

$$|F(\mathcal{C}) - f(\mathcal{C})| = |F(\mathcal{C}) - F(\mathcal{C} \cap C_k)| \leq d_{\mathcal{X}}(\mathcal{C}, \mathcal{C} \cap C_k) \leq \delta,$$

and k here depends only on δ, which concludes the proof of Lemma 2.2.5.

2.8.2 Proof of Lemma 2.2.11

Let us denote as in Section 2.2.2 by $X = (x, y)$ the coordinates in $\mathbb{R}^d \times \mathbb{R}^k$. We also recall that $\gamma \in (-1, 1)$. Let E_1 and E_2 be elements of $\mathcal{A}_m$ such that $\mathrm{Conf}_m E_1 = \mathrm{Conf}_m E_2$. Then we have $E_1 - E_2 = \nabla u$ where u solves $-\mathrm{div}\,(|y|^\gamma \nabla u) = 0$. We can also observe that $\nabla_x u$ (where ∇_x denote the vector of derivatives in the x directions only, is also a solution to the same equation (this should be understood component by component). This is a divergence form equation with a weight $|y|^\gamma$ which belongs to the so-called Muckenhoupt class A_2. The result of [FKS82, Theorem 2.3.12] then says that there exists $\lambda > 0$ such that for $0 < r < R$,

$$\mathrm{osc}(\nabla_x u, B(X, r)) \leq C \left(\frac{1}{\int_{B(X,R)} |y|^\gamma} \int_{B(X,R)} |y|^\gamma |\nabla_x u|^2 \right)^{1/2} (r/R)^\lambda, \tag{2.8.1}$$

where $\mathrm{osc}(u, B(X, r)) = \max_{B(X,r)} u - \min_{B(X,r)} u$. We note that the condition that $\mathcal{W}(E_1)$ and $\mathcal{W}(E_2)$ imply without difficulty that

$$\limsup_{R \to \infty} \frac{1}{R^d} \int_{K_R \times \mathbb{R}^d} |y|^\gamma |\nabla u|^2 < +\infty. \tag{2.8.2}$$

Applying (2.8.1) to X which belongs to a fixed compact set, and inserting (2.8.2) we find that

$$\mathrm{osc}(\nabla_x u, B(X, r)) \leq C \left(R^{-(d+1+\gamma)} R^d \right)^{1/2} (r/R)^\lambda$$

in the case $k = 1$, and respectively

$$\mathrm{osc}(\nabla_x u, B(X, r)) \leq C \left(R^{-d} R^d \right)^{1/2} (r/R)^\lambda$$

in the case $k = 0$. In both cases, letting $R \to \infty$, we deduce that $\mathrm{osc}(\nabla_x u, B(X, r)) = 0$, which means that $\nabla_x u$ is constant on every compact set of $\mathbb{R}^{d+k}$.

In the case $k = 0$, this concludes the proof that u is affine, and then E_1 and E_2 differ by a constant vector.

In the case $k = 1$, this implies that u is an affine function of x, for each given y. We may thus write $u(x, y) = a(y) \cdot x + b(y)$. Inserting into the equation $\mathrm{div}\,(|y|^\gamma \nabla u) = 0$, we find that $\partial_y(|y|^\gamma (a'(y)x + b'(y)) = 0$, i.e. $a'(y)x + b'(y) = \frac{c(x)}{|y|^\gamma}$. But the fact that $\int_{\mathbb{R}} |y|^\gamma |\partial_y u|^2 \, dy$ is convergent implies that $\int \frac{c(x)^2}{|y|^\gamma} \, dy$ must be, which implies that $c(x) = 0$ and thus $\partial_y u = 0$. This means that $u(x, y) = f(x)$. But then again $\int |y|^\gamma |\nabla u|^2 \, dy$ is convergent so we must have $\nabla f(x) = 0$ and u is constant. Thus $E_1 = E_2$ as claimed.

In the case $k = 1$, it follows that $\mathbb{W}_m(\mathcal{C})$ (if it is not infinite) becomes an inf over a singleton, hence is achieved.

Let us now turn to the case $k = 0$ (in that case we note that we must have $s = d - 2$ or (2.1.2)). Let $E \in \mathcal{A}_m$ be such that $\mathrm{Conf}_m E = \mathcal{C}$ and $\mathcal{W}(E) < \infty$ (if it exists), and let c be a constant vector in $\mathbb{R}^d$, then

$$\fint_{K_R} |E_\eta + c|^2 - mc_{d,s}g(\eta) = \fint_{K_R} |E_\eta|^2 - mc_{d,s}g(\eta) + |c|^2 + 2c \cdot \fint_{K_R} E_\eta. \tag{2.8.3}$$

We claim that $\fint_{K_R} E_\eta$ is bounded independently of η and R. So the right-hand side of (2.8.3) is a quadratic function of c, with fixed quadratic coefficients and linear and constant coefficients which are bounded with respect to R and η. A little bit of convex analysis implies that $c \mapsto \mathcal{W}(E+c)$ being a limsup (over R and η) of such functions is strictly convex, coercive and locally Lipschitz, hence it achieves its minimum for a unique c. This means that the infimum defining $\mathbb{W}_m$ is a uniquely achieved minimum.

To conclude the proof, we just need to justify that $\fint_{K_R} E_\eta$ is bounded independently of η and R. We may write

$$\fint_{K_R} E_\eta = \fint_{K_R} E_1 + \fint_{K_R} ((\nabla f_1 - \nabla f_\eta) * \mathcal{C},$$

where f_η is as in (2.2.16). Because we are in the case $s = d - 2$ or (2.1.2), ∇f_η and ∇f_1 are integrable and we may check that $\int_{K_R}((\nabla f_1 - \nabla f_\eta) * \mathcal{C}$ is bounded by $C\mathcal{C}(K_R)$ where C is independent of R and η. But since $\mathcal{W}(E) < \infty$ and $E \in \mathcal{A}_m$, we have $\lim_{R\to\infty} \frac{1}{|K_R|}\mathcal{C}(K_R) = m$ (cf. [PS15, Lemma 2.1]). It follows that

$$\left|\fint_{K_R} E_\eta\right| \leq C(1 + \mathcal{W}_1(E) + m)$$

and by almost monotonicity of $\mathcal{W}$ (Lemma 2.3.4) the claim follows.

2.8.3 Proof of Lemma 2.2.12

Let $\mathcal{X}_1^o$ be the image of $\mathcal{A}_1$ by $\mathcal{X}_1$ i.e. the set of point configuration of "mean density" 1 for which one can define a corresponding electric field. Let $\mathcal{C} \mapsto E(\mathcal{C})$ be a measurable map from $\mathcal{X}_1^o$ to $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ such that for any $\mathcal{C}$ we have $\mathcal{X}_1(E(\mathcal{C})) = \mathcal{C}$ and $\mathcal{W}(E(\mathcal{C})) = \mathbb{W}_1(\mathcal{C})$ (such a map can be chosen measurable because the set of electric fields E satisfying these conditions is closed, since it is a singleton according to Lemma 2.2.11).

For any $\mathcal{C}$ in $\mathcal{X}_1^o$ let us define the following sequence of random electric fields

$$P^{\mathrm{elec}}_{k,\mathcal{C}} := \fint_{C_k} \delta_{\theta_x \cdot E(\mathcal{C})}.$$

We claim that if $\mathbb{W}_1(\mathcal{C})$ is finite then the sequence $\{P^{\mathrm{elec}}_{k,\mathcal{C}}\}_k$ is relatively compact in $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k}))$ for the weak topology on $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$. Indeed for any integer m we have

$$\int \left[\int_{C_m} |y|^\gamma |E_\eta|^2\right] dP^{\mathrm{elec}}_{k,\mathcal{C}} = \frac{1}{|C_k|}\int_{C_k} dx \int_{C_m} |y|^\gamma |\theta_x \cdot E_\eta|^2 \leq \frac{1}{|C_k|}\int_{C_{k+m}} |y|^\gamma |E_\eta|^2 \tag{2.8.4}$$

and by definition of $\mathcal{W}$ we have

$$\lim_{k\to\infty} \frac{1}{|C_k|}\int_{C_{k+m}} |y|^\gamma |E_\eta|^2 = \mathcal{W}_\eta(E) + c_{d,s}g(\eta). \tag{2.8.5}$$

This implies that the sequence $\left\{\int \left[\int_{C_m} |y|^\gamma |E_\eta|^2\right] dP^{\text{elec}}_{k,\mathcal{C}}\right\}_k$ is bounded. Arguing as in the proof of Lemma 2.3.7 we get the existence of a limit point for $\{P^{\text{elec}}_{k,\mathcal{C}}\}_k$.

Let $\mathcal{C} \mapsto P^{\text{elec}}_{\infty,\mathcal{C}}$ be a measurable choice of a weak limit point (see e.g. [Coh72]) on the (measurable) set $\{\mathcal{C}, \mathbb{W}_1(\mathcal{C}) \text{ is finite}\}$. It is easy to see that $P^{\text{elec}}_{\infty,\mathcal{C}}$ is stationary (since we average $\theta_x \cdot E(\mathcal{C})$ on large hypercubes), concentrated on $\mathcal{A}_1$. Moreover, in view of (2.8.4)–(2.8.5) we have

$$\mathcal{W}_\eta(E^\varepsilon(\mathcal{C})) \geq \liminf_{k\to\infty} \int \left[\int_{C_m} |y|^\gamma |E_\eta|^2\right] dP^{\text{elec}}_{k,\mathcal{C}}(E) - c_{d,s} g(\eta)$$
$$= \int \left[\int_{C_m} |y|^\gamma |E_\eta|^2\right] dP^{\text{elec}}_{\infty,\mathcal{C}}(E) - c_{d,s} g(\eta)$$

and the right-hand side is $\widetilde{\mathcal{W}}_\eta(P^{\text{elec}}_{\infty,\mathcal{C}})$ by stationarity of $P^{\text{elec}}_{\infty,\mathcal{C}}$ and Lemma 2.2.8. Letting $\eta \to 0$ we deduce that $\widetilde{\mathcal{W}}(P^{\text{elec}}_{\infty,\mathcal{C}}) \leq \mathcal{W}(E)(\mathcal{C}))$.

Since $\widetilde{\mathbb{W}}_1(P)$ is finite, $\mathbb{W}_1(\mathcal{C})$ is finite P-a.s. and we may define the probability measure

$$P^{\text{elec}}_\infty := \int \delta_{P^{\text{elec}}_{\infty,\mathcal{C}}} dP(\mathcal{C}).$$

We check that

— P^{elec}_∞ is stationary, because $P^{\text{elec}}_{\infty,\mathcal{C}}$ is stationary for P-almost every $\mathcal{C}$.

— We have

$$\widetilde{\mathcal{W}}(P^{\text{elec}}_\infty) = \int \widetilde{\mathcal{W}}(P^{\text{elec}}_{\infty,\mathcal{C}}) dP(\mathcal{C}) \leq \int \mathcal{W}(E(\mathcal{C})) dP(\mathcal{C}) \leq \int (\mathbb{W}_1(\mathcal{C})) dP(\mathcal{C}) \leq \widetilde{\mathbb{W}}_1(P) + \varepsilon.$$

— The push-forward of P^{elec}_∞ by Conf_1 is P, because this the case of $\int \delta_{P^{\text{elec}}_{k,c}} dP(\mathcal{C})$ for all $k \geq 1$.

Hence we get that

$$\widetilde{\mathbb{W}}_1(P) \geq \min\{\widetilde{\mathcal{W}}(P^{\text{elec}}) \mid P^{\text{elec}} \text{ is stationary and the push-forward of } P^{\text{elec}} \text{ by } \text{Conf}_1 \text{ is } P\}.$$

The reverse inequality is obvious by definition of $\widetilde{\mathbb{W}}_1$.

2.8.4 Proof of Lemma 2.4.4

Let $k > 0$, $\varepsilon > 0$ and N be fixed. Let us consider $u_N \in \mathbb{R}^d$ such that $C_k \cup (u_N + C_k) \subset B(0, \sqrt{(1-\varepsilon)N})$. We first bound the number of points in $(C_k \cup C_k + u_N)$ with overwhelming probability, uniformly on the choice of u_N. For $k_N = N^{1/2+1/10}$ we have:

$$\mathbb{P}_{N,2}\left(\mathcal{N}(0,k) + \mathcal{N}(u_n,k) \geq k_N\right) = o\left(N^{-N}\right),$$

uniformly on the choice of u_N. This can be deduced e.g. from discrepancy estimates as in Lemma 2.3.8, which imply that

$$\mathbb{P}_{N,2}\left(\mathcal{N}(0,k) + \mathcal{N}(u_N,k) \geq k_N\right)$$

is bounded above by $\exp(-k_N^2)$ pour $k_N \gg \sqrt{N}$ (and uniformly on the choice of u_N). We may then neglect the event $\{\mathcal{N}(0,k) + \mathcal{N}(u_N,k) \geq k_N\}$ which contributes only with order $o(N^{-N})$ to (2.4.11).

Conditioning on the number of points in $C_k \cup (u_N + C_k)$ we may then restrict ourselves to quantify the translation-invariance of $\rho_{(N,2),k}$ (k-point correlation function of the determinantal point process $\mathbf{P}_{N,2}$) for all $k \leq k_N$. The determinantal nature of $\mathbf{P}_{N,2}$ implies that

$$\rho_{(N,2),k} = \det\left[K_N(x_i, x_j)\right]_{1\leq i,j\leq k}.$$

We know that $\rho_{(N,2),k}$ converges the correlation function $\rho_{(\infty,2),k}$ of Gin_2 which are translation-invariant (see e.g. [HKPV09, 4.3.7]). Thus we are left to bound the difference between $\rho_{(N,2),k}$ and $\rho_{(\infty,2),k}$. Let us compare the kernels K_∞ and K_N :

$$\begin{aligned} K_N(x_i, x_j) = \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \sum_{l=0}^{N-1} \frac{(x_i\bar{x_j})^l}{l!} &= \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \left((e^{x_i\bar{x_j}}) - \sum_{l=N}^{+\infty} \frac{(x_i\bar{x_j})^l}{l!} \right) \\ &= K_\infty(x_i, x_j) - \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \left(\sum_{l=N}^{+\infty} \frac{(x_i\bar{x_j})^l}{l!} \right). \end{aligned}$$

To bound the error term let us observe that

$$\begin{aligned} \left| \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \left(\sum_{l=N}^{+\infty} \frac{(x_i\bar{x_j})^l}{l!} \right) \right| &\leq \frac{1}{\pi} e^{-|x_i\bar{x_j}|} \sum_{l=N}^{+\infty} \frac{1}{l!} \left(\frac{|x_i\bar{x_j}|}{N} \right)^l N^l \\ &\leq \frac{1}{\pi} e^{-|x_i\bar{x_j}|} \left(\frac{|x_i\bar{x_j}|}{N} \right)^N \left(\sum_{l=N}^{+\infty} \frac{1}{l!} N^l \right). \end{aligned}$$

We may now use the well-known equivalent

$$\sum_{l=N}^{+\infty} \frac{1}{l!} N^l \sim \frac{e^N}{2}.$$

We deduce that

$$\frac{1}{\pi} e^{-|x_i\bar{x_j}|} \left(\frac{|x_i\bar{x_j}|}{N} \right)^N \left(\sum_{l=N}^{+\infty} \frac{1}{l!} N^l \right) \leq C \exp\left(-N \left(\frac{|x_i\bar{x_j}|}{N} - \log \frac{|x_i\bar{x_j}|}{N} - 1 \right) \right).$$

It is elementary that $\log(1-t) \leq t + \frac{t^2}{2}$ for all $t \in [0,1]$. We deduce that for all x_i, x_j in the disk of radius $\sqrt{(1-\varepsilon)N}$, since $1 - \frac{|x_i\bar{x_j}|}{N} \geq \varepsilon$, we have

$$\left| \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \left(\sum_{l=N}^{+\infty} \frac{(x_i\bar{x_j})^l}{l!} \right) \right| \leq C \exp(-\frac{\varepsilon^2}{2} N).$$

We thus obtain $K_N(x_i, x_j) = K_\infty(x_i, x_j) + O(\exp(-\frac{\varepsilon^2}{2}N))$ uniformly for x_i, x_j in the disk of radius $\sqrt{(1-\varepsilon)N}$.

An explicit computation yields ($\mathfrak{S}_k$ denotes the group of permutation of k elements and s the signature morphism)

$$\begin{aligned} \det\left[K_N(x_i, x_j)\right] = \sum_{\sigma\in\mathfrak{S}_k} s(\sigma) \prod_{i=1}^{k} \left(K_\infty(x_i, x_{\sigma(j)}) + O(\exp(-\frac{\varepsilon^2}{2}N)) \right) \\ = \det\left[K_\infty(x_i, x_j)\right] + k! \times R_N \quad (2.8.6) \end{aligned}$$

with an error term R_N satisfying

$$|R_N| \leq \sum_{l=0}^{k-1} \binom{N}{k} \left(\sup_{i,j} |K_N(x_i,x_j)|\right)^l \times O(\exp(-\frac{\varepsilon^2}{2}N))^{k-l}$$
$$= \left(\sup_{i,j} |K_N(x_i,x_j)| + O(\exp(-\frac{\varepsilon^2}{2}N))\right)^k - \sup_{i,j} |K_N(x_i,x_j)|$$

but K_N is uniformly bounded by 1 so that for $k \leq k_N$ we have

$$|R_N| \leq O(\exp(-\frac{\varepsilon^2}{2}N)). \tag{2.8.7}$$

Since $k! \leq (k_N)!$ is bounded above by

$$\left(N^{\frac{1}{2}+\frac{1}{10}}\right)! \ll \exp(N^{\frac{1}{2}+\frac{1}{5}}). \tag{2.8.8}$$

Combining (2.8.6), (2.8.7) and (2.8.8) yields

$$\left|\det\left[K_n(x_i,x_j)\right]_{1\leq i,j\leq k} - \det\left[K_\infty(x_i,x_j)\right]_{1\leq i,j\leq k}\right| = O(\exp(-\frac{\varepsilon^2}{2}N)). \tag{2.8.9}$$

Equation (2.8.9) together with the invariance property of $\rho_{(\infty,2),k}$ concludes the proof of the lemma.

2.8.5 Proof of Lemma 2.5.1

Let $m = \fint_K \mu$ be the average of μ over K. We may (see e.g. [PS15, Lemma 6.3]) partition K into $n_{K,\mu}$ hyperrectangles $\mathcal{R}_i$, which all have volume $1/m$, and whose sidelengths are in $[2^{-d}m^{-1/d}, 2^d m^{1/d}]$. In each of these hyperrectangles we solve

$$\begin{cases} \operatorname{div}\left(|y|^\gamma \nabla h_i\right) = c_{d,s}\left(\delta_{X_i} - m\delta_{\mathbb{R}^d}\right) & \text{in } \mathcal{R}_i \times [-1,1]^k \\ \nabla h_i \cdot \vec{\nu} = 0 & \text{on } \partial(\mathcal{R}_i \times [-1,1]^k) \end{cases}$$

According to [PS15, Lemma 6.5], if $X_i \subset \mathbb{R}^d \times \{0\}$ is at distance $\leq 2^{-(d+1)}m^{-1/d}$ from the center p_i of $\mathcal{R}_i$ then we have

$$\lim_{\eta\to 0} \left| \int_{\mathcal{R}_i \times [-1,1]^k} |y|^\gamma |\nabla (h_i)_\eta|^2 - c_{d,s} g(\eta) \right| \leq C$$

where C depends only on d and m. We may then define $E_i = \nabla h_i \mathbf{1}_{\mathcal{R}_i \times [-1,1]^k}$, and by compatibility of the normal components, the vector field $E^{\mathrm{gen}} = \sum_i E_i$ satisfies

$$\begin{cases} \operatorname{div}\left(|y|^\gamma E^{\mathrm{gen}}\right) = c_{d,s}\left(\sum_i \delta_{X_i} - m\delta_{\mathbb{R}^d}\right) & \text{in } K \times \mathbb{R}^k \\ E^{\mathrm{gen}} \cdot \vec{\nu} = 0 & \text{on } \partial(K \times \mathbb{R}^k) \end{cases}$$

and if $\eta < \eta_0 < 2^{-(d+2)}m^{-1/d}$,

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E^{\mathrm{gen}}_\eta|^2 - c_{d,s} n_{K,\mu} g(\eta) \leq C n_{K,\mu} \tag{2.8.10}$$

with C depending only on d and m. The last step is to rectify for the error made by replacing μ by m. For that, we use the following

Lemma 2.8.1 ([PS15], Lemma 6.4)**.** *Let K_R be a hyperrectangle whose sidelengths are in $[R, 2R]$, and μ a bounded measurable function such that $\int_{K_R} \mu$ is an integer, and let $m = \fint_{K_R} \mu$. The solution (unique up to constant) to*

$$\begin{cases} \operatorname{div}(|y|^\gamma \nabla h) = c_{d,s}(\mu - m)\delta_{\mathbb{R}^d} & \text{in } K_R \times [-R, R]^k \\ \nabla h \cdot \vec{\nu} = 0 & \text{on } \partial(K_R \times [-R, R]^k), \end{cases}$$

exists and satisfies

$$\int_{K_R \times [-R,R]^k} |y|^\gamma |\nabla h|^2 \leq C R^{d+1-\gamma} \|\mu - m\|^2_{L^\infty(K_R)}. \tag{2.8.11}$$

Applying this lemma provides a function h, and we let

$$\hat{E} = E + \nabla h \mathbf{1}_{K \times [-R,R]^k}.$$

It is obvious that $\hat{E}$ solves

$$\begin{cases} \operatorname{div}(|y|^\gamma \hat{E}) = c_{d,s}\left(\sum_i \delta_{X_i} - \mu\delta_{\mathbb{R}^d}\right) & \text{in } K \times \mathbb{R}^k \\ \hat{E} \cdot \vec{\nu} = 0 & \text{on } \partial(K \times \mathbb{R}^k). \end{cases}$$

Combining (2.8.11) and (2.8.10) and using the Cauchy-Schwarz inequality, we obtain

$$\int_{K \times \mathbb{R}^k} |y|^\gamma |\hat{E}_\eta|^2 \leq c_{d,s} n_{K,\mu}(g(\eta) + C) + C R^{d+1-\gamma} \|\mu - m\|^2_{L^\infty(K_R)} + C\left(n_{K,\mu} g(\eta)\right)^{\frac{1}{2}} R^{\frac{d+1-\gamma}{2}} \|\mu - m\|_{L^\infty(K_R)}.$$

Letting then $\mathcal{R}(K, \mu)$ be the family of configurations $\{X_i\}_{i=1}^{n_{K,\mu}}$ above where each X_i varies in $B(p_i, 2^{-(d+1)} m^{-1/d})$ (p_i being the center of $\mathcal{R}_i$), and with all possible permutations of the labels, we have thus obtained that for every $\mathcal{C} \in \mathcal{R}(K, \mu)$ the desired results hold.

Chapitre 3

Lois locales pour le gaz de Coulomb bi-dimensionnel

Ce chapitre est constitué de l'article "Local microscopic behavior for 2D Coulomb gases" [Leb15a].

Contents

3.1 Introduction

3.1.1 General setting

We consider a system of N points in the Euclidean space $\mathbb{R}^2$ with pairwise logarithmic interaction, in a confining potential V, and associate to any N-tuple $\vec{X}_N = (x_1, \dots, x_N)$ the energy

$$h_N(\vec{X}_N) := \sum_{1 \le i \ne j \le N} -\log|x_i - x_j| + N \sum_{i=1}^{N} V(x_i), \quad x_i \in \mathbb{R}^2. \tag{3.1.1}$$

We only impose mild conditions on the potential V (see Assumption 1).

For any value of the *inverse temperature* parameter $\beta > 0$ we consider the associated N-point Gibbs measure, which is absolutely continuous with respect to the Lebesgue measure on $(\mathbb{R}^2)^N$ with a density given by

$$d\mathbb{P}_N^\beta(\vec{X}_N) := \frac{1}{Z_{N,\beta}} e^{-\frac{1}{2}\beta h_N(\vec{X}_N)} d\vec{X}_N, \tag{3.1.2}$$

where we denote a N-tuple of points by $\vec{X}_N = (x_1, \dots, x_N)$ and $d\vec{X}_N := dx_1 \dots dx_N$. The constant $Z_{N,\beta}$ is a normalizing constant, also called the *partition function*, so that the total mass of $\mathbb{P}_N^\beta$ is 1.

a. Motivations

The model described by (3.1.1) and (3.1.2) is known in statistical physics as a *two-dimensional Coulomb gas*, *two-dimensional log-gas* or *two-dimensional one-component plasma*, we refer e.g. to [AJ81], [JLM93], [SM76] for a physical treatment of its main properties.

When $\beta = 2$ and V is quadratic, the probability measure (3.1.2) coincides with the joint law of eigenvalues of a non-Hermitian matrix model known as the *complex Ginibre ensemble*, which is obtained by sampling a $N \times N$ matrix whose coefficients are (properly normalized) i.i.d. complex Gaussians, see [Gin65]. For $\beta = 2$, more general potentials can be considered, which are associated to "random normal matrices" (see e.g. [AHM15]). Systems of particles with a logarithmic interaction as in (3.1.1), called *log-gases*, have been also (and mostly) been studied on the real line, motivated by their link with Hermitian random matrix theory. We refer to [For10] for a survey of the connection between log-gases and random matrix theory, and in particular to [For10, Chap.15] for the two-dimensional (non-Hermitian) case.

The Ginibre case (and the case $\beta = 2$ in general) has the special property that the point process associated to $\mathbb{P}_N^\beta$ becomes *determinantal*, which allows for an exact computation of many interesting quantities, e.g. the n-point correlation functions. The existence of a matrix model also allows for universality results at the microscopic scale as in [BYY14a, BYY14b]. In the present paper we rather work with general $\beta > 0$ and potential V, thus dealing with what could be called *two-dimensional β-ensembles* by analogy with the one-dimensional β-ensembles

which generalize the laws of eigenvalues of random Hermitian matrices (see e.g. [DE02]). The microscopic behavior of one-dimensional β-ensembles has been recently investigated in [BEY12, BEY14] and we aim at a similar understanding in the two-dimensional case.

b. First-order results: the macroscopic behavior

Let us first recall some results about the macroscopic behavior of the particle system as $N \to \infty$.

If the potential V has some regularity and grows fast enough at infinity (see Assumption 1) there is an associated *equilibrium measure* μ_{eq}, such that the sequence $\{\mu_N\}_N$ (where $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ denotes the *empirical measure* of the points) converges almost surely to μ_{eq}. Moreover the law of $\{\mu_N\}_N$ satisfies a Large Deviation Principle (LDP) at speed $\frac{\beta}{2}N^2$ on the space $\mathcal{P}(\mathbb{R}^2)$ of probability measures, with good rate function given by

$$I(\mu) := \iint -\log|x-y| d\mu(x) d\mu(y) + \int V(x) d\mu(x). \tag{3.1.3}$$

This characterizes the *first-order* or *macroscopic* behavior of the interacting particle system. Typically, as N becomes large, the N points $x_1, \dots, x_N$ arrange themselves according to the probability density $d\mu_{\text{eq}}$, which has compact support Σ. Events that deviate from this prediction occur only with $\mathbb{P}_N^\beta$-probability of order $\exp(-N^2)$. We refer to [Ser15, Chap.2] and the references therein for a detailed exposition.

c. Microscopic behavior with macroscopic average

In this section we summarize the main result of [LS15], which describes the behavior as $N \to \infty$ of a *microscopic* quantity obtained through a *macroscopic average*.

Let $\mathcal{X}$ be the set of locally finite point configurations in $\mathbb{R}^2$, endowed with the topology of vague convergence, and let us denote by $\mathcal{P}(\mathcal{X})$ the set of Borel probability measures on $\mathcal{X}$ i.e. the set of random point processes on $\mathbb{R}^2$ (we refer to Section 3.2.2 for more details).

In [LS15] (following the line of work [SS12], [SS15b], [SS15a], [RS15], [PS15]) S. Serfaty and the author have investigated the microscopic behavior of the system by making a statement on the point processes arising when zooming in by a factor $N^{1/2}$ (which is the typical inter-particle distance) and averaging over translations in a way that we now briefly present.

For any N-tuple $\vec{X}_N$, let $x_i' = N^{1/2}x_i$, $\nu_N' := \sum_{i=1}^N \delta_{x_i'}$, let $\Sigma' := N^{1/2}\Sigma$ denote the support of μ_{eq} after rescaling, and let i_N be the map $i_N : (\mathbb{R}^2)^N \to \mathcal{P}(\mathcal{X})$ defined by

$$i_N(\vec{X}_N) := \frac{1}{|\Sigma'|}\int_{\Sigma'} \delta_{\theta_{z'}\cdot\nu_N'}\, dz',$$

where $\theta_{z'}\cdot$ denotes the action of translation by $z' \in \Sigma'$, and where δ is the Dirac mass. The map i_N transforms a N-tuple of points into the data of all the blown-up point configurations obtained by zooming in by a factor $N^{1/2}$ around any $z \in \Sigma$. Such quantities are called *empirical fields*.

We let $\mathfrak{P}_{N,\beta}$ be the push-forward of $\mathbb{P}_N^\beta$ by i_N. The main result of [LS15] gives a large deviation principle for $\{\mathfrak{P}_{N,\beta}\}_N$ at speed N, on the space of stationary random point processes. The rate function on this subset of $\mathcal{P}(\mathcal{X})$ is given by

$$\mathcal{F}_\beta(P) := \frac{\beta}{2}\mathbf{E}_P[\mathbb{W}] + \mathsf{ent}[P|\mathbf{\Pi}^1],$$

where $\mathbb{W}$ is an energy functional which will be defined later, $\mathbf{E}_P$ denotes the expectation under P, and $\mathsf{ent}[P|\mathbf{\Pi}^1]$ is the *specific relative entropy* of P with respect to the Poisson point process of intensity 1 in $\mathbb{R}^2$ (see Section 3.2.5).

This LDP characterizes the microscopic behavior only in an *averaged* way, because of the average over translations in the definition of i_N. In fact (this is still a consequence of [LS15, Theorem 1]) this description can be enhanced by replacing the average over translations in Σ' by an average over translations in arbitrary small macroscopic regions (seen in blown-up scale), for example the square $C(z_0', \varepsilon N^{1/2})$, where $\varepsilon > 0$ is fixed and $z_0' = N^{1/2} z_0$ for some z_0 in the interior of Σ (the bulk). Let us emphasize that the average still takes place at the macroscopic scale $N^{1/2}$. This is done in [LS15] by considering "tagged" empirical fields which are elements of $\mathcal{P}(\Sigma \times \mathcal{X})$ keeping track of the point around which the configuration has been zoomed, thus allowing for a macroscopic localization.

d. Microscopic behavior with mesoscopic averages

The goal of this paper is to push further the analysis of [LS15] at finer scales and to consider *mesoscopic* versions of the map i_N. In other terms we look at the empirical fields obtained by averaging over translations in $C(z_0', N^\delta)$ for $0 < \delta < 1/2$, and we obtain a LDP at speed $N^{2\delta}$ with essentially the same rate function as above. It is crucial to average over a relatively large set and although one might hope to go down to even finer scales (e.g. $O(\log^k N)$ for k large enough) we do not expect a similar result to hold for a strictly speaking *microscopic* average at scale $O(1)$ (in blown-up coordinates).

The first-order results show that the empirical measure $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ converges to the equilibrium measure μ_{eq} almost surely. As a consequence of our analysis we get a "local law" (borrowing the terminology of [BYY14a] and [TV15, Theorem 20]) which implies that μ_N and μ_{eq} are close at small scales with very high probability. A very similar local law was obtained independently in [BBNY15].

3.1.2 Preliminary notation and definitions

a. Notations

For $R > 0$ we denote by C_R the square $[-R/2, R/2]^2$ and by $C(z, R)$ the translate of C_R by $z \in \mathbb{R}^2$. We denote by $D(p, r)$ the disk of center p and radius $r > 0$. If N is fixed and $\vec{X}_N \in (\mathbb{R}^2)^N$ we denote by $\nu_N := \sum_{i=1}^N \delta_{x_i}$ and $\nu_N' := \sum_{i=1}^N \delta_{x_i'}$ (where $x_i' = N^{1/2} x_i$).

Let $0 < \delta < 1/2$. We say that an event $\mathcal{A}$ occurs with δ-overwhelming probability if

$$\limsup_{N\to\infty} N^{-2\delta} \log \mathbb{P}_N^\beta(\mathcal{A}^c) = -\infty,$$

where $\mathcal{A}^c$ is the complement of $\mathcal{A}$. In particular, for any event $\mathcal{B}$, if $\mathcal{A}$ occurs with δ-overhelming probability we have

$$\limsup_{N\to\infty} N^{-2\delta} \log \mathbb{P}_N^\beta(\mathcal{B}) = \limsup_{N\to\infty} N^{-2\delta} \log \mathbb{P}_N^\beta(\mathcal{B} \cap \mathcal{A}),$$

and the same goes for the lim inf. In other terms, when evaluating probabilities of (logarithmic) order $N^{2\delta}$ we may restrict ourselves to the intersection with any event of δ-overhelming probability.

If $\{a_N\}_N, \{b_N\}_N$ are two sequences of non-negative real numbers, we will write $a_N \preceq b_N$ if if there exists $C > 0$ such that $a_N \le C b_N$ ($\mathbb{P}_N^\beta$-a.s. if the numbers are random), and we will write $a_N \preceq_\delta b_N$ if there exists $C > 0$ such that $a_N \le C b_N$ with δ-overhelming probability.

We will write $a_N \ll N^\delta$ if there exists $\tau > 0$ such that $a_N \le N^{\delta-\tau}$ ($\mathbb{P}_N^\beta$-a.s. if the numbers are random) and $a_N \ll_{\delta'} N^\delta$ if there exists $\tau > 0$ such that $a_N \le N^{\delta-\tau}$ with δ'-overwhelming probability.

b. Equilibrium measure and splitting of the energy

Under mild hypotheses on V (see Assumption 1) it is known (see e.g. [ST97, Chap.1]) that there exists a probability measure $\mu_{\rm eq}$ with compact support Σ which is the unique minimizer of I (as in (3.1.3)) over $\mathcal{P}(\mathbb{R}^2)$ (the set of probability measures). Defining ζ as

$$\zeta(x) := \int_{\mathbb{R}^2} -\log|x-y| d\mu_{\rm eq}(x) + \frac{V}{2} - \left(\iint_{\mathbb{R}^2} -\log|x-y| d\mu_{\rm eq}(x) d\mu_{\rm eq}(y) - \frac{1}{2}\int_{\mathbb{R}^2} V d\mu_{\rm eq} \right),$$

we have $\zeta \geq 0$ quasi-everywhere (q.e.) in $\mathbb{R}^2$ and $\zeta = 0$ q.e. on Σ, and in fact this characterizes $\mu_{\rm eq}$ uniquely, see [Fro35]. If $N \geq 1$ is fixed we let $\mu'_{\rm eq}(x) := \mu_{\rm eq}(xN^{-1/2})$.

If $\mathcal{C}$ is a finite point configuration we define the second-order energy functional $w_N(\mathcal{C})$ as

$$w_N(\mathcal{C}) := \iint_{\triangle^c} -\log|x-y| (d\mathcal{C} - d\mu'_{\rm eq})(x)(d\mathcal{C} - d\mu'_{\rm eq})(y), \tag{3.1.4}$$

where $\triangle^c$ denotes the complement of the diagonal $\triangle$. It computes the electrostatic interaction of the electric system made of the point charges in $\mathcal{C}$ and a negatively charged background of density $\mu'_{\rm eq}$, without the infinite self-interactions of the point charges.

Let $\tilde{\zeta}(\mathcal{C}) := \int \zeta d\mathcal{C}$. It was proven in [SS15b] (see also [Ser15, Chap.3]) that the following *exact splitting formula* holds:

Lemma 3.1.1. *For any $N \geq 1$ and any $\vec{X}_N \in (\mathbb{R}^2)^N$ we have, with I as in* (3.1.3)

$$h_N(\vec{X}_N) = N^2 I(\mu_{\rm eq}) - \frac{N\log N}{2} + w_N(\nu'_N) + 2N\tilde{\zeta}(\nu_N).$$

We may thus re-write the Gibbs measure $\mathbb{P}_N^\beta$ as

$$d\mathbb{P}_N^\beta(\vec{X}_N) = \frac{1}{K_{N,\beta}} e^{-\frac{1}{2}\beta(w_N(\nu'_N)+2N\tilde{\zeta}(\nu_N))} d\vec{X}_N, \tag{3.1.5}$$

where $K_{N,\beta}$ is a new normalizing constant. The exponent $(w_N(\nu'_N)+2N\tilde{\zeta}(\nu_N))$ is expected to be typically of order N, and it was proven in [LS15, Cor. 1.5] that $\log K_{N,\beta} = -N\min \bar{\mathcal{F}}_\beta + o(N)$, where $\bar{\mathcal{F}}_\beta$ is closely related to the function $\mathcal{F}_\beta$ mentioned above.

c. Energy and entropy

Renormalized energy. In [LS15], following [SS12,SS15b,RS15,PS15], an energy functional is defined at the level of random stationary point processes (see also [Ser15, Chap.3-6]), which is the Γ-limit of $\frac{1}{N}w_N$ as $N \to \infty$. We will define it precisely in Section 3.2.4 and we denote it by $\mathbb{W}_m$ (where $m \geq 0$ is a parameter - the notation differs slightly from that of [LS15] where it corresponds to $\widetilde{\mathbb{W}}_m$). It can be thought of as the infinite-volume limit of (3.1.4) and as a way of computing the interaction energy of an infinite configuration of point charges $\mathcal{C}$ together with a negatively charged background of constant density m.

Specific relative entropy. For any $m \geq 0$ we let $\mathbf{\Pi}^m$ be the law of a Poisson point process of intensity m in $\mathbb{R}^2$. Let P be a stationary random point process on $\mathbb{R}^2$. The relative specific entropy $\mathsf{ent}[P|\mathbf{\Pi}^m]$ of P with respect to $\mathbf{\Pi}^m$ is defined by

$$\mathsf{ent}[P|\mathbf{\Pi}^m] := \lim_{R\to\infty} R^{-2}\mathrm{Ent}\left(P_{|C_R}|\mathbf{\Pi}^m_{|C_R}\right), \tag{3.1.6}$$

where $P_{|C_R}$ denotes the random point process induced in C_R, and $\mathrm{Ent}(\cdot|\cdot)$ denotes the usual relative entropy (or Kullbak-Leibler divergence) of two probability measures defined on the same probability space. We take the appropriate sign convention for the entropy so that it is non-negative: if μ, ν are two probability measures defined on the same space we let $\mathrm{Ent}\,(\mu|\nu) := \int \log\frac{d\mu}{d\nu} d\mu$ if μ is absolutely continuous with respect to ν and $+\infty$ otherwise. For more details we refer to Section 3.2.5.

d. Good control on the energy

In this paragraph we define the notion of "good control at scale δ", which expresses the fact that our particle system has good properties in any square of sidelength N^δ (after blow-up). The assumption that good control at scale δ holds will be a key point in order to prove the LDP at slightly smaller scales. Moreover we will see that the "good control" assumption can be bootstrapped, i.e. good control at scale δ implies good control at scale δ_1 for $\delta_1 < \delta$ large enough.

Let us first introduce the local electric field E^{loc} and its truncation E^{loc}_η, we will come back to these definitions in more detail in Section 3.2.3. If $\vec{X}_N$ is a N-tuple of points in $\mathbb{R}^2$, for any $0 < \eta < 1$ we denote by $\nu'_{N,\eta}$ the measure $\nu'_{N,\eta} := \sum_{i=1}^N \delta^{(\eta)}_{x'_i}$, where $\delta^{(\eta)}_{x'_i}$ denotes the uniform probability measure on the circle of center x'_i and radius η. We let E^{loc} be the associated "local electric field" $E^{\mathrm{loc}}(x) := (-\nabla \log) * (\nu'_N - \mu'_{\mathrm{eq}})$ and E^{loc}_η its truncation at scale η, defined by $E^{\mathrm{loc}}_\eta(x) := (-\nabla \log) * (\nu'_{N,\eta} - \mu'_{\mathrm{eq}})$. Finally, we denote by $\mathring{\Sigma}$ the interior of Σ.

Definition 3.1.2. *For any $0 < \delta \leq \frac{1}{2}$ we say that a good control at scale δ holds if for any $z_0 \in \mathring{\Sigma}$ and any $0 < \delta_1 < \delta$ we have with δ_1-overwhelming probability:*

1. *The number of (blown-up) points $\mathcal{N}^{z_0}_\delta$ in the square $C(z'_0, N^\delta)$ is of order $N^{2\delta}$*

$$\mathcal{N}^{z_0}_\delta \preceq_{\delta_1} N^{2\delta}. \tag{3.1.7}$$

2. *For any $0 < \eta < 1$ we have*

$$\int_{C^{z_0}_\delta} |E^{\mathrm{loc}}_\eta|^2 + \mathcal{N}^{z_0}_\delta \log \eta \preceq_{\delta_1} N^{2\delta}, \tag{3.1.8}$$

which expresses the fact that the energy in the square $C(z'_0, N^\delta)$ (after blow-up) is of order $N^{2\delta}$.

Let us emphasize that (3.1.7), (3.1.8) control quantities at scale δ by looking at probabilities at scale $\delta_1 < \delta$.

3.1.3 Rate function

Let us define the *local* rate function as

$$\mathcal{F}^m_\beta(P) := \frac{\beta}{2}\mathbb{W}_m(P) + \mathsf{ent}[P|\mathbf{\Pi}^m].$$

It is a *good rate function* because both terms are good rate functions (see e.g. [LS15, Lemma 4.1]).

For any $m > 0$ we let $\mathcal{P}_{s,m}(\mathcal{X})$ be the set of random stationary point processes of intensity m. Let us define a scaling map $\sigma_m : \mathcal{P}_s(\mathcal{X}) \to \mathcal{P}_s(\mathcal{X})$ such that $\sigma_m(P)$ is the push-forward of P by $\mathcal{C} \mapsto m^{-1/2}\mathcal{C}$. It is easy to see that σ_m induces a bijection from $\mathcal{P}_{s,m}(\mathcal{X})$ to $\mathcal{P}_{s,1}(\mathcal{X})$ for any $m > 0$.

It is proven (see [LS15, Def. 2.4, Lemma 4.2]) that

Lemma 3.1.3. *The map σ_m induces a bijection between the minimizers of $\mathcal{F}^m_\beta$ over $\mathcal{P}_{s,m}(\mathcal{X})$ and the minimizers of $\mathcal{F}^1_\beta$ over $\mathcal{P}_{s,1}(\mathcal{X})$.*

We may now state our main results.

3.1.4 Statement of the results

If $z_0 \in \mathring{\Sigma}$ and $0 < \delta_1 < 1/2$ are fixed, let us define the map $i^{z_0}_{N,\delta_1} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ as

$$i^{z_0}_{N,\delta_1}(\mathcal{C}) := N^{-2\delta_1} \int_{C(z_0', N^{\delta_1})} \delta_{\theta'_z \cdot (\mathcal{C} \cap C(z_0', N^{\delta_1}))} dz'.$$

Such quantities are called empirical fields. We denote by $\mathfrak{P}^{z_0}_{N,\beta,\delta_1}$ the law of the push-forward of $\mathbb{P}^{\beta}_N$ by the map $i^{z_0}_{N,\delta_1}$ - in other words: the empirical field observed around z_0 by averaging at the mesoscopic scale N^{δ_1}. Finally we denote by m_{eq} the density of μ_{eq} (see Assumption 1).

Theorem 11. *For any $0 < \delta_1 < 1/2$, for any z_0 in $\mathring{\Sigma}$, the sequence $\{\mathfrak{P}^{z_0}_{N,\beta,\delta_1}\}_N$ obeys a large deviation principle at speed $N^{2\delta_1}$ with good rate function $\left(\mathcal{F}^{m_{\mathrm{eq}}(z_0)}_{\beta} - \min \mathcal{F}^{m_{\mathrm{eq}}(z_0)}_{\beta}\right)$.*

Moreover a good control at scale δ_1 holds in the sense of Definition 3.1.2.

Theorem 11 tells us in particular that the behavior around $z_0 \in \mathring{\Sigma}$ depends on V only through the value $m_{\mathrm{eq}}(z_0)$, and in view of Lemma 3.1.3 it has only the effect of scaling the configurations. This yields another example of the *universality* phenomenon: the small scale behavior of the particle system is essentially independent of the choice of V.

The first consequence of Theorem 11 is a bound on the discrepancy i.e. the difference between the number of points of ν'_N in a given square and the mass given by μ'_{eq}.

Corollary 3.1.4. *Let $z_0 \in \mathring{\Sigma}$, let $0 < \delta < 1/2$ and $\delta_1 \in (\delta/2, \delta)$. We have*

$$\left| \int_{C(z_0', N_1^{\delta})} d\nu'_N - d\mu'_{\mathrm{eq}} \right| \ll_{\delta_1} N^{\frac{4}{3}\delta}.$$

For $\delta_1 < \delta$ close to δ, the bound $N^{\frac{4}{3}\delta}$ on the difference is much smaller than the typical value of each term, of order $N^{2\delta_1}$. It allows us to prove a *local law* in the following sense:

Corollary 3.1.5. *Let $z_0 \in \mathring{\Sigma}$ and $0 < \delta \leq 1/2$ be fixed. Let f be a C^1 function (which may depend on N) such that f is supported in $C(z_0', N^{\delta})$. Then for any $\delta/2 < \delta_1 < \delta$ we have*

$$N^{-2\delta} \left| \int_{C(z_0', N^{\delta})} f(d\nu'_n - d\mu'_{\mathrm{eq}}) \right| \preceq_{\delta_1} \|\nabla f\|_{\infty} N^{\delta_1} + \|f\|_{\infty} N^{-2\delta/3}.$$

In particular if $f(z) = \tilde{f}(N^{-\delta}(z - z_0'))$ for some compactly supported C^1 function $\tilde{f}$ then $\|\nabla f\|_{\infty} \preceq N^{-\delta}$ and $\|f\|_{\infty} \preceq 1$, thus we get

$$N^{-2\delta} \left| \int_{C(z_0', N^{\delta})} f(d\nu'_n - d\mu'_{\mathrm{eq}}) \right| \ll_{\delta_1} 1.$$

a. Comments and open questions.

In the statement of the results we restrict ourselves to the following setting: we first pick a point z_0 in the interior of Σ (called the *bulk*) and then look at the point process in $C(z', N^{\delta})$ with $N^{-1/2} z' = z_0$. A careful inspection of the proof shows that we might have taken z' depending on N more finely, e.g. by considering a sequence z' with $N^{-1/2} z' \to z_0 \in \mathring{\Sigma}$, while keeping the same conclusions. It does not seem possible to take $z' \to z_0 \in \partial\Sigma$ (the "edge case") in general because the density m_{eq} may vanish near the boundary - however, this does not happen in the standard example of the quadratic potential, in which the density is constant up to the boundary of the

support. Our analysis might be done in the edge case at a scale $\delta \geq \delta_c$ depending on the speed at which $m_{\mathrm{eq}}(z)$ vanishes, but we do not pursue this goal here.

The minimizers of the rate function are unknown in general, however it is proven in [LS15, Corollary 1.4] that the Ginibre point process minimizes $\mathcal{F}^1_\beta$ over $\mathcal{P}_{s,1}(\mathcal{X})$ for $\beta = 2$. We do not know whether uniqueness of the minimizers holds for $\beta = 2$, nor for any value of β. Uniqueness of the minimizers for some $\beta > 0$ would imply that the empirical fields have a limit in law as $N \to \infty$, which would heuristically correspond to some "β-Ginibre" random point process. In that case, our results shows that the hypothetical convergence

$$\text{Empirical field averaged at scale } N^\delta \to \beta\text{-Ginibre}$$

holds at arbitrarily fine scales $\delta > 0$, which would hint at the convergence in law of the non-averaged point process ν'_N to the conjectural β-Ginibre point process.

Another open question is the behavior of the minimizers as $\beta \to \infty$ (the low-temperature limit). The *crystallization conjecture* (see e.g. [BL15] for a review) predicts that the minimum of $\mathbb{W}_1$ on $\mathcal{P}_{s,1}(\mathcal{X})$ is (uniquely) attained by the random stationary point process associated to the triangular lattice. In the high-temperature limit, it is proven in [Leb15b, Theorem 2] that minimizers of $\mathcal{F}^1_\beta$ converge (in a strong sense) to $\mathbf{\Pi}^1$ as $\beta \to 0$.

The result of [LS15] and most of the methods used in this paper are valid in a broader setting than the two-dimensional, logarithmic case, in particular we could think of treating the $1d$ log-gas (i.e. the β-ensembles). It turns out that an adaptation of the present method in the one-dimensional case allows one to improve the result of [LS15] to finer, mesoscopic scales, however, we have been unable so far to go down to the finest scale $N^{-1+\varepsilon}$ and we hope to return on this question in a subsequent work.

3.1.5 Plan of the paper and sketch of the proof

In Section 3.2 we introduce some notation and we give the definitions of the main objects used throughout the paper, as well as their key properties. In Section 3.3 we gather preliminary results about the energy w_N and we prove the main technical tool, called the "screening lemma". Section 3.4 is devoted to the proof of a LDP upper bound and Section 3.4 to the lower bound. We combine these two steps to prove Theorem 11 in Section 3.6, together with Corollary 3.1.4 and 3.1.5. Section 3.7 is devoted to intermediate results which we postpone there.

Let us now sketch how the proof of Theorem 11 goes. The basic idea is a bootstrap argument, we find that there exists $t < 1$ such that

$$\text{Good control at scale } \delta \longrightarrow \begin{cases} \text{Large deviations at scale } \delta_1 \\ \text{Good control at scale } \delta_1 \end{cases} \quad \text{for all } t\delta \leq \delta_1 < \delta. \tag{3.1.9}$$

Once good control at scale $\delta = 1/2$ is established, Theorem 11 follows. A similar bootstrap argument was used in [RNS15] for studying the minimizers of w_N (which corresponds to the $\beta = +\infty$, or zero temperature case).

The main obstruction to obtaining LDP for empirical fields in our context is the non-locality of the energy (3.1.4): due to the long-range nature of the interactions, it is hard to localize the energy in a given square in such a way that it only depends of the point configuration in this square. Another way of seeing it is that $E^{\mathrm{loc}}(x)$ depends *a priori* on the whole configuration $\vec{X}_N$ and not only on the points close to x.

To prove (3.1.9) we rely on the following steps: let $z_0 \in \mathring{\Sigma}$ be fixed. For the sake of simplicity let us assume that $w_N(\vec{X}_N) = \frac{1}{2\pi}\int_{\mathbb{R}^2} |E^{\mathrm{loc}}|^2$, where E^{loc} is the local electric field defined in Section d. (see also Section 3.2.3).

1. For any $\vec{X}_N$, we split the energy $w_N(\vec{X}_N)$ as

$$\int_{\mathbb{R}^2} |E^{\mathrm{loc}}|^2 = \int_{C(z_0', N^{\delta_1})} |E^{\mathrm{loc}}|^2 + \int_{C(z_0', N^{\delta_1})^c} |E^{\mathrm{loc}}|^2,$$

   and we split $\vec{X}_N$ as $\vec{X}^{\mathrm{in}} + \vec{X}^{\mathrm{out}}$ where $\vec{X}^{\mathrm{in}}$ is the point configuration in $C(z_0', N^{\delta_1})$ and $\vec{X}^{\mathrm{out}}$ is the point configuration in $C(z_0', N^{\delta_1})^c$.

2. We define $F^{\mathrm{in}}(\vec{X}^{\mathrm{in}})$ (resp. $F^{\mathrm{out}}(\vec{X}^{\mathrm{out}})$) as the minimal energy of an electric field associated to $\vec{X}^{\mathrm{in}}$ (resp. $\vec{X}^{\mathrm{out}}$). We thus have

$$w_N(\vec{X}_N) \geq F^{\mathrm{in}}(\vec{X}^{\mathrm{in}}) + F^{\mathrm{out}}(\vec{X}^{\mathrm{out}}).$$

   The two terms in the right-hand side become independent (they depend from two distinct sets of variables).

3. Inserting the previous inequality into the expression of the Gibbs measure (3.1.5) we obtain, for any event $\mathcal{A}$ "concerning" $\vec{X}^{\mathrm{in}}$

$$\mathbb{P}_N^\beta(\mathcal{A}) \leq \frac{1}{K_{N,\beta}} \left(\int_{\mathcal{A}} e^{-\frac{1}{2}\beta F^{\mathrm{in}}(\vec{X}^{\mathrm{in}})} d\vec{X}^{\mathrm{in}} \right) \left(\int e^{-\frac{1}{2}\beta (F^{\mathrm{out}}(\vec{X}^{\mathrm{out}}) + 2N\tilde{\zeta}(\vec{X}^{\mathrm{out}}))} d\vec{X}^{\mathrm{out}} \right). \quad (3.1.10)$$

   This can be used to prove a first LDP upper bound (taking $\mathcal{A} = \{i_{N,\delta_1}^{z_0}(\vec{X}^{\mathrm{in}}) \in B(P, \varepsilon)\}$) or a first "good control" estimate (taking $\mathcal{A} = \{F^{\mathrm{in}}(\vec{X}^{\mathrm{in}}) \gg N^{2\delta_1}\}$).

4. Then we need to prove that (3.1.10) is sharp (at scale δ_1). Given $\vec{X}^{\mathrm{out}}$ and $\vec{X}^{\mathrm{in}}$, it amounts to be able to reconstruct (a family of) point configurations $\vec{X}_N \approx \vec{X}^{\mathrm{out}} + \vec{X}^{\mathrm{in}}$ such that $w_N(\vec{X}_N) \leq F^{\mathrm{in}}(\vec{X}^{\mathrm{in}}) + F^{\mathrm{out}}(\vec{X}^{\mathrm{out}}) + o(N^{2\delta_1})$. This is where the screening procedure is used: we modify $\vec{X}^{\mathrm{out}}$ and the associated electric field a little bit (this procedure follows the line of work [SS15b, SS15a, RS15, PS15] and is called *screening* for reasons that will appear later) so that we may glue together $\vec{X}^{\mathrm{in}}$ and the new $\vec{X}^{\mathrm{out}}$ and create an electric field compatible with the new (slightly modified) point configuration $\vec{X}_N$. It is then a general fact that $w_N(\vec{X}_N)$ (the energy of the *local* electric field associated to $\vec{X}_N$) is the smallest energy in a wide class of compatible electric fields.

   In particular, proving a partial converse to (3.1.10) allows us to estimate the "local partition function"

$$\frac{1}{K_{N,\beta}} \left(\int e^{-\frac{1}{2}\beta (F^{\mathrm{out}}(\vec{X}^{\mathrm{out}}) + 2N\tilde{\zeta}(\vec{X}^{\mathrm{out}}))} d\vec{X}^{\mathrm{out}} \right),$$

   and also to show a LDP lower bound. Combined with the estimates of the previous step, it proves (3.1.9).

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3.2 Notations, assumptions and main definitions

3.2.1 Assumption on the potential

Assumption 1. *The potential V is such that*

1. *V is lower semi-continuous (l.s.c.) and bounded below.*
2. *The set $\{x \in \mathbb{R}^2 \mid V(x) < \infty\}$ has positive logarithmic capacity.*

3. We have $\lim_{|x|\to\infty} \frac{V(x)}{2} - \log|x| = +\infty$.

These first three conditions ensure that the equilibrium measure μ_{eq} *is well-defined and has compact support* Σ. *Furthermore we ask that the measure* μ_{eq} *has a density* m_{eq} *which is* κ*-Hölder in* Σ, *for some* $0 < \kappa \leq 1$

$$|m_{\text{eq}}(x) - m_{\text{eq}}(y)| \leq |x-y|^{\kappa}. \tag{3.2.1}$$

If V is C^2, it is known that μ_{eq} is absolutely continous with respect to the Lebesgue measure on $\mathbb{R}^2$ and its density coincides with ΔV in Σ. Thus in particular (3.2.1) is satisfied as soon as V is $C^{2,\kappa}$. Let us observe that the third assumption (that V is "strongly confining") could be slightly relaxed into

$$\liminf_{|x|\to\infty} \frac{V(x)}{2} - \log|x| > -\infty,$$

i.e. V is only "weakly confining", in which case the support Σ might not be compact (see [Har12] for a proof of the first-order LDP in this case). We believe that Theorem 11 should extend to the non-compact case as well, since it is really a *local* result, but we do not pursue this goal here.

3.2.2 Point configurations and point processes

a. Point configurations.

If B is a Borel set of $\mathbb{R}^2$ we denote by $\mathcal{X}(B)$ the set of locally finite point configurations in B or equivalently the set of non-negative, purely atomic Radon measures on B giving an integer mass to singletons. We will often write $\mathcal{C}$ for $\sum_{p\in\mathcal{C}} \delta_p$. We endow the set $\mathcal{X} := \mathcal{X}(\mathbb{R}^2)$ (and the sets $\mathcal{X}(B)$ for B Borel) with the topology induced by the topology of weak convergence of Radon measure (also known as vague convergence or convergence against compactly supported continuous functions), these topologies are metrizable and we fix a compatible distance $d_{\mathcal{X}}$.

b. Volume of configurations.

Let B be a Borel set of $\mathbb{R}^2$. For any $N \geq 1$, let $\sim_N$ be the equivalence relation on B^N defined as $(x_1,\dots,x_N) \sim_N (y_1,\dots,y_N)$ if and only if there exists a permutation $\sigma \in \mathfrak{S}_N$ (the symmetric group on N elements) such that $x_i = y_{\sigma(i)}$ for $i = 1,\dots,N$. We denote by $B^N/\mathfrak{S}_N$ the quotient set and by π_N the canonical projection $B^N \to B^N/\mathfrak{S}_N$. The set of finite point configurations in B can be identified to $\{\emptyset\} \cup \bigcup_{N=1}^{+\infty} B^N/\mathfrak{S}_N$.

If $\mathcal{A} \subset B^N/\mathfrak{S}_N$ we define $\hat{\mathcal{A}} \subset B^N$ as $\hat{\mathcal{A}} := \bigcup_{\mathcal{C}\in\mathcal{A}} \mathcal{C}$. It is easy to see that $\hat{\mathcal{A}}$ is the largest subset of B^N such that the (direct) image of $\hat{\mathcal{A}}$ by π_N is $\mathcal{A}$.

We will call "the volume of $\mathcal{A}$" and write (with a slight abuse of notation) $\mathbf{Leb}^{\otimes N}(\mathcal{A})$ the quantity $\mathbf{Leb}^{\otimes N}(\hat{\mathcal{A}})$.

c. Random point process.

A random point process is a probability measure on $\mathcal{X}$. We denote by $\mathcal{P}_s(\mathcal{X})$ the set of stationary random point processes i.e. those which are invariant under (push-forward by) the natural action of $\mathbb{R}^2$ on $\mathcal{X}$ by translations. We endow $\mathcal{P}_s(\mathcal{X})$ with the topology of weak convergence of probability measures, and we fix a compatible distance $d_{\mathcal{P}(\mathcal{X})}$, e.g. the 1-Wasserstein distance. Throughout the text we will denote by $B(P,\varepsilon)$ the closed ball of center P and radius ε for $d_{\mathcal{P}(\mathcal{X})}$.

3.2.3 Electric systems and electric fields

a. Finite electric system.

We will call an "electric system" a couple $(\mathcal{C}, \mu)$ where $\mathcal{C}$ is a point configuration and μ is a non-negative measurable bounded function in $\mathbb{R}^2$. We say that the system is finite if $\mathcal{C}$ is finite and μ is compactly supported. We say that the system is neutral if it is finite and $\int_{\mathbb{R}^2} d\mathcal{C} = \int_{\mathbb{R}^2} \mu(x)dx$.

b. Electric fields.

Let $1 < p < 2$ be fixed. We define the set of electric fields Elec as the set of vector fields in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$-\mathrm{div}\, E = 2\pi\,(\mathcal{C} - \mu) \text{ in } \mathbb{R}^2 \tag{3.2.2}$$

for some electric system $(\mathcal{C}, \mu)$. When (3.2.2) holds we say that E is compatible with $(\mathcal{C}, \mu)$ in $\mathbb{R}^2$ and we denote it by $E \in \mathsf{Elec}(\mathcal{C}, \mu)$. If K is a compact subset of $\mathbb{R}^2$ with piecewise C^1 boundary we let $\mathsf{Elec}(\mathcal{C}, \mu, K)$ be the set of electric fields which are compatible with $(\mathcal{C}, \mu)$ in K i.e. such that

$$-\mathrm{div}\, E = 2\pi\,(\mathcal{C} - \mu) \text{ in } K.$$

We denote by Elec^0 the set of decaying electric fields, such that $E(z) = O(|z|^{-2})$ as $|z| \to \infty$. We let $\mathsf{Elec}^0(\mathcal{C}, \mu, K)$ be the set of electric fields which are compatible with $\mathcal{C}, \mu$ in K *and* decay.

c. Local electric fields.

If $(\mathcal{C}, \mu)$ is a finite electric system there is a natural compatible electric field, namely the local electric field defined as $E^{\mathrm{loc}} := -\nabla \log * (\mathcal{C} - \mu)$. We also define the "local electric potential" $H^{\mathrm{loc}} := -\log * (\mathcal{C} - \mu)$. The scalar field H^{loc} corresponds physically to the electrostatic potential generated by the point charges of $\mathcal{C}$ together with a background of "density" μ. The vector field E^{loc} can be thought of as the associated electrostatic field. It is easy to see that E^{loc} fails to be in L^2_{loc} because it blow ups like $|x|^{-1}$ near each point of $\mathcal{C}$, however E^{loc} is in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for any $1 < p < 2$.

Truncation procedure. The renormalization procedure of [RS15, PS15] uses a truncation of the singularities which we now recall. We define the truncated Coulomb kernel as follows: for $0 < \eta < 1$ and $x \in \mathbb{R}^2$, let $f_\eta(x) = (-\log|x| - \log\eta)_+$. If $(\mathcal{C}, \mu)$ is an electric system and $E \in \mathsf{Elec}(C, \mu)$ we let

$$E_\eta(X) := E(X) - \sum_{p \in \mathcal{C}} \nabla f_\eta(X - p).$$

3.2.4 Renormalized energy

a. For finite point configurations.

It follows from the definition, and the fact that $-\log$ is (up to a constant) the Coulomb kernel in dimension 2, that $-\Delta H^{\mathrm{loc}} = 2\pi(\mathcal{C} - \mu)$, where H^{loc} denotes the local electric potential associated to a finite electric system $(\mathcal{C}, \mu)$. We may thus observe that $w_N(\mathcal{C})$ (defined in (3.1.4)) can be written

$$w_N(\mathcal{C}) \approx -\frac{1}{2\pi} \int H^{\mathrm{loc}} \Delta H^{\mathrm{loc}}$$

(up to diagonal terms). Using $E^{\mathrm{loc}} = \nabla H^{\mathrm{loc}}$ and integrating by parts we obtain heuristically $w_N(\mathcal{C}) = \int_{\mathbb{R}^2} |E^{\mathrm{loc}}|^2$. However, this computation does not make sense because E^{loc} fails to be in

L^2 around each point charge (and indeed the diagonal is excluded in the definition of w_N). The correct way of giving a sense to "$w_N(\mathcal{C}) \approx \int_{\mathbb{R}^2} |E^{\mathrm{loc}}|^2$" is to use a *renormalization* procedure, using the truncation at scale η defined above. The following is proven in [SS15b].

Lemma 3.2.1. *For any $N \geq 1$ and any $\vec{X}_N \in (\mathbb{R}^2)^N$, we have*

$$w_N(\nu'_N) = \frac{1}{2\pi} \lim_{\eta \to 0} \int_{\mathbb{R}^2} \left(|E^{\mathrm{loc}}_\eta|^2 + N \log \eta \right). \tag{3.2.3}$$

Moreover w_N is bounded below on $(\mathbb{R}^2)^N$ by $O(N)$.

b. For infinite electric fields.

Let $(\mathcal{C}, \mu)$ be an electric system and $E \in \mathsf{Elec}(\mathcal{C}, \mu)$. We let $\mathcal{W}_\eta(E)$ be

$$\mathcal{W}_\eta(E) := \frac{1}{2\pi} \limsup_{R \to \infty} R^{-2} \int_{C_R} \left(|E_\eta|^2 + \mu \log \eta \right).$$

The renormalized energy of E is then defined as $\mathcal{W}(E) := \limsup_{\eta \to 0} \mathcal{W}_\eta(E)$.

c. For (random) infinite point configurations.

If $(\mathcal{C}, \mu)$ is an electric system with μ constant equal to some $m > 0$ we define

$$\mathbb{W}_m(\mathcal{C}) = \inf_{E \in \mathsf{Elec}(\mathcal{C}, \mu)} \mathcal{W}(E).$$

Similarly if P is a random point process we let $\mathbb{W}_m(P) = \mathbf{E}_P[\mathbb{W}_m]$ for any $m > 0$.

The following lower semi-continuity result was proven in [LS15, Lemma 4.1].

Lemma 3.2.2. *For any $m > 0$ the map $P \mapsto \mathbb{W}_m(P)$ is lower semi-continuous on $\mathcal{P}_s(\mathcal{X})$. Moreover its sub-level sets are compact. In particular, $\mathbb{W}_m$ is bounded below.*

3.2.5 Specific relative entropy

For any $P \in \mathcal{P}_s(\mathcal{X})$, the specific relative entropy $\mathsf{ent}[P|\mathbf{\Pi}^m]$ is defined as in (3.1.6) (see e.g. [Geo93]).

Lemma 3.2.3. *For any $m \geq 0$ the map $P \mapsto \mathsf{ent}[P|\mathbf{\Pi}^m]$ is well-defined on $\mathcal{P}_s(\mathcal{X})$, it is affine lower semi-continous and its sub-level sets are compact.*

Proof. We refer to [RAS09, Chap. 6] for a proof of these statements. □

a. Large deviations for the reference measure.

For $\delta > 0$, $N \geq 1$ and $S_N \in \mathbb{N}$, let $\mathbf{B}_{S_N,N,\delta}$ be the law of the Bernoulli point process with S_N points in $C(0, N^\delta)$.

Proposition 3.2.4. *Let $P \in \mathcal{P}_{s,c}(\mathcal{X})$ and $\delta > 0$, let $\{S_N\}_N$ be such that $S_N \sim_{N\to\infty} mN^{2\delta}$ for some $m > 0$. We have*

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} N^{-2\delta} \log \boldsymbol{B}_{S_N,N,\delta}(\{i^{z_0}_{N,\delta}(\mathcal{C}) \in B(P, \varepsilon)\}) = -\mathsf{ent}[P|\mathbf{\Pi}^m]. \tag{3.2.4}$$

Proof. First, let us replace $\mathbf{B}_{S_N,N,\delta}$ by $\mathbf{\Pi}^m$, the law of a Poisson point process of intensity m. It follows from [Geo93, Theorem 3.1] that

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} N^{-2\delta}\log \mathbf{\Pi}^m(\{i^{z_0}_{N,\delta}(\mathcal{C})\in B(P,\varepsilon)\}) = -\mathsf{ent}[P|\mathbf{\Pi}^m]$$

The probability under $\mathbf{\Pi}^m$ of having S_N points in $C(0,N^\delta)$ is given by $e^{-mN^{2\delta}}\frac{(mN^{2\delta})^{S_N}}{S_N!}$ and Stirling's formula yields

$$N^{-2\delta}\log\left(e^{-mN^{2\delta}}\frac{(mN^{2\delta})^{S_N}}{S_N!}\right) = -m + \frac{S_N}{N^{2\delta}}\log\frac{mN^{2\delta}}{S_N} + \frac{S_N}{N^{2\delta}} + o(1).$$

Since $S_N \sim_N mN^{2\delta}$ the right-hand side is $o(1)$. It is not hard to conclude that (3.2.4) holds, using the fact that P itself is of intensity m. For more details we refer to [LS15, Section 7.2] where a similar result is proven. □

3.3 Preliminary considerations on the energy

3.3.1 Monotonicity estimates

a. Almost monotonicity in η of the local energy.

The next lemma expresses the fact that the limit $\eta\to 0$ in (3.2.3) is almost monotonous.

Lemma 3.3.1. *Let $(\mathcal{C},\mu)$ be a neutral electric system with N points and E^{loc} be the associated local electric field. We have, for any $0<\eta<\eta_1<1$,*

$$\left(\int_{\mathbb{R}^2}|E^{\mathrm{loc}}_\eta|^2 + N\log\eta\right) - \left(\int_{\mathbb{R}^2}|E^{\mathrm{loc}}_{\eta_1}|^2 + N\log\eta_1\right) \succeq -N\|\mu\|_\infty\eta_1. \tag{3.3.1}$$

Proof. This is [PS15, Lemma 2.3]. □

Let us note that, integrating by parts, we may re-write $\int_{\mathbb{R}^2}|E^{\mathrm{loc}}_\eta|^2$ as $\int_{\mathbb{R}^2} -H^{\mathrm{loc}}_\eta\Delta H^{\mathrm{loc}}_\eta$ and (3.3.1) is really a monotonicity estimate for $\left(\int_{\mathbb{R}^2} -H^{\mathrm{loc}}_\eta\Delta H^{\mathrm{loc}}_\eta + N\log\eta\right)$ as η varies.

b. A localized monotonicity estimate.

Lemma 3.3.2. *Let $(\mathcal{C},\mu)$ be an electric system and E^{loc} be the associated local electric field. Let $R_2>10$ and let N^{in} be the number of points of $\mathcal{C}$ in C_{R_2}. We let also N^{bou} be the number of points of $\mathcal{C}$ in $C_{R_2}\backslash C_{R_2-5}$. We have for any $0<\eta_1<\eta_0<1$,*

$$\left(\int_{C_{R_2}}|E^{\mathrm{loc}}_{\eta_1}|^2 - N^{\mathrm{in}}\log\eta_1\right) - \left(\int_{C_{R_2}}|E^{\mathrm{loc}}_{\eta_0}|^2 - N^{\mathrm{in}}\log\eta_0\right) \succeq -N^{\mathrm{in}}||\mu||_\infty\eta_0$$
$$+ N^{\mathrm{bou}}\log\eta_1 + (\log\eta_1 - 1)\int_{C_{R_2}\backslash C_{R_2-5}}\left(|E^{\mathrm{loc}}_{\eta_0}|^2 + |E^{\mathrm{loc}}_{\eta_1}|^2\right).$$

Proof. It follows from the proof of [PS15, Lemma 2.4], see e.g. [PS15, Equation 2.29] and the one immediatly after. □

c. Almost monotonicity with no points near the boundary.

Lemma 3.3.3. *Let $(\mathcal{C},\mu)$ be an electric system in $\mathbb{R}^2$ and E^{loc} be the associated local electric field. Let $0<R_2$ and let $0<\eta_1<1$ be such that the smeared out charges at scale η_1 do not intersect ∂C_{R_2} i.e. $\bigcup_{p\in\mathcal{C}} B(p,\eta_1)\cap\partial C_{R_2}=\emptyset$. Let us denote by N^{in} the number of points in C_{R_2}. Then we have for any $\eta\leq\eta_1$*

$$\left(\int_{C_{R_2}}|E^{\mathrm{loc}}_{\eta}|^2+N^{\mathrm{in}}\log\eta\right)-\left(\int_{C_{R_2}}|E^{\mathrm{loc}}_{\eta_1}|^2+N^{\mathrm{in}}\log\eta_1\right)\succeq -N^{\mathrm{in}}\eta_1||\mu||_\infty \tag{3.3.2}$$

Proof. We postpone the proof to Section 3.7.1. □

3.3.2 Discrepancy estimates

Lemma 3.3.4. *Let $N\geq 1$, and let $\mathcal{C}$ be a finite point configuration in $\mathbb{R}^2$. Let E be a gradient electric field in $\mathsf{Elec}(\mathcal{C},\mu'_{\mathrm{eq}})$. For any $R>0$, let $\mathcal{D}_R$ be the discrepancy $\mathcal{D}_R:=\int_{C_R}(d\mathcal{C}-d\mu'_{\mathrm{eq}})$ in C_R.*

For any $\eta\in(0,1)$ we have

$$\mathcal{D}_R^2\min\left(1,\frac{\mathcal{D}_R}{R^2}\right)\preceq\int_{C_{2R}}|E_\eta|^2.$$

Proof. This follows from [RS15, Lemma 3.8]. □

As a corollary, we see that if a good control holds at scale δ, then the discrepancies are controlled at smaller scales.

Lemma 3.3.5. *Let $0<\delta\leq\frac{1}{2}$ and let us assume that a good control holds at scale δ. Then for any $R\in(\frac{1}{2}N^{\delta_1},2N^{\delta_1})$ with $\frac{\delta}{2}<\delta_1<\delta$ we have*

$$|\mathcal{D}_R|\ll_{\delta_1} N^{\frac{4}{3}\delta}. \tag{3.3.3}$$

Proof. Let us apply Lemma 3.3.4 with $\eta=1/2$, taking E to be the local electric field E^{loc}. It yields

$$\mathcal{D}_R^2\min\left(1,\frac{\mathcal{D}_R}{R^2}\right)\preceq\int_{C_{2R}}|E^{\mathrm{loc}}_{1/2}|^2.$$

Using the good control on the energy at scale δ (in the sense of Definition 3.1.2) we have $\int_{C_{2R}}|E^{\mathrm{loc}}_{1/2}|^2+\mathcal{N}_{2R}\log(1/2)\preceq N^{2\delta}$, and $\mathcal{N}_{2R}$ is itself $\preceq_{\delta_1}N^{2\delta}$. We thus obtain that $\mathcal{D}_R^2\min\left(1,\frac{\mathcal{D}_R}{R^2}\right)\preceq_{\delta_1}N^{2\delta}$. Then, elementary considerations imply that if $\frac{\delta}{2}<\delta_1<\delta$, we have $\mathcal{D}_R^3\preceq_{\delta_1}N^{2(\delta_1+\delta)}$ which yields (3.3.3). □

Application: number of points in a square.

Lemma 3.3.6. *Let $0<\delta\leq\frac{1}{2}$ and let us assume that a good control holds at scale δ. Then we have, for any $R\in(\frac{1}{2}N^{\delta_1},2N^{\delta_1})$ with $\frac{2}{3}\delta<\delta_1<\delta$, and for any $z_0\in\mathring{\Sigma}$, letting $\mathcal{N}_R^{z_0}:=\int_{C(z_0',R)}d\mathcal{C}$*

$$\left|\mathcal{N}_R^{z_0}-m_{\mathrm{eq}}(z_0)R^2\right|\ll_{\delta_1}N^{2\delta_1}. \tag{3.3.4}$$

Proof. We have by definition $\int_{C(z_0',R)}d\mu'_{\mathrm{eq}}=\int_{C(z_0',R)}m_{\mathrm{eq}}(z_0+tN^{-1/d})dt$. Using the Hölder assumption (3.2.1) we get $\left|\int_{C(z_0',R)}d\mu'_{\mathrm{eq}}-R^2m_{\mathrm{eq}}(z_0)\right|\preceq N^{2\delta_1+\kappa(\delta_1-\frac{1}{2})}$. Since $\kappa>0$ and $\delta_1<1/2$ we get $\left|\int_{C(z_0',R)}d\mu'_{\mathrm{eq}}-R^2m_{\mathrm{eq}}(z_0)\right|\ll N^{2\delta_1}$. Lemma 3.3.5 yields $\left|\mathcal{N}_R^{z_0}-\int_{C(z_0',R)}d\mu'_{\mathrm{eq}}\right|\ll_{\delta_1}N^{\frac{4}{3}\delta}$. Combining these two inegalities we see that if $\delta_1>\frac{2}{3}\delta$ then (3.3.4) holds. □

3.3.3 Minimality of local energy against decaying fields

Lemma 3.3.7. *Let $K \subset \mathbb{R}^2$ be a compact set with piecewise C^1 boundary. Let $(\mathcal{C}, \mu)$ be a neutral electric system in K. Let E^{loc} be the local electric field associated to $(\mathcal{C}, \mu)$ and let $E \in \mathsf{Elec}^0(\mathcal{C}, \mu)$. For any $0 < \eta < 1$ we have*

$$\int_{\mathbb{R}^2} |E_\eta^{\mathrm{loc}}|^2 \leq \int_{\mathbb{R}^2} |E_\eta|^2. \tag{3.3.5}$$

The proof is very similar that of [LS15, Lemma 3.12] and we postpone it to Section 3.7.

3.3.4 The screening lemma

Lemma 3.3.8. *There exists $C > 0$ universal such that the following holds.*

Let $z \in \mathbb{R}^2$ and for any N let $(\mathcal{C}, \mu)$ be an electric system in $\mathbb{R}^2$, let $0 < \delta_3 < \delta_2 < \delta_1 < 1/2$ and let R_1, R_2 positive be such that, letting $\eta_1 := N^{-10}$, we have

1. *$R_2 \in [N^{\delta_1} + N^{\delta_2}, N^{\delta_1} + 2N^{\delta_2}]$ and $R_1 \in [R_2 - 3N^{\delta_3}, R_2 - 2N^{\delta_3}]$.*
2. *The smeared out charges at scale η_1 do not intersect $\partial C(z, R_2)$, i.e.*

$$\bigcup_{p \in \mathcal{C}}^{N} D(p, \eta_1) \cap \partial C(z, R_2) = \emptyset.$$

3. *N^{gen} is an integer, where $N^{\mathrm{gen}} := \int_{C(z,R_1)} d\mu$.*
4. *Letting $N^{\mathrm{mid}} := \int_{C(z,R_2)\setminus C(z,R_1)} d\mathcal{C}$, it holds $N^{\mathrm{mid}} \ll N^{2\delta_1}$.*

Let us assume that μ satisfies $0 < \underline{m} \leq \mu \leq \overline{m}$ on $C_{R_2} \setminus C_{R_1}$ and that furthermore there exists $C_\mu > 0$ such that

$$\forall (x, y) \in (C_{R_2} \setminus C_{R_1})^2, |\mu(x) - \mu(y)| \leq C_\mu |x - y|^\kappa, \tag{3.3.6}$$

where κ is as in (3.2.1). Let E be in $\mathsf{Elec}(\mathcal{C}, \mu, \mathbb{R}^2 \setminus C_{R_1})$ and let $M := \int_{\partial C_{R_2}} |E_\eta|^2$.

If the following inequality is satisfied

$$M \leq C \min(\underline{m}^2, 1) N^{3\delta_3}, \tag{3.3.7}$$

then there exists a measurable family A_N^{tran} of point configurations such that for any $\mathcal{C}^{\mathrm{tran}} \in A_N^{\mathrm{tran}}$

1. *The configuration $\mathcal{C}^{\mathrm{tran}}$ is supported in $C_{R_2} \setminus C_{R_1}$.*
2. *The configuration $\mathcal{C}^{\mathrm{tran}}$ has N^{tran} points where (with $\vec{n}$ the unit normal vector)*

$$N^{\mathrm{tran}} := \int_{\partial C_{R_2}} E_{\eta_1} \cdot \vec{n} - \int_{C_{R_2} \setminus C_{R_1}} d\mu. \tag{3.3.8}$$

3. *The points of $\mathcal{C}^{\mathrm{tran}}$ are well-separated from each other and from the boundaries*

$$\min_{p_1 \neq p_2 \in \mathcal{C}^{\mathrm{tran}}} |p_1 - p_2| \succeq \overline{m}^{-1/2}, \quad \min_{p \in \mathcal{C}^{\mathrm{tran}}} \mathrm{dist}(p, \partial C_{R_2} \cup \partial C_{R_1}) \succeq \overline{m}^{-1/2}. \tag{3.3.9}$$

4. *There exists an electric field $E^{\mathrm{tran}} \in \mathsf{Elec}(\mathcal{C}^{\mathrm{tran}}, \mu, C_{R_2} \setminus C_{R_1})$ such that*
   (a) *We have*

$$E_{\eta_1}^{\mathrm{tran}} \cdot \vec{n} = \begin{cases} E_{\eta_1} \cdot \vec{n} & \text{on } \partial C_{R_2} \\ 0 & \text{on } \partial C_{R_1} \end{cases}. \tag{3.3.10}$$

(b) The energy of E^{tran} *is bounded by*

$$\int_{C_{R_2}\setminus C_{R_1}} |E^{\mathrm{tran}}_{\eta_1}|^2 \preceq N^{\delta_3} M + N^{\delta_1+3\delta_3} C_\mu N^{\kappa\delta_3} + N^{\delta_1+\delta_3}\log N. \tag{3.3.11}$$

Moreover the volume of A_N^{tran} *is bounded below as follows*

$$\log \mathbf{Leb}^{\otimes N^{\mathrm{tran}}}(A_N^{\mathrm{tran}}) \succeq -(\overline{m} N^{\delta_1+\delta_3} + N^{\delta_1} + M)\log \overline{m}. \tag{3.3.12}$$

Proof. The result is inspired from the "screening lemmas" of [SS15b, Prop. 6.4], [PS15, Prop 6.1] and [LS15, Prop 5.2]. Our setting is slightly simpler because we have ruled out the possibility of having point charges close to the boundary ∂C_{R_2}. Here we screen the electric field "from the inside" (a similar procedure is used in [RNS15]) whereas the aforementioned screening results were constructing a field E such that $E \equiv 0$ outside a certain hypercube. Another difference is that in the present lemma we really need to deal with a variable background μ.

In the rest of the proof we set $l = N^{\delta_3}$.

Step 1. *Subdividing the domain.* We claim that we may split $C_{R_2}\setminus C_{R_1}$ into a finite family of rectangles $\{H_i\}_{i\in I}$ with sidelengths in $[l/2, 2l]$ such that letting $\bar{m}_i := \frac{1}{|H_i|}\int_{H_i} d\mu$ and

$$m_i := \bar{m}_i + \frac{1}{|H_i|}\frac{1}{2\pi}\int_{\partial C_{R_2}\cap \partial H_i} E_\eta \cdot \vec{n},$$

we have for any $i \in I$

$$|m_i - \bar{m}_i| < \frac{1}{2}\underline{m}, \quad m_i|H_i| \in \mathbb{N}.$$

The fact that we may split $C_{R_2}\setminus C_{R_1}$ into a finite family of rectangles $\{H_i\}_{i\in I}$ with sidelengths in $[l/2, 2l]$ is elementary. Using Cauchy-Schwarz's inequality we see that

$$\left|\frac{1}{|H_i|}\int_{\partial C_{R_2}\cap\partial H_i} E_\eta\cdot\vec{n}\right| \preceq \frac{1}{l^{3/2}}\left(\int_{\partial C_{R_2}} |E_\eta|^2\right)^{1/2} = M^{1/2} l^{-3/2}$$

hence assuming (3.3.7) (with C large enough) we have $|m_i - \bar{m}_i| < \frac{1}{2}\underline{m}$ for any tiling of $C_{R_2}\setminus C_{R_1}$ by rectangles of sidelengths $\in [l/2, 2l]$. It remains to show that we may obtain a tiling such that $m_i|H_i| \in \mathbb{N}$. We have by definition

$$m_i|H_i| = \int_{H_i} d\mu + \frac{1}{2\pi}\int_{\partial C_{R_2}\cap\partial H_i} E_\eta\cdot\vec{n}.$$

Increasing the sidelengths of H_i by t (with $t \le l/10$) increases $\int_{H_i} d\mu$ by a quantity $\succeq \underline{m} l t$ whereas it changes $\int_{\partial C_{R_2}\cap\partial H_i} E_\eta\cdot\vec{n}$ by a quantity $\preceq \sqrt{tM}$. We thus see that if (3.3.7) holds, up to modifying the boundaries of H_i a little bit (e.g. changing the sidelengths by a quantity less than $l/10$) we can ensure that each $m_i|H_i| \in \mathbb{N}$.

We may then subdivide further each rectangle H_i into a finite family of rectangles $\{R_\alpha\}_{\alpha\in I_i}$ which all have an area m_i^{-1} and sidelengths bounded above and below by universal constants times $m_i^{-1/2}$ (for a proof of this fact, see [PS15, Lemma 6.3.]). Let us emphasize that since μ is bounded above we have $m_i^{-1} \preceq 1$.

Step 2. *Defining the transition field and configuration.* For $i \in I$ we let $E^{(1,i)} := \nabla h^{(1,i)}$ where $h^{(1,i)}$ is the solution to

$$-\Delta h^{(1,i)} = 2\pi(m_i - \mu) \text{ in } H_i,$$

such that $\nabla h^{(1,i)}\cdot\vec{n} = -E_{\eta_1}\cdot\vec{n}$ on each side of H_i which is contained in ∂C_{R_2} and $\nabla h^{(1,i)}\cdot\vec{n} = 0$ on the other sides.

We also let, for any $\alpha \in I_i$, $h^{(2,\alpha)}$ be the solution to

$$-\Delta h^{(2,\alpha)} = 2\pi \left(\delta_{p_\alpha} - m_i\right) \text{ in } R_\alpha, \nabla h^{(2,\alpha)} \cdot \vec{n} = 0 \text{ on } \partial R_\alpha,$$

where p_α is the center of R_α. We define $E^{(2,i)}$ as $E^{(2,i)} := \sum_{\alpha \in I_i} \nabla h^{(2,\alpha)}$. Finally we define the *transition field* E^{tran} as $E^{\text{tran}} := \sum_{i \in I} E^{(1,i)} + E^{(2,i)}$, and the *transition configuration* $\mathcal{C}^{\text{tran}}$ as $\mathcal{C}^{\text{tran}} := \sum_{i \in I} \sum_{\alpha \in I_i} \delta_{p_\alpha}$. It is easy to see that

$$-\text{div}\left(E^{\text{tran}}\right) = 2\pi \left(\mathcal{C}^{\text{tran}} - \mu\right) \text{ in } C_{R_2} \backslash C_{R_1}, \quad E^{\text{tran}}_{\eta_1} \cdot \vec{n} = \begin{cases} E_{\eta_1} \cdot \vec{n} & \text{on } \partial C_{R_2} \\ 0 & \text{on } \partial C_{R_1} \end{cases}. \tag{3.3.13}$$

In particular (3.3.8) and (3.3.10) hold.

Step 3. *Controlling the energy.* For any $i \in I$ the energy of $E^{(1,i)}$ can be bounded using Lemma 3.7.1 as follows

$$\int_{H_i} |E^{(1,i)}|^2 \preceq l \int_{\partial H_i \cap \partial C_{R_2}} |E_\eta|^2 + l^4 ||\mu - \bar{m}_i||^2_{L^\infty(H_i)},$$

and using the Hölder assumption (3.3.6) on μ we have $||\mu - \bar{m}_i||^2_{L^\infty(H_i)} \preceq C_\mu l^\kappa$ hence

$$\int_{H_i} |E^{(1,i)}|^2 \preceq l \int_{\partial H_i \cap \partial C_{R_2}} |E_\eta|^2 + C_\mu l^{4+\kappa}.$$

For any $\alpha \in I_i$ we also have, again by standard estimates

$$\int_{R_\alpha} |\nabla h^{(2,\alpha)}_{\eta_1}|^2 \preceq -\log \eta_1 \preceq \log N \text{ by choice of } \eta_1.$$

The number of rectangles R_α for $\alpha \in I_i$, $i \in I$ is bounded by the volume of $C_{R_2} \backslash C_{R_1}$ hence is $\preceq N^{\delta_1 + \delta_3}$. We deduce that

$$\int_{C_{R_2} \backslash C_{R_1}} |E^{\text{tran}}_{\eta_1}|^2 \preceq l \int_{\partial C_{R_2}} |E_\eta|^2 + \#I C_\mu l^{4+\kappa} + N^{\delta_1 + \delta_3} \log N,$$

where $\#I$ denotes the cardinality of I. We may observe that $\#I \preceq N^{\delta_1 + \delta_3} l^{-2}$ and get (using that $l = N^{\delta_3}$)

$$\int_{C_{R_2} \backslash C_{R_1}} |E^{\text{tran}}_{\eta_1}|^2 \preceq N^{\delta_3} M + N^{\delta_1 + 3\delta_3} C_\mu N^{\kappa \delta_3} + N^{\delta_1 + \delta_3} \log N.$$

which yields (3.3.11).

Step 4. *Constructing a family.* As was observed in [PS15, Remark 6.7], [LS15, Proposition 5.2], we may actually construct a whole family of configurations $\mathcal{C}^{\text{tran}}$ and associated electric fields E^{tran} such that (3.3.13) and (3.3.11) hold. Indeed, since $\mu \leq \overline{m}$ the sidelengths of R_α are $\succeq \overline{m}^{-1/2}$ hence we may move each of the points p_α (for $\alpha \in I_i$, $i \in I$) arbitrarily within a disk of radius $\frac{1}{10}\overline{m}^{-1/2}$ and proceed as above (so that (3.3.9) is conserved). This creates a volume of configurations of order $\overline{m}^{-N^{\text{tran}}}$. To get (3.3.12) it suffices to observe that

$$N^{\text{tran}} \preceq \overline{m} N^{\delta_1 + \delta_3} + N^{\delta_1} + M,$$

which follows from (3.3.8) by applying the Cauchy-Schwarz inequality and using the fact that $R_2^2 - R_1^2 \preceq N^{\delta_1 + \delta_3}$. □

3.4 A large deviation upper bound

Given a point z_0 and a scale δ_1 we localize the energy $w_N(\vec{X}_N)$ by splitting it between the "interior part" (which, roughly speaking, corresponds to the L^2-norm of E^{loc} on $C(z_0', N^{\delta_1})$) and the "exterior part", then we replace both quantities by smaller ones which are independent (this corresponds to separating variables). Roughly speaking, it allows us to localize the problem on $C(z_0', N^{\delta_1})$ by deriving a "local energy" (which corresponds to the interior part) and a "local partition function" (the exponential sum of contributions of the exterior part). This lower bound on the energy will be complemented by a matching upper bound in Section 3.5.

3.4.1 Definition of good interior and exterior boundaries and energies

The decomposition between interior and exterior part will be done at the boundary of some "good" square, not much larger than $C(z_0', N^{\delta_1})$. We give the definition of good exterior and interior boundaries, with an abuse of notation which is discussed in the next subsection.

Definition 3.4.1 (Exterior boundary)**.** *Let $1/2 > \delta > \delta_1 > \delta_2 > 0$ and $\eta_0 > 0$ be fixed, let $N \geq 1$ and let $\eta_1 := N^{-10}$. Let $z_0 \in \mathring{\Sigma}$, let $R_2 > 0$ and let $\vec{X}^{\mathrm{out}}$ be a point configuration in $C(z_0', R_2)^c$.*

Exterior fields. *We say that E is in $\mathsf{Elec}^{\mathrm{out}}(\vec{X}^{\mathrm{out}})$ if E is in $\mathsf{Elec}^0(\mathcal{C}, \mu_{\mathrm{eq}}', \mathbb{R}^2)$ for some point configuration $\mathcal{C}$ with N points such that $\mathcal{C} = \vec{X}^{\mathrm{out}}$ on $C(z, R_2)^c$.*

Good exterior boundary. *Let $E \in \mathsf{Elec}^{\mathrm{out}}(\vec{X}^{\mathrm{out}})$. We say that $\partial C(z_0', R_2)$ is a good exterior boundary for E if the following conditions are satisfied:*

1. *We have*
$$R_2 \in [N^{\delta_1} + N^{\delta_2}, N^{\delta_1} + 2N^{\delta_2}]. \tag{3.4.1}$$
2. *The smeared out charges at scale η_1 do not intersect $\partial C(z_0', R_2)$*
$$\bigcup_{p \in \mathcal{C}} D(p, \eta_1) \cap \partial C(z_0', R_2) = \emptyset. \tag{3.4.2}$$
3. *The energy near $\partial C(z_0', R_2)$ is controlled as follows*
$$\int_{C(z_0', R_2) \setminus C(z_0', R_2 - 5)} |E_{\eta_0}|^2 \preceq N^{2\delta - \delta_2} |\log \eta_0|, \tag{3.4.3}$$
$$\int_{C(z_0', R_2) \setminus C(z_0', R_2 - 5)} |E_{\eta_1}|^2 \preceq N^{2\delta - \delta_2} \log N, \tag{3.4.4}$$
$$\int_{\partial C(z_0', R_2)} |E_{\eta_1}|^2 \preceq N^{2\delta - \delta_2} (\log N)^2. \tag{3.4.5}$$
4. *We have, letting $N^{\mathrm{bou}} := \int_{C(z_0', R_2) \setminus C(z_0', R_2 - 5)} d\mathcal{C}$*
$$N^{\mathrm{bou}} \ll N^{2\delta_1}. \tag{3.4.6}$$
5. *We have, letting $N^{\mathrm{in}} := \int_{C(z_0', R_2)} d\mathcal{C}$*
$$N^{\mathrm{in}} \preceq N^{2\delta_1}. \tag{3.4.7}$$
6. *We have, letting $\mathcal{N}_{\delta_1}^{z_0} := \int_{C(z_0', N^{\delta_1})} d\mathcal{C}$*
$$N^{\mathrm{in}} - \mathcal{N}_{\delta_1}^{z_0} \ll N^{2\delta_1} \text{ and } |\mathcal{N}_{\delta_1}^{z_0} - m_{\mathrm{eq}}(z_0) N^{2\delta_1}| \ll N^{2\delta_1}. \tag{3.4.8}$$
This very last inequality does not depend on R_2, but it is convenient to include it in the definition of the exterior boundary.

Best exterior energy. *Let N^{out} denote the number of points of $\vec{X}^{\mathrm{out}}$ in $C(z_0', R_2)^c$. We define $F^{\mathrm{out}}(\vec{X}^{\mathrm{out}})$ as*

$$F^{\mathrm{out}}(\vec{X}^{\mathrm{out}}) := \frac{1}{2\pi} \min_E \lim_{\eta \to 0} \left(\int_{C(z_0', R_2)^c} |E_\eta|^2 + N^{\mathrm{out}} \log \eta \right)$$

where the min *is taken over the set of electric fields E satisfying $E \in \mathsf{Elec}^{\mathrm{out}}(\vec{X}^{\mathrm{out}})$ and such that $\partial C(z_0', R_2)$ is a good exterior boundary for E (if there is no such field we set $F^{\mathrm{out}}(\vec{X}^{\mathrm{out}}) = +\infty$). The minimum is achieved because on the one hand $E \mapsto |E_\eta|^2$ is coercive for the weak L^p_{loc} topology, and on the other hand the L^2 norm is coercive and lower semi-continuous for the weak L^2 topology.*

Definition 3.4.2 (Interior boundary)**.** *Let $1/2 > \delta_1 > \delta_2 > \delta_3 > 0$ and $\eta_0 > 0$ be fixed. Let $N \geq 1$ and let $z_0 \in \mathring{\Sigma}$, let $R_1 > 0$, let $R_2 > 0$ such that* (3.4.1) *holds and let $\vec{X}^{\mathrm{in}}$ be a point configuration in $C(z_0', R_2)$.*

Interior fields. *Let $E \in \mathsf{Elec}(\mathbb{R}^2)$. We say that E is in $\mathsf{Elec}^{\mathrm{in}}(\vec{X}^{\mathrm{in}})$ if E is in $\mathsf{Elec}^0(\mathcal{C}, \mu'_{\mathrm{eq}}, \mathbb{R}^2)$ for some $\mathcal{C} \in \mathcal{X}(\mathbb{R}^2)$ such that $\mathcal{C} = \vec{X}^{\mathrm{in}}$ on $C(z_0', R_2)$.*

Good interior boundary. *We say that $\partial C(z_0', R_1)$ is a good interior boundary for $\vec{X}^{\mathrm{in}}$ if*

1. *We have*
$$R_1 \in [R_2 - 3N^{\delta_3}, R_2 - 2N^{\delta_3}]. \tag{3.4.9}$$
2. *N^{gen} is an integer, where $N^{\mathrm{gen}} := \int_{C(z_0', R_1)} d\mu'_{\mathrm{eq}}$.*
3. *Letting $N^{\mathrm{mid}} := \int_{C(z_0', R_2) \setminus C(z_0', R_1)} d\mathcal{C}$, it holds*
$$N^{\mathrm{mid}} \ll N^{2\delta_1}. \tag{3.4.10}$$

Best interior energy. *Let N^{in} denote the number of points of $\vec{X}^{\mathrm{in}}$ in $C(z_0', R_2)$. For any $0 < \eta_0 < 1$ we define $F^{\mathrm{in}}_{\eta_0}$ as*

$$F^{\mathrm{in}}_{\eta_0}(\vec{X}^{\mathrm{in}}) := \frac{1}{2\pi} \min_E \left(\int_{C(z_0', R_2)} |E^{\mathrm{loc}}_{\eta_0}|^2 + N^{\mathrm{in}} \log \eta_0 \right),$$

where the minimum is taken over the set of electric fields E such that $E \in \mathsf{Elec}^{\mathrm{in}}(\vec{X}^{\mathrm{in}})$ (if $\mathsf{Elec}^{\mathrm{in}}(\vec{X}^{\mathrm{in}})$ is empty we set $F^{\mathrm{in}}(\vec{X}^{\mathrm{in}}) = +\infty$).

3.4.2 Finding good boundaries

The conditions (3.4.3), (3.4.4), (3.4.5), (3.4.6), (3.4.7), (3.4.8), (3.4.10) are *asymptotic* as $N \to \infty$, in particular they do not make sense for a finite N (nonetheless, (3.4.1), (3.4.2) and (3.4.9) do). Strictly speaking one thus has to consider sequences $R_2 = R_2(N)$ and $R_1 = R_1(N)$.

Lemma 3.4.3. *Let $1/2 \geq \delta > \delta_1 > \delta_2 > \delta_3 > 0$ and $\eta_0 > 0$ be fixed, with $\delta_1 > 2\delta/3$. Let $z_0 \in \mathring{\Sigma}$ and $\eta_0 > 0$ be fixed. Assume that good control at scale δ holds and let $\delta_1 \in (\delta/2, \delta)$. With δ_1-overhelming probability, there exists R_1, R_2 such that, letting $\vec{X}^{\mathrm{in}} = \vec{X}'_N \cap C(z_0', R_2)$ and $\vec{X}^{\mathrm{out}} = \vec{X}'_N \cap C(z_0', R_2)^c$, we have*

1. *$\partial C(z_0', R_2)$ is a good exterior boundary for $\vec{X}^{\mathrm{out}}, E^{\mathrm{loc}}$.*
2. *$\partial C(z_0', R_1)$ is a good interior boundary for $\vec{X}^{\mathrm{in}}$.*

Proof of Lemma 3.4.3. First we look for a good exterior boundary $\partial C(z_0', R_2)$. The good control at scale δ implies that

$$\int_{C(z_0',N^\delta)} |E_{\eta_1}^{\mathrm{loc}}|^2 \preceq_{\delta_1} N^{2\delta}(1+|\log\eta_1|) \preceq N^{2\delta}\log N, \tag{3.4.11}$$

$$\int_{C(z_0',N^\delta)} |E_{\eta_0}^{\mathrm{loc}}|^2 \preceq_{\delta_1} N^{2\delta}(1+|\log\eta_0|) \preceq N^{2\delta}|\log\eta_0|. \tag{3.4.12}$$

In view of (3.4.11), (3.4.12), by the pigeonhole principle, we may find (with δ_1-overhelming probability) an interval $[R-10, R+10]$ included in $[N^{\delta_1}+N^{\delta_2}, N^{\delta_1}+2N^{\delta_2}]$ such that

$$\int_{C(z_0',R+10)\setminus C(z_0',R-10)} \left(|E_{\eta_1}^{\mathrm{loc}}|^2 + |E_{\eta_0}^{\mathrm{loc}}|^2\right) \preceq N^{2\delta-\delta_2}(\log N + |\log\eta_0|). \tag{3.4.13}$$

We may find N^2 disjoint strips of width $2N^{-2}(\log N)^{-1}$ in $[R-8, R+8]$. In view of (3.4.13), there are at least $N^2/2$ such strips on which the integral of $|E_{\eta_1}^{\mathrm{loc}}|^2+|E_{\eta_0}^{\mathrm{loc}}|^2$ is $\preceq N^{2\delta-\delta_2-2}(\log N+|\log\eta_0|)$. On the other hand there are at most N point charges, thus since $\eta_1 = N^{-10}$ by the pigeonhole principle we may moreover assume that no smeared out charge (at scale η_1) intersects the strips. Finally a mean value argument on one of these strips shows that we may find R_2 such that (3.4.5) and (3.4.2) holds. By (3.4.13) we also have (3.4.3) and (3.4.4).

Next, we look for a good interior boundary $\partial C(z_0', R_1)$. Since z_0 is in the interior of Σ, the density m_{eq}' is bounded above and below by positive constants on $C(z_0', N^\delta)$ (for N large enough) and thus the derivative of $R \mapsto \int_{C(z_0',R)} dm_{\mathrm{eq}}'$ is bounded above and below by (positive) constants times N^{δ_1} on $C(z_0', 2N^{\delta_1})$. Hence we may find $R_1 \in [R_2 - 3N^{\delta_3}, R_2 - 2N^{\delta_3}]$ such that $\int_{C(z_0',R_1)} d\mu_{\mathrm{eq}}'$ is an integer, hence the first two points of the definition of a good interior boundary are satisfied.

We have (3.4.7) with δ_1-overhelming probability according to the good control at scale δ. Since $\delta_1 > 2\delta/3$, the discrepancy estimates of Lemma 3.3.6 imply that, up to an error $\ll_{\delta_1} N^{2\delta_1}$ we have

$$N^{\mathrm{bou}} = m_{\mathrm{eq}}(z_0)(R_2^2 - (R_2-10)^2) \preceq_{\delta_1} N^{\delta_1} \ll N^{2\delta_1}$$
$$N^{\mathrm{tran}} = m_{\mathrm{eq}}(z_0)(R_2^2 - R_1^2) \preceq_{\delta_1} N^{\delta_1+\delta_2} \ll N^{2\delta_1},$$

which proves (3.4.6) and (3.4.10). We obtain (3.4.8) with similar arguments. □

3.4.3 A first LDP upper bound

Proposition 3.4.4. *Let $1/2 \geq \delta > \delta_1 > \delta_2 > \delta_3 > 0$ be fixed with $\delta_1 > 2\delta/3$ and $2\delta - \delta_2 < \delta_1$. Let $z_0 \in \mathring{\Sigma}$ and $0 < \eta_0 < 1$ be fixed. Assume that a good control holds at scale δ.*

For any $P \in \mathcal{P}_s(\mathcal{X})$ and any $\varepsilon > 0$ we have

$$\log \mathfrak{P}_{N,\beta,\delta_1}^{z_0}(B(P,\varepsilon)) \leq -\log K_{N,\beta}$$
$$+ \log \max_{R_1,R_2,N^{\mathrm{out}}} \binom{N}{N^{\mathrm{out}}} \left(\int_{i_{N,\delta_1}^{z_0}(\vec{X}^{\mathrm{in}})\in B(P,\varepsilon)} e^{-\frac{1}{2}\beta F_{\eta_0}^{\mathrm{in}}(\vec{X}^{\mathrm{in}})} d\vec{X}^{\mathrm{in}}\right)\left(\int e^{-\frac{1}{2}\beta(F^{\mathrm{out}}(\vec{X}^{\mathrm{out}})+N\tilde{\zeta}(\vec{X}^{\mathrm{out}}))} d\vec{X}^{\mathrm{out}}\right)$$
$$+ N^{2\delta_1} O(\eta_0). \tag{3.4.14}$$

Let us first give some precisions about (3.4.14). The max on $R_1, R_2, N^{\mathrm{out}}$ is restricted to the set of $\{R_1, R_2\}$ such that (3.4.1) and (3.4.9) hold, with N^{out} between 1 and N. Once N^{out} is fixed, we let $d\vec{X}^{\mathrm{in}} = dx_1 \dots dx_{N^{\mathrm{in}}}$ and $d\vec{X}^{\mathrm{out}} = dx_1 \dots dx_{N^{\mathrm{out}}}$, with $N^{\mathrm{out}} + N^{\mathrm{in}} = N$.

Proof. Using the definition of $\mathfrak{P}^{z_0}_{N,\beta,\delta_1}$ and of the Gibbs measure $\mathbb{P}^\beta_N$ we have

$$\mathfrak{P}^{z_0}_{N,\beta,\delta_1}\left(B(P,\varepsilon)\right) = \frac{1}{K_{N,\beta}} \int_{(i^{z_0}_{N,\delta_1})^{-1}(B(P,\varepsilon))} e^{-\frac{1}{2}\beta(w_N(\nu'_N)+N\tilde{\zeta}(\nu_N))} d\vec{X}_N. \tag{3.4.15}$$

Step 1. *Finding good boundaries.* We apply Lemma 3.4.3. With δ_1-overhelming probability we obtain R_1, R_2 such that (3.4.1) and (3.4.9) holds, and such that, letting $\vec{X}^{\rm in} = \vec{X}'_N \cap C(z'_0, R_2)$ and $\vec{X}^{\rm out} = \vec{X}'_N \cap C(z'_0, R_2)^c$, we have

1. $\partial C(z'_0, R_2)$ is a good exterior boundary for $\vec{X}^{\rm out}, E^{\rm loc}$.
2. $\partial C(z'_0, R_1)$ is a good interior boundary for $\vec{X}^{\rm in}$.

In order to prove (3.4.14) we may restrict ourselves, in the right-hand side of (3.4.15), to any event of δ_1-overhelming probability, and we will henceforth assume that good boundaries exist.

Step 2. *Splitting the energy w_N.* For any $\vec{X}_N$, let R_1, R_2 be as above. We have, using (3.2.3)

$$w_N(\nu'_N) = \lim_{\eta\to 0}\left(\int_{C(z'_0,R_2)} |E^{\rm loc}_\eta|^2 + N^{\rm in}\log\eta\right) + \lim_{\eta\to 0}\left(\int_{C(z'_0,R_2)^c} |E^{\rm loc}_\eta|^2 + N^{\rm out}\log\eta\right). \tag{3.4.16}$$

Since (3.4.2) holds we may apply Lemma 3.3.3 to $(\vec{X}'_N, \mu'_{\rm eq})$ and R_2, η_1 as above. It yields, since $\eta_1 = N^{-10}$,

$$\lim_{\eta\to 0}\left(\int_{C(z'_0,R_2)} |E^{\rm loc}_\eta|^2 + N^{\rm in}\log\eta\right) \geq \left(\int_{C(z'_0,R_2)} |E^{\rm loc}_{\eta_1}|^2 + N^{\rm in}\log\eta_1\right) + o(1), \text{ as } N\to\infty.$$

We may then apply Lemma 3.3.2 to $(\nu'_N, \mu'_{\rm eq})$ and R_2, with η_0, η_1 as above. The number $N^{\rm in}$ of points in $C(z'_0, R_2)$ is controlled by (3.4.7), the number $N^{\rm bou}$ of points near the boundary is controlled by (3.4.6) and the energy near the boundary is controlled by (3.4.3) and (3.4.4). We obtain

$$\left(\int_{C(z'_0,R_2)} |E^{\rm loc}_{\eta_1}|^2 + N^{\rm in}\log\eta_1\right) - \left(\int_{C(z'_0,R_2)} |E^{\rm loc}_{\eta_0}|^2 + N^{\rm in}\log\eta_0\right) \\ \succeq -N^{2\delta_1}\eta_0 - N^{\rm bou}\log N - N^{2\delta-\delta_2}(\log N)^2,$$

which may be simplied (assuming that $2\delta - \delta_2 < \delta_1$, which will be later ensured by the choice (3.5.4)) as

$$\left(\int_{C(z'_0,R_2)} |E^{\rm loc}_{\eta_1}|^2 + N^{\rm in}\log\eta_1\right) - \left(\int_{C(z'_0,R_2)} |E^{\rm loc}_{\eta_0}|^2 + N^{\rm in}\log\eta_0\right) \succeq -N^{2\delta_1}\eta_0. \tag{3.4.17}$$

Using Definition 3.4.2 we thus get

$$\frac{1}{2\pi}\lim_{\eta\to 0}\left(\int_{C(z'_0,R_2)} |E^{\rm loc}_\eta|^2 + N^{\rm in}\log\eta\right) - F^{\rm in}_{\eta_0}(\vec{X}^{\rm in}) \succeq -\eta_0 N^{2\delta_1}. \tag{3.4.18}$$

On the other hand, we may write, using Definition 3.4.1

$$\frac{1}{2\pi}\lim_{\eta\to 0}\left(\int_{C(z'_0,R_2)^c} |E^{\rm loc}_\eta|^2 + N^{\rm out}\log\eta\right) \geq F^{\rm out}(\vec{X}^{\rm out}). \tag{3.4.19}$$

Combining (3.4.16), (3.4.18), (3.4.19) and inserting them into (3.4.15) yields (3.4.14). □

3.4.4 Good control upper bound

Lemma 3.4.5. *Let $1/2 \geq \delta > \delta_1 > \delta_2 > \delta_3 > 0$ be fixed with $\delta_1 > 2\delta/3$. Let $z_0 \in \mathring{\Sigma}$ and $0 < \eta_0 < 1$ be fixed. Assume that a good control holds at scale δ.*

Let us denote by $\mathcal{E}_M$ the event

$$\mathcal{E}_M := \left\{ \int_{C(z_0', N^{\delta_1})} |E^{\mathrm{loc}}_{\eta_0}|^2 + \mathcal{N}^{z_0}_{\delta_1} \log \eta_0 \geq 2\pi N^{2\delta_1} M \right\}. \tag{3.4.20}$$

We have

$$\log \mathbb{P}^{\beta}_N(\mathcal{E}_M) \leq -\log K_{N,\beta} - \frac{\beta}{2} M + \log \max_{R_1, R_2, N^{\mathrm{out}}} \binom{N}{N^{\mathrm{out}}} \left(\int e^{-\beta(F^{\mathrm{out}}(\vec{X}^{\mathrm{out}}) + N\tilde{\zeta}(\vec{X}^{\mathrm{out}}))} d\vec{X}^{\mathrm{out}} \right) + N^{2\delta_1} O(\eta_0). \tag{3.4.21}$$

Proof. We follow the same steps as in the proof of Proposition 3.4.4, replacing the event $B(P, \varepsilon)$ by $\mathcal{E}_M$.

Let us write

$$\left(\int_{C(z_0', R_2)} |E^{\mathrm{loc}}_{\eta_0}|^2 + N^{\mathrm{in}} \log \eta_0 \right) \geq \left(\int_{C(z_0', R_1)} |E^{\mathrm{loc}}_{\eta_0}|^2 + \mathcal{N}^{z_0}_{\delta_1} \log \eta_0 \right) + (N^{\mathrm{in}} - \mathcal{N}^{z_0}_{\delta_1}) \log \eta_{\eta_0}$$

Using (3.4.17) and the definition of $\mathcal{E}_M$ we get, conditionally to $\mathcal{E}_M$

$$\left(\int_{C(z_0', R_2)} |E^{\mathrm{loc}}_{\eta_1}|^2 + N^{\mathrm{in}} \log \eta_1 \right) \geq N^{2\delta_1} M + (N^{\mathrm{in}} - \mathcal{N}^{z_0}_{\delta_1}) \log \eta_0 + O(N^{2\delta_1}).$$

Since we have $|N^{\mathrm{in}} - \mathcal{N}^{z_0}_{\delta_1}|_{\delta_1} \ll N^{2\delta_1}$ we deduce that (conditionally to $\mathcal{E}_M$), using Lemma 3.3.3

$$\lim_{\eta \to 0} \left(\int_{C(z_0', R_2)} |E^{\mathrm{loc}}_{\eta}|^2 + N^{\mathrm{in}} \log \eta \right) \geq N^{2\delta_1} M + O(N^{2\delta_1}). \tag{3.4.22}$$

Using (3.4.22) instead of (3.4.18) and arguing as in the proof of Proposition 3.4.4 gives (3.4.21). □

3.4.5 Large deviation upper bound for the interior part

In (3.4.14) we have separated the Gibbs measure into its interior and exterior parts. In the next lemma, we give a large deviation upper bound for the interior part, namely

$$\int_{i^{z_0}_{N,\delta_1}(\vec{X}^{\mathrm{in}}) \in B(P,\varepsilon)} e^{-\frac{1}{2}\beta F^{\mathrm{in}}_{\eta_0}(\vec{X}^{\mathrm{in}})} d\vec{X}^{\mathrm{in}}.$$

Up to technical details, this is a classical application of Varadhan's lemma: the large deviations for the reference point process (without the exponential term) are known from Proposition 3.2.4, and on the other hand the lower semi-continuity of the energy near a random stationary point process P implies that $F^{\mathrm{in}}_{\eta_0}(\vec{X}^{\mathrm{in}}) \geq \mathbb{W}_{m_{\mathrm{eq}}(z_0)}(P)$ on $\{i^{z_0}_{N,\delta_1}(\vec{X}^{\mathrm{in}}) \in B(P,\varepsilon)\}$, hence we obtain

Lemma 3.4.6. *Let $1/2 \geq \delta > \delta_1 > \delta_2 > \delta_3 > 0$ be fixed with $\delta_1 > 2\delta/3$, and let us assume that good control at scale δ holds. We have, for any R_1, R_2 satisfying* (3.4.1), (3.4.9), *and any N^{in}*

$$\limsup_{\eta_0 \to 0, \varepsilon \to 0, N \to \infty} N^{-2\delta_1} \log \int_{i^{z_0}_{N,\delta_1}(\vec{X}^{\mathrm{in}}) \in B(P,\varepsilon)} e^{-\frac{1}{2}\beta F^{\mathrm{in}}_{\eta_0}(\vec{X}^{\mathrm{in}})} d\vec{X}^{\mathrm{in}} \leq -\mathcal{F}^{m_{\mathrm{eq}}(z_0)}_{\beta}(P). \tag{3.4.23}$$

Proof. **Step 1.** *Lower semi-continuity of the energy.* We claim that

$$\liminf_{\eta_0\to 0,\varepsilon\to 0,N\to\infty} N^{-2\delta_1}\inf\left\{F^{\rm in}_{\eta_0}(\vec{X}^{\rm in}),\vec{X}^{\rm in}\in\mathcal{X}(C(z_0',R_2)),i^{z_0}_{N,\delta_1}(\vec{X}^{\rm in})\in B(P,\varepsilon)\right\}\geq \mathbb{W}_{m_{\rm eq}(z_0)}(P). \quad (3.4.24)$$

To prove (3.4.24), let $E=E(N)$ be a minimizing sequence in (3.4.24), let $\mathcal{C}=\mathcal{C}(N)$ be the associated point configuration in $C(z_0',N^{\delta_1})$ and let us define

$$P_N^{\rm elec}:=N^{-2\delta_1}\int_{C(z_0',N^{\delta_1})}\delta_{\theta'_z\cdot E}dz',\quad P_N:=N^{-2\delta_1}\int_{C(z_0',N^{\delta_1})}\delta_{\theta'_z\cdot\mathcal{C}}dz'.$$

We have, for any $m\geq 1$,

$$\mathbf{E}_{P_N^{\rm elec}}\left[\frac{1}{|C_m|}\int_{C_m}|E_{\eta_0}|^2\right]\leq N^{-2\delta_1}\int_{C(z_0',R_2)}|E_{\eta_0}|^2,$$

which proves that the push-forward of $P_N^{\rm elec}$ by $E\mapsto E_{\eta_0}$ is tight in $L^2(C_m,\mathbb{R}^2)$ for the weak topology, whereas the sequence $\{P_N^{\rm elec}\}$ itself is tight in $L^p(C_m,\mathbb{R}^2)$ (indeed when η is fixed, the L^2-norm of E_η controls the L^p-norm of E, this follows easily from Hölder's inequality, see [RS15, Lemma 3.9]). On the other hand, the sequence of random point processes $\{P_N\}$ is also tight because the expectation of the number of points in any square is bounded and, up to subsequence extraction, it converges to some $Q\in B(P,\varepsilon)$. Denoting by $Q^{\rm elec}$ a limit point of $\{P_N^{\rm elec}\}$ it is not hard to see that $Q^{\rm elec}$ is compatible with Q, and by lower semi-continuity of the L^2-norm with respect to weak convergence we have

$$\frac{1}{|C_m|}\int_{C_m}|E_{\eta_0}|^2dQ^{\rm elec}\leq\liminf_{N\to\infty}N^{-2\delta_1}\int_{C(z_0',R_2}|E_{\eta_0}|^2.$$

Up to applying a standard diagonal extraction procedure we may assume that it holds for any $m\geq 1$, hence

$$\mathbf{E}_{Q^{\rm elec}}[|E_{\eta_0}|^2]\leq\liminf_{N\to\infty}N^{-2\delta_1}\int_{C(z_0',R_2)}|E_{\eta_0}|^2.$$

Letting $\varepsilon\to 0$ and arguing as above concerning the tightness of $Q^{\rm elec}$ and Q (in $L^p_{\rm loc}$ and in $\mathcal{X}$) and for the lower semi-continuity of the norm with respect to weak convergence we obtain

$$\mathbf{E}_{P^{\rm elec}}[|E_{\eta_0}|^2]\leq\lim_{\varepsilon\to 0}\liminf_{N\to\infty}N^{-2\delta_1}\int_{C(z_0',N^{\delta_1})}|E_{\eta_0}|^2$$

where $P^{\rm elec}$ is some random electric field compatible with P. Letting $\eta_0\to 0$, using (3.4.8) and the definition of $\mathbb{W}_{m_{\rm eq}(z_0)}(P)$ we thus obtain

$$\mathbb{W}_{m_{\rm eq}(z_0)}(P)\leq\lim_{\eta_0\to 0,\varepsilon\to 0}\liminf_{N\to\infty}N^{-2\delta_1}\left(\int_{C(z_0',N^{\delta_1})}|E_{\eta_0}|^2+\mathcal{N}^{z_0}_{\delta_1}\log\eta_0\right).$$

Step 2. *Large deviations without interactions.* We claim that, on the other hand

$$\limsup_{\varepsilon\to 0,N\to\infty}N^{-2\delta_1}\log\int_{i^{z_0}_{N,\delta_1}(\vec{X}^{\rm in})\in B(P,\varepsilon)}\mathbf{1}_{C(z_0',R_2)}(\vec{X}^{\rm in})d\vec{X}^{\rm in}\leq-\mathsf{ent}[P|\mathbf{\Pi}^{m_{\rm eq}(z_0)}]. \quad (3.4.25)$$

The configuration $\vec{X}^{\rm in}$ has $N^{\rm in}$ points. Among them, $N^{\rm in}-\mathcal{N}^{z_0}_{\delta_1}$ are not affected by the constraint $i^{z_0}_{N,\delta}(\vec{X}^{\rm in})\in B(P,\varepsilon)$ because they belong to $C(z_0',N^{\delta_1})^c$, and are free to move in

$C(z_0', R_2)\backslash C(z_0', N^{\delta_1})$, but we know from (3.4.8) that $N^{\rm in} - \mathcal{N}_{\delta_1}^{z_0} \ll N^{2\delta_1}$, thus the volume contribution of these points is negligible because $N^{-2\delta_1}\log|C(z_0', R_2)\backslash C(z_0', N^{\delta_1})|^{N^{\rm in}-\mathcal{N}_{\delta_1}^{z_0}} = o(1)$. On the other hand we know, using (3.4.8), that $\mathcal{N}_{\delta_1}^{z_0} \sim_{N\to\infty} m_{\rm eq}(z_0)N^{2\delta_1}$ and then (3.4.25) follows from Proposition 3.2.4.

Step 3. *Conclusion.* Combining (3.4.24) and (3.4.25) yields (3.4.23). □

3.4.6 A second LDP upper bound

Combining Proposition 3.4.4 and Lemma 3.4.6 and letting $\eta_0 \to 0$ we obtain

Proposition 3.4.7.

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty} N^{-2\delta_1}\log \mathfrak{P}^{z_0}_{N,\beta,\delta_1}\left(B(P,\varepsilon)\right) \le -\mathcal{F}_\beta^{m_{\rm eq}(z_0)}(P) + \limsup_{N\to\infty} N^{-2\delta_1}\log K^\beta_{N,z,\delta_1} \tag{3.4.26}$$

where we let K^β_{N,z,δ_1} *be such that*

$$\log K^\beta_{N,z,\delta_1} := -\log K_{N,\beta} + \log \int_{R_1,R_2} \sum_{N^{\rm out}} \binom{N}{N^{\rm out}} \int e^{-\frac{1}{2}\beta(F^{\rm out}(\vec{X}^{\rm out})+N\tilde{\zeta}(\vec{X}^{\rm out}))} d\vec{X}^{\rm out}. \tag{3.4.27}$$

3.5 Large deviation lower bound

In this section, we derive a converse estimate to (3.4.26), by showing that splitting the energy as in Proposition 3.4.4 is essentially sharp as far as probabilities of order $\exp(-N^{2\delta_1})$ are concerned.

3.5.1 Generating microstates

In the next lemma, we recall a tool which was introduced in [LS15]. Given a stationary point process P and a large square C_R, Lemma 3.5.1 can be thought of as a way of generating a family of point configurations in C_R whose empirical field is close to P, whose interaction energy is close to the renormalized energy of P, and such that the volume of the family is optimal in view of the specific relative entropy of P.

Lemma 3.5.1. *Let* $z_0 \in \mathring{\Sigma}$ *and* $0 < \delta_2 < \delta_1 < 1/2$ *be fixed. Let* $P \in \mathcal{P}_{s,m_{\rm eq}(z_0)}(\mathcal{X})$ *such that* $\mathbb{W}_{m_{\rm eq}(z_0)}(P)$ *and* $\mathsf{ent}[P|\mathbf{\Pi}^{m_{\rm eq}(z_0)}]$ *are finite.*

For any $N \ge 1$, *let* $R_1 > 0$ *be such that* $R_1 \in (N^{\delta_1}, N^{\delta_1} + 2N^{\delta_2})$ *and* $N^{\rm gen} := \int_{C(z_0',R_1)} dm'_{\rm eq}$ *is an integer. Moreover let us assume that* $N^{\rm gen} \sim_{N\to\infty} m_{\rm eq}(z_0)N^{2\delta_1}$.

Then there exists a family $A_N^{\rm int}$ *of point configurations in* $C(z_0', R_1)$ *such that the following properties hold for any* $\mathcal{C}^{\rm int} \in A_N^{\rm int}$*:*

1. *The configuration* $\mathcal{C}^{\rm int}$ *has* $N^{\rm gen}$ *points in* $C(z_0', R_1)$.
2. *The continuous average of* $\mathcal{C}^{\rm int}$ *on* $C(z_0', N^{\delta_1})$ *is close to* P, *i.e.*

$$i^{z_0}_{N,\delta_1}(\mathcal{C}^{\rm int}) \in B(P, o(1)) \text{ as } N\to\infty. \tag{3.5.1}$$

3. *There exists an electric field* $E^{\rm int} \in \mathsf{Elec}\left(\mathcal{C}^{\rm int}, \mu'_{\rm eq}, C(z_0', R_1)\right)$ *such that*
   (a) $E^{\rm int}\cdot\vec{n} = 0$ *on* $\partial C(z_0', R_1)$, *where* $\vec{n}$ *is the unit normal vector.*

(b) The energy of E^{int} is controlled by $\mathbb{W}_{m_{\mathrm{eq}}(z_0)}(P)$

$$\frac{1}{2\pi}N^{-2\delta_1}\lim_{\eta\to 0}\left(\int_{C(z_0',R_1)}|E_\eta^{\mathrm{int}}|^2+N^{\mathrm{gen}}\log\eta\right)\leq \mathbb{W}_{m_{\mathrm{eq}}(z_0)}(P)+o(1)\ \text{as } N\to\infty, \tag{3.5.2}$$

uniformly on A_N^{int}.

4. The (logarithmic) volume of the family is close to the relative specific entropy of P

$$N^{-2\delta_1}\log \mathbf{Leb}^{\otimes N^{\mathrm{gen}}}(A_N^{\mathrm{int}})\geq -\mathsf{ent}[P|\mathbf{\Pi}^{m_{\mathrm{eq}}(z_0)}]+o(1),\ \text{as } N\to\infty. \tag{3.5.3}$$

Proof. This follows from the analysis of [LS15, Section 6]. Let us sketch the main steps here.

We fix $R>0$ and we tile $C(z_0',R_1)$ by squares of sidelength $\approx R$. We let $\{C_i\}_{i\in I}$ be this collection of squares and x_i be the center of C_i. We sample a point configuration $\mathcal{C}$ in $C(z_0',R_1)$ according to the law $\mathbf{B}_{N^{\mathrm{gen}}}$ of a Bernoulli point process with N^{gen} points, and we decompose $\mathcal{C}$ as $\mathcal{C}=\sum_{i\in I}\mathcal{C}_i$ where $\mathcal{C}_i:=\mathcal{C}\cap C_i$ is the point configuration in C_i. We form two points processes, the continuous average $M_1:=i^{z_0}_{N,\delta_1}(\mathcal{C})$ and the discrete average $M_2:=\frac{1}{\#I}\sum_{i\in I}\delta_{\mathcal{C}_i}$. Classical large deviations arguments (similar to Section 3.2.5) show that both M_1 and M_2 bleong to $B(P,\varepsilon)$ with probability $\approx\exp(-N^{2\delta_1}\mathsf{ent}[P|\mathbf{\Pi}^{\mu_{\mathrm{eq}}(z)}])$.

Then we apply to each point configuration $\mathcal{C}_i$ the "screening-then-regularization" procedure of [LS15, Section 5]. The screening procedure is similar in spirit to the one described in Lemma 3.3.8, except that here we change $\mathcal{C}_i$ to $\mathcal{C}_i^{\mathrm{scr}}$ by modifying the configuration only in a thin layer near ∂C_i and we construct an electric field E_i^{scr} compatible with $\mathcal{C}_i^{\mathrm{scr}}$ and which is screened outside C_i (whereas in Lemma 3.3.8 we rather "screen the configurations from the inside"). By gluing the fields E_i^{scr} together we define E^{scr} which is compatible with $\mathcal{C}^{\mathrm{scr}}:=\sum_{i\in I}\mathcal{C}_i^{\mathrm{scr}}$. The next task is to "regularize" the point configurations, which means to separate the pairs of points which are too close from each other. This changes $\mathcal{C}^{\mathrm{scr}}$ into $\mathcal{C}^{\mathrm{mod}}$ (which is very much like $\mathcal{C}^{\mathrm{scr}}$) and E^{scr} into an electric field E^{mod} which is still screened outside $C(z_0',R_1)$.

The energy of E^{scr} can be seen to satisfy, for any $0<\eta<1$

$$\int_{C(z_0',R_1)}|E_\eta^{\mathrm{scr}}|^2+N^{\mathrm{gen}}\log\eta=\sum_{i\in I}\int_{C_i}|(E_i^{\mathrm{scr}})_\eta|^2+N^{\mathrm{gen}}\log\eta,$$

and a certain continuity property of the energy shows that the right-hand side is smaller than $N^{2\delta_1}(\mathbb{W}_{m_{\mathrm{eq}}(z_0)}(P)+o(1))$ often enough. Passing from E^{scr} to E^{mod} does not affect this estimate, on the contrary the regularization procedure allows to bound the difference between the truncated energy $\int_{C(z_0',R_1)}|E_\eta^{\mathrm{mod}}|^2+N^{\mathrm{gen}}\log\eta$ and its limit as $\eta\to 0$. □

3.5.2 Choice of the deltas

In the rest of the proof, given $0<\delta\leq 1/2$ we will need to fix $\delta_3,\delta_2,\delta_1$ satifying some inequalities.

Lemma 3.5.2. *Let $\gamma:=\sqrt{\frac{1+\kappa/2}{1+\kappa/3}}$, with κ as in* (3.2.1)*. Since $0<\kappa\leq 1$ we have $1<\gamma\leq\frac{3\sqrt{2}}{4}\approx 1.06$. Let also $\alpha:=\frac{\gamma-1}{1-\frac{\gamma}{3}}$, we have $\alpha\in(0,1)$.*

For any $0<\delta\leq 1/2$ and any $\delta_1,\delta_2,\delta_3$ such that

$$\delta>\delta_1>\max\left(\frac{3}{4}\delta,\delta\frac{1-\alpha}{1-\alpha^2},\delta(1+\kappa/2)-\kappa/2\right),\delta_3=\frac{1}{3}\delta\gamma,\quad \delta_2=\alpha^2\delta_3+(1-\alpha^2)\delta_1, \tag{3.5.4}$$

we have $0<\delta_3<\delta_2<\delta_1<\delta$, $\delta_1>\frac{2}{3}\delta$, $3\delta_3>\delta$, $\delta_1+3\delta_3+\kappa(\delta_3-1/2)<2\delta_1$, $2\delta<\delta_2+3\delta_3$ and $2\delta-\delta_2<2\delta_1$. Moreover, if we consider the lower bound on δ_1 as a function $f(\delta)$, we have $f^{\circ k}(\delta)\to 0$ as $k\to\infty$, where $f^{\circ k}$ denotes the k-th iteration of f.

Proof. It is clear that $\delta_3 > 0$. From the fact that $\delta_1 \geq \delta(1+\kappa/2) - \kappa/2$ and the expression of δ_3 we get $\delta_1 + 3\delta_3 + \kappa(\delta_3 - 1/2) < 2\delta_1$. Since $\kappa > 0$ we have $3\delta_3 > \delta$. Since $\kappa \leq 1$ we also have $\delta_3 \leq \frac{1}{3}\sqrt{\frac{9}{8}}\delta < \frac{3}{4}\delta$ hence $\delta_3 < \delta_1$. On the other hand from the definition of δ_1 it is clear that $\frac{2}{3}\delta < \delta_1 < \delta$ because $\kappa > 0$ and $\delta \leq 1/2$. Since $\delta_1 > \delta\frac{1-\alpha}{1-\alpha^2}$ and since $\alpha < \alpha^2$, the inegality $2\delta < \delta_2 + 3\delta_3$ follows from checking that $\delta(2-(1+\alpha/3)\gamma) \leq (1-\alpha)\delta$. The inequality $2\delta < \delta_2 + 3\delta_3$ implies the last one, $2\delta - \delta_2 < 2\delta_1$, because it can be easily checked that $3\delta_3 = \delta\gamma < \frac{6}{4}\delta < 2\delta_1$.

Finally, we may observe that $f(\delta) \leq \max(\frac{3}{4}\delta, \frac{1-\alpha}{1-\alpha^2}\delta, \delta - \kappa/4)$ hence $f^{\circ k}(\delta)$ is decreasing and tends to 0 as $k \to \infty$. □

3.5.3 A LDP lower bound

We use Lemma 3.5.1 and the screening result of Lemma 3.3.8 to prove a first LDP lower bound.

Proposition 3.5.3. *Let $0 < \delta \leq 1/2$ and $z_0 \in \mathring{\Sigma}$ be fixed. Assume that a good control holds at scale δ and let us fix $\delta_1, \delta_2, \delta_3$ as in* (3.5.4)*. For any $P \in \mathcal{P}_{s,m_{\mathrm{eq}}(z_0)}(\mathcal{X})$ we have*

$$\log \mathfrak{P}^{z_0}_{N,\beta,\delta_1}\left(B(P,\varepsilon)\right) \geq -N^{2\delta_1}\mathcal{F}^{m_{\mathrm{eq}}(z_0)}_{\beta}(P) - \log K_{N,\beta} + \max_{R_1,R_2,N^{\mathrm{out}}} \log \binom{N}{N^{\mathrm{out}}} \left(\int e^{-\beta(F^{\mathrm{out}}(\vec{X}^{\mathrm{out}})+N\tilde{\zeta}(\vec{X}^{\mathrm{out}}))} d\vec{X}^{\mathrm{out}}\right) + o(N^{2\delta_1}). \quad (3.5.5)$$

The maximum $\max_{R_1,R_2,N^{\mathrm{out}}}$ is taken among $\{R_1, R_2\}$ satisfying (3.4.1) and (3.4.9) and with N^{out} between 1 and N.

Proof. By definition of $\mathfrak{P}^{z_0}_{N,\beta,\delta_1}$ and $\mathbb{P}^{\beta}_N$ it is enough to prove

$$\log \int_{(i^{z_0}_{N,\delta_1})^{-1}(B(P,\varepsilon))} e^{-\beta(w_N(\nu'_N)+N\tilde{\zeta}(\nu_N))} d\vec{X}_N \geq -N^{2\delta_1}\mathcal{F}^{m_{\mathrm{eq}}(z_0)}_{\beta}(P) + \max_{R_1,R_2,N^{\mathrm{out}}} \log \binom{N}{N^{\mathrm{out}}} \left(\int e^{-\beta(F^{\mathrm{out}}(\vec{X}^{\mathrm{out}})+N\tilde{\zeta}(\vec{X}^{\mathrm{out}}))} d\vec{X}^{\mathrm{out}}\right) + o(N^{2\delta_1}). \quad (3.5.6)$$

Let $R_1, R_2, N^{\mathrm{out}}$ be fixed. Let $\vec{X}^{\mathrm{out}}$ be a finite point configuration in $C(z'_0, R_2)^c$ such that $F^{\mathrm{out}}(\vec{X}^{\mathrm{out}})$ is finite. Let $E \in \mathsf{Elec}^{\mathrm{out}}(\vec{X}^{\mathrm{out}})$ be a minimizer in the definition of F^{out}. We claim that there exists a set $\mathcal{A}^{\mathrm{tot}}$ of N-tuples $\vec{X}_N$ such that $\vec{X}_N = \vec{X}^{\mathrm{out}}$ on $C(z'_0, R_2)^c$, that the energy is controlled uniformly on A^{tot} as follows

$$w_N(\nu'_N) \leq -(\mathbb{W}_{m_{\mathrm{eq}}(z_0)}(P) + F^{\mathrm{out}}(\vec{X}^{\mathrm{out}})) + o(N^{2\delta_1}),$$

and that the volume of A^{tot} is almost optimal

$$\log \mathbf{Leb}^{\otimes N}(A^{\mathrm{tot}}) \geq -\mathsf{ent}[P|\mathbf{\Pi}^{m_{\mathrm{eq}}(z_0)}] + \log \binom{N}{N^{\mathrm{out}}} + o(N^{2\delta_1}).$$

Step 1. *Screening E.* We may apply Lemma 3.3.8 to the point configuration $\vec{X}^{\mathrm{out}}$ and the electric field E, with $\mu = \mu'_{\mathrm{eq}}$. Let us check that the assumptions of Lemma 3.3.8 are satisfied:

1. The first condition on R_1, R_2 is satisfied by assumption (see (3.4.1) and (3.4.9)).
2. Since $F^{\mathrm{out}}(\vec{X}^{\mathrm{out}})$ is finite, (3.4.2) holds i.e. the smeared out charges at scale η_1 do not intersect $\partial C(z'_0, R_2)$.

3. The third and fourth condition on $N^{\mathrm{gen}}, N^{\mathrm{mid}}$ follow from the fact that R_1 is a good interior boundary.

From Assumption 1 we know that $\mu'_{\mathrm{eq}} \preceq 1$, and since we are blowing-up the configuration around $z_0 \in \mathring{\Sigma}$ the density m'_{eq} is bounded below on $C(z'_0, N^\delta)$ by some $\underline{m} > 0$ depending on z_0. We deduce from (3.2.1) that $|\mu'_{\mathrm{eq}}(x) - \mu'_{\mathrm{eq}}(y)| \preceq N^{-\kappa/2}|x-y|^\kappa$, hence we may chose $C_\mu = N^{-\kappa/2}$ in (3.3.6).

By definition of a good exterior boundary (see (3.4.5)) we have $\int_{\partial C(z'_0,R_2)} |E_{\eta_1}|^2 \preceq N^{2\delta-\delta_2}\log^2 N$, thus (3.3.7) is satisfied (for N large enough) as long as $2\delta < \delta_2 + 3\delta_3$ (which is ensured by the choice (3.5.4)).

We obtain a family A_N^{tran} of point configurations such that the conclusions of Lemma 3.3.8 hold.

Step 2. *Generating microstates.* Now we apply Lemma 3.5.1 with R_1 as above and obtain a family A_N^{int} of point configurations with N^{gen} points in $C(z'_0, R_1)$ together with screened electric fields E^{int} such that (3.5.1), (3.5.2), (3.5.3) are satisfied.

Step 3. *Gluing pieces together and bounding the energy.* For any $\mathcal{C}^{\mathrm{tran}} \in A_N^{\mathrm{tran}}$ and $\mathcal{C}^{\mathrm{int}} \in A_N^{\mathrm{int}}$ we form the configuration

$$\mathcal{C}^{\mathrm{tot}} := \mathcal{C}^{\mathrm{tran}} + \mathcal{C}^{\mathrm{int}} + \vec{X}^{\mathrm{out}}.$$

It is easy to check that $\mathcal{C}^{\mathrm{tot}}$ always has N points. Indeed we know that

— By Lemma 3.5.1, $\mathcal{C}^{\mathrm{int}}$ always has $\int_{C(z'_0,R_1)} d\mu'_{\mathrm{eq}}$ points.

— By construction, $\mathcal{C}^{\mathrm{tran}}$ has $N^{\mathrm{tran}} = \int_{\partial C(z'_0,R_2)} E_{\eta_1}\cdot\vec{n} - \int_{C(z'_0,R_2)\setminus C(z'_0,R_1)} d\mu'_{\mathrm{eq}}$ points.

— By integrating the compatibility relation of E and $\vec{X}^{\mathrm{out}}$, we get

$$N^{\mathrm{out}} = N - \int_{\partial C(z'_0,R_2)} E_{\eta_1}\cdot\vec{n} + \int_{C(z'_0,R_2)} d\mu'_{\mathrm{eq}}.$$

If E^{tran} and E^{int} are the electric fields associated to $\mathcal{C}^{\mathrm{tran}}$ and $\mathcal{C}^{\mathrm{int}}$ we also define

$$E^{\mathrm{tot}} := E^{\mathrm{tran}} + E^{\mathrm{int}} + E\mathbf{1}_{C(z'_0,R_2)^c}$$

By construction the normal derivatives of E^{tran} and E^{int} coincide on $\partial C(z'_0,R_1)$ (they both vanish), and the normal derivatives of E_η^{tran} and E_η coincide on $\partial C(z'_0,R_2)$ for any $\eta \le \eta_1$ (they coincide for η_1 by construction, but since there are no points at distance $\le \eta_1$ of $\partial C(z'_0,R_2)$ the value of the fields E_η and E_η^{tran} on $\partial C(z'_0,R_2)$ do not depend on η for $\eta \le \eta_1$). Thus E^{tot} satisfies

1. $-\mathrm{div}\, E^{\mathrm{tot}} = 2\pi(\mathcal{C}^{\mathrm{tot}} - \mu'_{\mathrm{eq}})$ in $\mathbb{R}^2$
2. E^{tot} coincides with E on $C(z'_0,R_2)^c$. In particular E^{tot} belongs to Elec^0, as E does.
3. The energy of E^{tot} is bounded as follows

$$\lim_{\eta\to 0}\left(\int_{\mathbb{R}^2} |E_\eta^{\mathrm{tot}}|^2 + N\log\eta\right) \le F^{\mathrm{out}}(\vec{X}^{\mathrm{out}}) + N^{2\delta_1}\mathbb{W}_{m_{\mathrm{eq}}(z_0)}(P) + o(N^{2\delta_1}) \text{ as } N\to\infty. \tag{3.5.7}$$

To show (3.5.7), let us split the energy of E^{tot} as

$$\begin{aligned}\int_{\mathbb{R}^2} |E_\eta^{\mathrm{tot}}|^2 + N\log\eta &= \left(\int_{C(z'_0,R_2)^c} |E_\eta|^2 + N^{\mathrm{out}}\log\eta\right)\\ &+ \left(\int_{C(z'_0,R_2)\setminus C(z'_0,R_1)} |E_\eta^{\mathrm{tran}}|^2 + N^{\mathrm{tran}}\log\eta\right) + \left(\int_{C(z'_0,R_1)} |E_\eta^{\mathrm{int}}|^2 + N^{\mathrm{gen}}\log\eta\right)\end{aligned} \tag{3.5.8}$$

By definition of $F^{\rm out}(\vec{X}^{\rm out})$ and by the choice of E we have

$$\frac{1}{2\pi}\lim_{\eta\to 0}\left(\int_{C(z_0',R_2)^c}|E_\eta|^2+N^{\rm out}\log\eta\right)=F^{\rm out}(\vec{X}^{\rm out}). \tag{3.5.9}$$

In view of (3.5.2) we have

$$\frac{1}{2\pi}\lim_{\eta\to 0}\left(\int_{C(z_0',R_1)}|E_\eta^{\rm int}|^2+N^{\rm gen}\log\eta\right)\le N^{2\delta_1}\mathbb{W}_{m_{\rm eq}(z_0)}(P)+o(N^{2\delta_1}). \tag{3.5.10}$$

Finally, the conclusions of Lemma 3.3.8 combined with the control (3.4.5) and the fact that $C_\mu=N^{-\kappa/2}$ ensure that

$$\lim_{\eta\to 0}\left(\int_{C(z_0',R_2)\setminus C(z_0',R_1)}|E_\eta^{\rm tran}|^2+N^{\rm tran}\log\eta\right)\preceq N^{\delta_3}N^{2\delta-\delta_2}(\log N)^2+N^{\delta_1+3\delta_3}N^{\kappa(\delta_3-1/2)}+N^{\delta_1+\delta_3}\log N.$$

The choice of $\delta_1,\delta_2,\delta_3$ as in (3.5.4) yields

$$\lim_{\eta\to 0}\left(\int_{C(z_0',R_2)\setminus C(z_0',R_1)}|E_\eta^{\rm tran}|^2+N^{\rm tran}\log\eta\right)\ll N^{2\delta_1}. \tag{3.5.11}$$

Inserting (3.5.9), (3.5.10) and (3.5.11) into (3.5.8) yields (3.5.7).

Now, using the minimality of local energy as stated in Lemma 3.3.7 and the formula (3.2.3) we conclude that

$$w_N(\mathcal{C}^{\rm tot})\le\frac{1}{2\pi}\lim_{\eta\to 0}\left(\int_{\mathbb{R}^2}|E_\eta^{\rm tot}|^2+N\log\eta\right)\le F^{\rm out}(\vec{X}^{\rm out})+N^{2\delta_1}\mathbb{W}_{m_{\rm eq}(z_0)}(P)+o(N^{2\delta_1})\text{ as }N\to\infty. \tag{3.5.12}$$

Step 4. *Volume considerations.* For any $\vec{X}^{\rm out}$ we let $A^{\rm tot}(\vec{X}^{\rm out})$ be the set of point configurations $\mathcal{C}^{\rm tot}$ obtained as above. Now, let $\mathcal{A}$ be a measurable set of finite point configurations $\vec{X}^{\rm out}$ with $N^{\rm out}$ points in $C(z_0',R_2)^c$ such that $F^{\rm out}(\vec{X}^{\rm out})$ is finite for all $\vec{X}^{\rm out}\in\mathcal{A}$. We let $\mathcal{A}^{\rm tot}$ be

$$\mathcal{A}^{\rm tot}:=\bigcup_{\vec{X}^{\rm out}\in\mathcal{A}}A^{\rm tot}(\vec{X}^{\rm out}).$$

Using the volume estimate (3.3.12) we obtain, with the choice (3.5.4)

$$\log\mathbf{Leb}^{\otimes N}(\mathcal{A}^{\rm tot})\ge\log\mathbf{Leb}^{\otimes N^{\rm out}}(\mathcal{A})+\log\mathbf{Leb}^{\otimes N^{\rm gen}}(A_N^{\rm int})+o(N^{2\delta_1})+\log\binom{N}{N^{\rm out}N^{\rm tran}N^{\rm gen}},$$

(where the last term denotes a multinomial coefficient). Using (3.5.3) and a straightforward combinatorial inequality yields

$$\log\mathbf{Leb}^{\otimes N}(\mathcal{A}^{\rm tot})\ge\log\mathbf{Leb}^{\otimes N^{\rm out}}(\mathcal{A})-N^{2\delta_1}{\rm Ent}[P|\mathbf{\Pi}^{m_{\rm eq}(z_0)}]+o(N^{2\delta_1})+\log\binom{N}{N^{\rm out}}. \tag{3.5.13}$$

This proves the claim made before Step 1.

Step 5. *Conclusion.* Combining (3.5.12) and (3.5.13), we obtain that

$$\log\int_{(i_{N,\delta_1}^{z_0})^{-1}(B(P,\varepsilon))}e^{-\frac{1}{2}\beta(w_N(\nu_N')+N\tilde{\zeta}(\nu_N))}d\vec{X}_N$$
$$\ge\log\binom{N}{N^{\rm out}}\left(\int e^{-\frac{1}{2}\beta(F^{\rm out}(\vec{X}^{\rm out})+N\tilde{\zeta}(\vec{X}^{\rm out}))}d\vec{X}^{\rm out}\right)-N^{2\delta_1}({\rm Ent}[P|\mathbf{\Pi}^{m_{\rm eq}(z_0)}]+\frac{1}{2}\beta\mathbb{W}_{m_{\rm eq}(z_0)}(P))+o(N^{2\delta_1}),$$

for any choice of R_1, R_2 and N^{out} as in the definitions 3.4.1 and 3.4.2. It yields (3.5.6). □

3.6 Conclusion

3.6.1 Proof of Theorem 11

a. Good control at macroscopic scale

Lemma 3.6.1. *Good control holds at scale $\delta = \frac{1}{2}$.*

Proof. Let $z_0 \in \mathring{\Sigma}$. Using (3.1.5) we see that

$$\log \mathbb{P}_N^\beta(w_N(\vec{X}_N) \geq NM) \leq -\log K_{N,\beta} + \log \int e^{-\frac{\beta}{2}(NM + N\tilde{\zeta}(\vec{X}_N))} d\vec{X}_N.$$

Since w_N is bounded below by $O(N)$ (see Lemma 3.2.1) we have $-\log K_{N,\beta} + \log\left(\int e^{-\frac{\beta}{2}N\tilde{\zeta}(\vec{X}_N)}\right) = O(N)$ (which may depend on β), hence we get

$$\log \mathbb{P}_N^\beta\left(w_N(\vec{X}_N) \geq M\right) \leq N\left(-\frac{\beta}{2}M + O(1)\right).$$

We deduce that for M_0 large enough, $\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N^\beta(w_N(\vec{X}_N) \geq M_0) < 0$, which in particular implies that $\limsup_{N\to\infty} \frac{1}{N^{2\delta}} \log \mathbb{P}_N^\beta(w_N(\vec{X}_N) \geq M_0)$ for any $\delta < \frac{1}{2}$, thus we have $w_N(\vec{X}_N) \preceq N$ with δ-overhelming probability. Using (3.2.3) and Lemma 3.3.1 we get for any $\eta \in (0,1)$ that

$$\lim_{\eta\to 0} \int_{\mathbb{R}^2} |E_\eta^{\text{loc}}|^2 + N\log\eta \preceq_\delta N,$$

which yields (3.1.8), and we deduce (3.1.7) from the discrepancy estimates of Lemma 3.3.4. □

b. Exponential tightness

Lemma 3.6.2. *For any $0 < \delta \leq 1/2$, if good control holds at scale δ then $\mathfrak{P}^{z_0}_{N,\beta,\delta_1}$ is exponentially tight (at speed $N^{2\delta_1}$) for any $z_0 \in \mathring{\Sigma}$ and $\frac{2}{3}\delta < \delta_1 < \delta$.*

Proof. Let $z_0 \in \mathring{\Sigma}$ and $\frac{2}{3}\delta < \delta_1 < \delta$ be fixed. The good control at scale δ, combined with Lemma 3.3.6, implies that there exists $C > 0$ such that the number of points in $C(z_0', N^{\delta_1})$ is bounded above by $CN^{2\delta_1}$ with δ_1-overhelming probability. It implies that $\mathfrak{P}^{z_0}_{N,\beta,\delta_1}$ is concentrated on the compact subset

$$\left\{P \in \mathcal{P}(\mathcal{X}), \mathbf{E}_P[\mathcal{N}(0,R)] \leq CR^2 \quad \forall R > 0\right\},$$

with δ_1-overhelming probability, which ensures exponential tightness at speed $N^{2\delta_1}$. □

c. Proof of the theorem

Proof of Theorem 11. **Step 1.** *Good control $\Longrightarrow$ LDP.*

In this first step we claim that if a good control holds at scale δ, then the LDP of Theorem 11 holds for δ_1 as in (3.5.4).

Indeed, comparing the right-hand side of (3.5.5) with the definition (3.4.27) of K^β_{N,z,δ_1} we see that $\lim_{N\to\infty} N^{-2\delta_1} \log K^\beta_{N,z,\delta_1}$ exists and that the following weak large deviation principle holds

$$\lim_{\varepsilon\to 0} \lim_{N\to\infty} N^{-2\delta_1} \log \mathfrak{P}^{z_0}_{N,\beta,\delta_1}\left(B(P,\varepsilon)\right) = -\mathcal{F}_\beta^{m_{\text{eq}}(z_0)}(P) - \lim_{N\to\infty} N^{-2\delta_1} \log K^\beta_{N,z,\delta_1}, \tag{3.6.1}$$

for any $P \in \mathcal{P}_{s,m_{\rm eq}(z_0)}(\mathcal{X})$, hence since $\mathbb{W}_{m_{\rm eq}(z_0)}(P) = +\infty$ as soon as P is not of intensity $m_{\rm eq}(z_0)$, we may write (3.6.1) for any $P \in \mathcal{P}_s(\mathcal{X})$. By exponential tightness we obtain a full large deviation inequality: for any measurable $A \subset \mathcal{P}_s(\mathcal{X})$ it holds

$$
\begin{aligned}
-\inf_{P\in \mathring{A}} \mathcal{F}_\beta^{m_{\rm eq}(z_0)}(P) - \lim_{N\to\infty} N^{-2\delta_1}\log K^\beta_{N,z,\delta_1} \\
\leq \liminf_{N\to\infty} N^{-2\delta_1}\log \mathfrak{P}^{z_0}_{N,\beta,\delta_1}(A) \leq \limsup_{N\to\infty} N^{-2\delta_1}\log \mathfrak{P}^{z_0}_{N,\beta,\delta_1}(A) \\
\leq -\inf_{P\in \bar{A}} \mathcal{F}_\beta^{m_{\rm eq}(z_0)}(P) - \lim_{N\to\infty} N^{-2\delta_1}\log K^\beta_{N,z,\delta_1}. \quad (3.6.2)
\end{aligned}
$$

In particular, taking $A = \mathcal{P}_s(\mathcal{X})$ we obtain

$$
\lim_{N\to\infty} N^{-2\delta_1}\log K^\beta_{N,z,\delta_1} = \inf \mathcal{F}_\beta^{m_{\rm eq}(z_0)}(P), \quad (3.6.3)
$$

and inserting (3.6.3) into (3.6.2) yields the LDP for $\{\mathfrak{P}^{z_0}_{N,\beta,\delta_1}\}_N$ as stated in Theorem 11.

Step 2. *Good control $\Longrightarrow$ good control.*

We now claim that if a good control holds at scale δ, then it holds at scale δ_1 with δ_1 as in (3.5.4). Combining the "good control upper bound" of Lemma 3.4.5 and the lower bound estimates which yield (3.6.3) we deduce that

$$
\log \mathfrak{P}^{z_0}_{N,\beta,\delta_1}(\mathcal{E}_M) \preceq -MN^{2\delta_1} + O(N^{2\delta_1}),
$$

where $\mathcal{E}_M$ is as in (3.4.20). In particular it implies that $\int_{C(z_0',N^{\delta_1})} |E^{\rm loc}_{\eta_0}|^2 + \mathcal{N}^{z_0}_{\delta_1}\log\eta_0 \preceq_{\delta'} N^{2\delta_1}$ for any $\delta' < \delta_1$ and any $\eta_0 \in (0,1)$. We also have $\mathcal{N}^{z_0}_{\delta_1} \preceq_{\delta_1} N^{2\delta_1}$ (since it was proven in Lemma 3.4.3 that (3.4.8) holds with δ_1-overhelming probability) hence in particular $\mathcal{N}^{z_0}_{\delta_1} \preceq_{\delta'} N^{2\delta_1}$ for $\delta' < \delta_1$.

Step 3. *Conclusion.*

Combining both steps with the initialization of Lemma 3.6.1 and the conclusions of Lemma 3.5.2 yields the proof of Theorem 11. □

3.6.2 Proof of Corollary 3.1.4

Proof. We simply combine the fact that a good control holds at any scale $0 < \delta \leq 1/2$ (which follows from Theorem 11) with Lemma 3.3.5. □

3.6.3 Proof of Corollary 3.1.5

Proof. We may split $C(z_0', N^\delta)$ into a family $\{C_i\}_{i\in I}$ of squares of sidelength $\approx N^{\delta_1}$, with $\#I \approx N^{2(\delta-\delta_1)}$. For any $i \in I$ we have, letting z_i be the center of C_i and $\mathcal{D}_i$ the discrepancy in C_i

$$
\int_{C_i} f(d\nu_N' - d\mu_{\rm eq}') = \mathcal{D}_i f(z_i) + \int_{C_i} (f(z) - f(z_i))(d\nu_N' - d\mu_{\rm eq}').
$$

Since good control holds at scale δ we have $|\mathcal{D}_i| \preceq_{\delta_1} N^{4\delta/3}$ (from the discrepancy estimates of Lemma 3.3.5) and $\int_{C(z_0',N^\delta)} d\nu_N' \preceq_{\delta_1} N^{2\delta}$. On the other hand $\int_{C(z_0',N^\delta)} d\mu_{\rm eq}' \preceq N^{2\delta}$ (because $\mu_{\rm eq}'$ is bounded above). Moreover the mean value theorem yields $|f(z) - f(z_i)| \leq N^{\delta_1}||\nabla f||_\infty$. We thus have

$$
\left|\sum_{i\in I}\int_{C_i} f(d\nu_N' - d\mu_{\rm eq}')\right| \preceq N^{2(\delta-\delta_1)}N^{4\delta/3}||f||_\infty + N^{2\delta}N^{\delta_1}||\nabla f||_\infty,
$$

hence we see that

$$N^{-2\delta}\left|\int_{C(z_0',N^\delta)} f(d\nu_n' - d\mu_{\rm eq}')\right| \preceq_{\delta_1} ||\nabla f||_\infty N^{\delta_1} + ||f||_\infty N^{-2\delta/3},$$

which concludes the proof since $\delta_1 < \delta$. □

3.7 Additional proofs

3.7.1 Proof of Lemma 3.3.3

Proof. We may decompose $E^{\rm loc}$ as $E^{\rm in} + E^{\rm out}$ where $E^{\rm in}$ is the local electric field generated by the electric system *inside* C_{R_2} and $E^{\rm out}$ is the local electric field generated by the electric system *outside* C_{R_2}. We have

$$\int_{C_{R_2}} |E_\eta^{\rm loc}|^2 = \int_{C_{R_2}} |E_\eta^{\rm in}|^2 + \int_{C_{R_2}} |E_\eta^{\rm out}|^2 + 2\int_{C_{R_2}} E_\eta^{\rm in}\cdot E_\eta^{\rm out}.$$

Since the charges outside C_{R_2} are at distance at least η_1 from ∂C_{R_2} we may replace $E_\eta^{\rm out}$ by $E^{\rm out}$ in the previous identity (in fact we have $E_\eta^{\rm out} = E^{\rm out}$ on C_R for $\eta \le \eta_1$). Integrating by parts we obtain

$$\int_{C_{R_2}} |E_\eta^{\rm loc}|^2 = \int_{C_{R_2}} -H_\eta^{\rm in}\Delta H_\eta^{\rm in} + \int_{\partial C_{R_2}} H_\eta^{\rm in} E_\eta\cdot\vec{n} + \int_{\partial C_{R_2}} H_\eta^{\rm in} E^{\rm out}\cdot\vec{n}$$

(up to additive terms which do not depend on $\eta \le \eta_1$), where $H^{\rm in}$ is the local electric potential generated by the electric system inside C_{R_2}. By assumption we have $H_\eta^{\rm in} = H_{\eta_1}^{\rm in}$ and $E_\eta^{\rm in} = E_{\eta_1}^{\rm in}$ on ∂C_{R_2}. Finally we see that

$$\int_{C_{R_2}} |E_\eta^{\rm loc}|^2 - |E_{\eta_1}^{\rm loc}|^2 = -\int_{\mathbb{R}^2}\left(H_\eta^{\rm in}\Delta H_\eta^{\rm in} - H_{\eta_1}^{\rm in}\Delta H_{\eta_1}^{\rm in}\right)$$

for any $\eta \le \eta_1$, and (3.3.2) is obtained as Lemma 3.3.1 (*cf.* the remark after the statement of Lemma 3.3.1). □

3.7.2 Proof of Lemma 3.3.7

Proof. The neutrality of the system implies that the local electric potential $H^{\rm loc}$ decays like $|x|^{-1}$ as $|x| \to \infty$ in $\mathbb{R}^2$ and $E^{\rm loc}$ decreases like $|x|^{-2}$. If the right-hand side of (3.3.5) is infinite then there is nothing to prove. If it is finite, given $M > 1$ and letting χ_M be a smooth nonnegative function equal to 1 in C_M and 0 at distance ≥ 1 from C_M, we may write

$$\begin{aligned}\int_{\mathbb{R}^2}\chi_M|E_\eta|^2 &= \int_{\mathbb{R}^2}\chi_M|E_\eta - E_\eta^{\rm loc}|^2 + \int_{\mathbb{R}^2}\chi_M|E_\eta^{\rm loc}|^2 + 2\int_{\mathbb{R}^2}\chi_M(E_\eta - E_\eta^{\rm loc})\cdot E_\eta^{\rm loc}\\ &\ge \int_{\mathbb{R}^2}\chi_M|E_\eta^{\rm loc}|^2 + 2\int_{\mathbb{R}^2}\chi_M(E_\eta - E_\eta^{\rm loc})\cdot(\nabla H_\eta^{\rm loc})\\ &= \int_{\mathbb{R}^2}\chi_M|E_\eta^{\rm loc}|^2 + 2\int_{\mathbb{R}^2} H_\eta^{\rm loc}(E_\eta - E_\eta^{\rm loc})\cdot\nabla\chi_M,\end{aligned}$$

where we have integrated by parts and we have used the fact that $E, E^{\rm loc}$ are compatible with the same configuration (hence $\operatorname{div}(E_\eta - E_\eta^{\rm loc}) = 0$). Letting $M \to \infty$, the last term tends to 0 by finiteness of the right-hand side of (3.3.5),the decay properties of $H^{\rm loc}$ and $E^{\rm loc}$ and the decay assumption on E. □

3.7.3 Auxiliary estimate for screening

Lemma 3.7.1. *Let $l > 0$ and let H be a rectangle of $\mathbb{R}^2$ with sidelengths in $(l/2, 3l/2)$. Let $g \in L^2(\partial H)$ and let m be a function on H of average $m_0 := \frac{1}{|H|}\int_H m$ such that*

$$-2\pi m_0 |H| = \int_{\partial H} g. \tag{3.7.1}$$

Then there exists a solution h to $-\Delta h = 2\pi m$ in H with $\nabla h \cdot \vec{n} = 0$ on ∂H satisfying

$$\int_H |\nabla h|^2 \preceq l \int_{\partial H} |g|^2 + l^4 ||m - m_0||^2_{L^\infty(H)}.$$

Proof. A solution exists thanks to the compatibility condition (3.7.1). We may split h as $h_1 + h_2$ where h_1 solves

$$-\Delta h_1 = 2\pi m_0 \text{ in } H, \quad \nabla h_1 \cdot \vec{n} = g \text{ on } \partial H,$$

and h_2 is the mean zero solution to

$$-\Delta h_2 = 2\pi (m - m_0) \text{ in } H, \quad \nabla h_2 \cdot \vec{n} = 0 \text{ on } \partial H.$$

In view of [RS15, Lemma 5.8] we may find h_1 satisfying

$$\int_H |\nabla h_1|^2 \preceq l \int_{\partial H} |g|^2. \tag{3.7.2}$$

We also claim that

$$\int_H |\nabla h_2|^2 \preceq l^4 ||m - m_0||^2_{L^\infty(H)}. \tag{3.7.3}$$

Indeed it is easy to check that (3.7.3) holds when $l = 1$, and the general case follows by a scaling argument.

Combining (3.7.2) and (3.7.3) concludes the proof. □

Chapitre 4

Limites de haute et basse température

Ce chapitre est constitué de l'article "Logarithmic, Coulomb and Riesz energy of point processes" [Leb15b].

Contents

4.1 Introduction

4.1.1 General setting

a. Logarithmic, Coulomb and Riesz interactions.

We consider a system of points (which we can think of as being point particles carrying a positive unit charge) in the Euclidean space $\mathbb{R}^d$ interacting via logarithmic, Coulomb or Riesz pairwise interactions

$$g(x) = -\log|x|, \quad \text{in dimension } d = 1, \tag{4.1.1}$$

$$g(x) = -\log|x|, \quad \text{in dimension } d = 2, \tag{4.1.2}$$

or in general dimension

$$g(x) = \frac{1}{|x|^s}, \quad \max(0, d-2) \leq s < d. \tag{4.1.3}$$

Cases (4.1.1) and (4.1.2) are known as one- and two-dimensional log-gases, and we will refer to them as the "logarithmic cases". One-dimensional log-gases have been extensively studied for their connection with important random matrix models known as the β-ensembles (see [For10]). The two-dimensional log-gas is known in the physics literature as a *two-dimensional one-component plasma* (see [AJ81]) and can also model non-Hermitian random matrices such as the Ginibre ensemble [Gin65]. The cases (4.1.3) correspond to higher-dimensional Coulomb gases (if $s = d-2$) or Riesz gases.

The statistical mechanics of N points $(x_1, \dots, x_N)$ interacting pairwise via g under a confining potential V at inverse temperature $\beta \in (0, +\infty)$ is given by the canonical Gibbs measure

$$d\mathbb{P}_{N,\beta}(x_1, \dots, x_N) := \frac{1}{Z_{N,\beta}} e^{-\beta\left(\sum_{i\neq j} g(x_i - x_j) + N\sum_{i=1}^N V(x_i)\right)} dx_1 \dots dx_N,$$

where $Z_{N,\beta}$ is a normalizing constant. The *macroscopic* behavior i.e. the behavior of the empirical measure $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ is well studied in the limit $N \to \infty$, see e.g. [Ser15] and the references therein. The limiting macroscopic arrangement is described by the equilibrium measure μ_{eq}, which is a probability measure on $\mathbb{R}^d$, depending on V, with compact support Σ, such that $\{\mu_N\}_N$ converges weakly to μ_{eq}, $\mathbb{P}_{N,\beta}$-a.s.

In order to study the microscopic arrangement of the particles, Sandier-Serfaty have derived in [SS15b] (see also [SS15a], [RS15], [PS15]) a *second-order* energy functional W_N which governs the fluctuations around μ_{eq}, together with an object $\mathbb{W}$ defined on infinite point configurations which is the limit of W_N as $N \to \infty$ in the sense of Γ-convergence of functionals (see [Ser15]). Let $x_i' := N^{1/d} x_i$, $\nu_N' := \sum_{i=1}^N \delta_{x_i'}$ and let μ_{eq}' be the push-forward of μ_{eq} by $x \mapsto N^{1/d}x$. The "blown-up" point configuration ν_N' encodes the position of particles at the microscopic (inter-particle) scale $N^{-1/d}$. We let

$$W_N(x_1, \dots, x_N) := \frac{1}{N} \iint_{\triangle^c} g(x-y)(d\nu_N' - d\mu_{eq}'(x)) \otimes (d\nu_N' - d\mu_{eq}'(y)), \tag{4.1.4}$$

where $\triangle$ is the diagonal. It can be seen that W_N computes (up to the factor $\frac{1}{N}$) the Coulomb/Riesz interaction of the electric system made of the finite (charged) point configuration ν_N' and of a negatively charged background of density $d\mu_{eq}'$, with itself, without the infinite self-interactions of the charges because the diagonal $\triangle$ is excluded.

An integration by parts shows that W_N may be re-written with the help of the associated electric field $E^{\text{loc}} := \nabla g * (\nu_N' - \mu_{eq}')$, whose norm is computed in a *renormalized* fashion to

take care of the singularities around each charge (we will come back to this procedure in Section 4.3.1).

Following the same procedure, a *renormalized energy* functional $\mathcal{W}$ is defined on the space of electric fields corresponding to infinite point configurations together with a uniform background of intensity 1. If $\mathcal{C}$ is a point configuration, its Coulomb/Riesz energy $\mathbb{W}(\mathcal{C})$ is then defined as

$$\mathbb{W}(\mathcal{C}) := \inf_E \mathcal{W}(E),$$

where the infimum is taken among the set of electric fields E which are compatible with $\mathcal{C}$. Finally if P is a random point process (a probability measure on point configurations) its energy is defined by

$$\mathbb{W}^{\text{elec}}(P) := \mathbf{E}_P[\mathbb{W}].$$

We refer to Section 4.3.1 for more details. The superscript "elec" is added by us and refers to this "electric" approach to the definition of a Coulomb/Riesz energy.

b. Free energy at microscopic scale.

In [LS15] S. Serfaty and the author have obtained a second-order (or *process level*) large deviation principle concerning the average microscopic behaviour of the particles under the canonical Gibbs measure at inverse temperature $\beta \in (0, +\infty)$. This behaviour is characterized by a certain random point process P (the law of a random point configuration) and it amounts to minimizing a free energy functional of the form

$$\mathcal{F}_\beta(P) := \beta \mathbb{W}^{\text{elec}}(P) + \mathsf{ent}[P|\mathbf{\Pi}] \tag{4.1.5}$$

on the space $\mathcal{P}_{s,1}(\mathcal{X})$ of translation-invariant random point processes whose mean density of points is 1. The term $\mathsf{ent}[P|\mathbf{\Pi}]$ denotes the *specific relative entropy* of P with respect to the Poisson point process of intensity 1 on $\mathbb{R}^d$, it is the infinite-volume analogue of the usual relative entropy.

From a physics perspective, knowing the minimizers of the free energy and how they behave as β varies allows one to retrieve some of the thermodynamic properties of the physical system at the microscopic scale (e.g. the existence of phase transitions). From the random matrix theory point of view, it was proven in [LS15] that an important family of point processes governing the microscopic behavior of eigenvalues, namely the Sine_β processes of Valko-Virag (see [VV09]), minimizes $\mathcal{F}_\beta$ for $\beta > 0$. Hence it would be very useful to have information on $\mathcal{F}_\beta$, its level sets and its minimizers (depending on β). A drawback of the free energy $\mathcal{F}_\beta$ is that computing it explicitly is hard. The energy term in particular is difficult to evaluate and except for the case of periodic configurations (for which exact formulas hold) no value of $\mathbb{W}$ is known and the mere finiteness of $\mathbb{W}(\mathcal{C})$ for a given point configuration $\mathcal{C}$ is unclear in general (see however [GS13] for some criteria). On the other hand, level sets of $\mathbb{W}$ are easily seen to be degenerate because small perturbations of any given configuration $\mathcal{C}$ typically do not change its energy. Concerning both issues, it turns out to be helpful to look for a definition of the energy directly at the level of stationary random point processes instead of averaging the energy computed configuration-wise.

c. The approach of Borodin-Serfaty.

In [BS13] a related notion of a renormalized energy for random point processes was introduced in the logarithmic setting (4.1.1) and (4.1.2). Given a stationary random point process P, Borodin and Serfaty proceed by periodizing the point process induced in a square (or interval) of sidelength R and computing its renormalized energy by the mean of explicit formulas valid

in the periodic setting. If $(a_1, \dots, a_N)$ is a configuration in a torus $\mathbb{T}$ of volume N in $\mathbb{R}^d$, the associated periodic point configuration has an energy

$$\mathbb{W}(a_1, \dots, a_N) = \frac{c_{d,s}^2}{N} \sum_{i \neq j} G^{\text{per}}(a_i - a_j) + c_{d,s}^2 \lim_{x \to 0} \left(G^{\text{per}}(x) + \frac{\log(x)}{c_{d,s}} \right),$$

where $c_{d,s}$ is a constant and G^{per} is a certain periodic Green function which has a logarithmic singularity at 0. Taking the expectation under P, using an expansion of G^{per} and sending $N \to \infty$ they obtain an energy $\mathbb{W}^{\text{BS}}(P)$ (our notation) which may be written, up to an additive constant, as

$$\mathbb{W}^{\text{BS}}(P) = \int_{\mathbb{R}^d} -\log|v|(\rho_{2,P}(v) - 1)dv, \tag{4.1.6}$$

where $\rho_{2,P}$ denotes the two-point correlation function of P, which can be seen as a function of one variable by stationarity (we abuse notation and let $\rho_{2,P}(v) := \rho_{2,P}(0, v)$). Using this explicit expression in terms of $\rho_{2,P}$, they are able to compute the energy $\mathbb{W}^{\text{BS}}$ for some specific point processes (e.g. the Sine_β processes for $\beta = 1, 2, 4$, and the Ginibre point process) as well as to solve minimization problems (the minimization of $\mathbb{W}^{\text{BS}}$ over a large class of determinantal point processes). However no general rigorous connection is drawn between $\mathbb{W}^{\text{BS}}$ and the electric definition $\mathbb{W}^{\text{elec}}$ which derives from the energy functional $\mathcal{W}_N$. Moreover the formulas of [BS13] only apply to random point processes for which $\rho_{2,P}(x, y) - 1$ decays fast enough as $|x - y| \to \infty$. The approach of the present paper is strongly inspired by the one of [BS13] and is an attempt to give a partial connection between $\mathbb{W}^{\text{BS}}$ and $\mathbb{W}^{\text{elec}}$.

4.1.2 Main results

The purpose of this paper is twofold. First we introduce an energy $\mathbb{W}^{\text{int}}$ ("int" as "intrinsic") defined on $\mathcal{P}_{s,1}(\mathcal{X})$ which is expressed only in terms of g and of the two-point correlation function, and we connect $\mathbb{W}^{\text{int}}$ with $\mathbb{W}^{\text{elec}}$. In a second part we use $\mathbb{W}^{\text{int}}$ to handle the energy term in $\mathcal{F}_\beta$, which allows us to describe the behavior of minimizers of $\mathcal{F}_\beta$ in the limiting cases $\beta \to 0$ (in any dimension) and $\beta \to \infty$ (in dimension 1).

a. New definition of the Coulomb/Riesz energy.

If P is a stationary random point process of intensity 1 and $\rho_{2,P}$ denotes its two-point correlation function, we define in Section 4.3.2 its "intrinsic" energy (with respect to the interaction g) as

$$\mathbb{W}^{\text{int}}(P) := \liminf_{R \to \infty} \frac{1}{R^d} \iint_{C_R^2 \setminus \triangle} g(x, y)(\rho_2(x, y) - 1)dxdy,$$

where C_R is the hypercube $[-\frac{R}{2}, \frac{R}{2}]^d$. An equivalent formulation is

$$\mathbb{W}^{\text{int}}(P) := \liminf_{R \to \infty} \frac{1}{R^d} \int_{[-R,R]^d \setminus \{0\}} g(v)(\rho_{2,P}(v) - 1) \prod_{i=1}^{d} (R - |v_i|)dv, \tag{4.1.7}$$

where $v = (v_1, \dots, v_d)$ and where we made again the abuse of notation $\rho_{2,P}(v) := \rho_{2,P}(0, v)$. The expression (4.1.7) shows similarities with (4.1.6) in the logarithmic cases.

Let us also define, in the logarithmic cases

$$\mathcal{D}^{\log}(P) := C^{\log} \limsup_{R \to \infty} \left(\frac{1}{R^d} \iint_{C_R^2} (\rho_{2,P}(x, y) - 1)dxdy + 1 \right) \log R, \tag{4.1.8}$$

where $C^{\log}$ is a constant whose value is irrelevant for our concerns.

Finally we introduce the free energy functional analogous to $\mathcal{F}_\beta$ (defined in (4.1.5))

$$\mathcal{F}'_\beta := \beta \mathbb{W}^{\text{int}} + \mathsf{ent}[\cdot|\mathbf{\Pi}],$$

or in the logarithmic cases

$$\mathcal{F}'_\beta := \beta(\mathbb{W}^{\text{int}} + \mathcal{D}^{\log}) + \mathsf{ent}[\cdot|\mathbf{\Pi}].$$

Let us recall the following definition: let X be a topological space and $f, g : X \to \mathbb{R}$ two functions. We say that g is the lower semi-continuous regularization of f if for any $x \in X$ we have

$$g(x) = \liminf_{y \to x} f(y).$$

Our first main result is

Theorem 12. *The functionals $\mathbb{W}^{\text{elec}}$ and $\mathbb{W}^{\text{int}}$ are related as follows.*

- *In the one-dimensional logarithmic case* (4.1.1)*, $\mathbb{W}^{\text{elec}}$ is the lower semi-continuous regularization of $\mathbb{W}^{\text{int}} + \mathcal{D}^{\log}$, and for any $\beta \in (0, +\infty)$, $\mathcal{F}_\beta$ is the lower semi-continuous regularization of $\mathcal{F}'_\beta$.*
- *In the non-Coulomb cases* (4.1.3) *with $s > d - 2$, $\mathbb{W}^{\text{elec}}$ is the lower semi-continuous regularization of $\mathbb{W}^{\text{int}}$, and for any $\beta \in (0, +\infty)$, $\mathcal{F}_\beta$ is the lower semi-continuous regularization of $\mathcal{F}'_\beta$.*
- *In the two-dimensional logarithmic (Coulomb) case* (4.1.2)*, we have*

$$\mathbb{W}^{\text{elec}} \leq \mathbb{W}^{\text{int}} + \mathcal{D}^{\log}$$

- *In the higher dimensional Coulomb cases* (4.1.3) *with $s = d - 2$, we have*

$$\mathbb{W}^{\text{elec}} \leq \mathbb{W}^{\text{int}}.$$

A first interest of Theorem 12 is that it provides a way of showing that a given random point process has finite energy. For example in the $d = 3$ Coulomb case, the Poisson point process of intensity 1 is easily seen to satisfy $\mathbb{W}^{\text{int}}(\mathbf{\Pi}) = 0$, hence $\mathbb{W}^{\text{elec}}(\mathbf{\Pi})$ is finite and nonpositive.

Let us emphasize that Theorem 12 is less precise in the Coulomb cases than in the cases (4.1.1) and (4.1.3) with $s > d-2$, to which we will henceforth refer as the "non-Coulomb cases".

In the following statement, by saying that two minimization problems are equivalent we mean that both functionals have exactly the same infima. If g is the lower semi-continuous regularization of f on X, then the minimization problems associated to f and g are equivalent, thus

Corollary 4.1.1. *We deduce from Theorem 12 that in the non-Coulomb cases*

1. *Minimizing $\mathbb{W}^{\text{elec}}$ on $\mathcal{P}_{s,1}(\mathcal{X})$ is equivalent to minimizing $\mathbb{W}^{\text{int}}$ (or $\mathbb{W}^{\text{int}}+\mathcal{D}^{\log}$) on $\mathcal{P}_{s,1}(\mathcal{X})$.*
2. *For any $\beta \in (0, +\infty)$, minimizing $\mathcal{F}_\beta$ on $\mathcal{P}_{s,1}(\mathcal{X})$ is equivalent to minimizing $\mathcal{F}'_\beta$ on $\mathcal{P}_{s,1}(\mathcal{X})$.*

b. Applications.

We use the explicit expression of $\mathbb{W}^{\text{int}}$ and its link with $\mathbb{W}^{\text{elec}}$ to perform simple computations, which allow us to prove the following two results concerning the minimization of $\mathcal{F}_\beta$ in the limit $\beta \to 0$ and $\beta \to \infty$.

High-temperature limit. As can be expected, when $\beta \to 0$ any minimizer of $\mathcal{F}_\beta$ gets close to the minimizer of the entropy term.

Theorem 13. *For all cases* (4.1.1), (4.1.2), (4.1.3), *the minimizers of* $\mathcal{F}_\beta$ *converge as* $\beta \to 0$ *to the law of the Poisson point process* $\mathbf{\Pi}$. *Moreover this convergence holds in entropy sense i.e.*

$$\lim_{\beta\to 0} \sup_{\mathcal{F}_\beta(P_\beta)=\min \mathcal{F}_\beta} \mathsf{ent}[P_\beta|\mathbf{\Pi}] = 0. \tag{4.1.9}$$

In the special case of one-dimensional log-gases, as proven in [LS15] a minimizer of $\mathcal{F}_\beta$ is the Sine_β process of Valko-Virag [VV09]. Hence our method yields another proof for a recent result of Allez and Dumaz [AD14]:

Corollary 4.1.2. *As* $\beta \to 0$ *the* Sine_β *point process converges weakly in the space of Radon measure (endowed with the topology of vague convergence) to the law of a Poisson point process on* $\mathbb{R}$.

Low-temperature limit. In dimension 1 we may also characterize the limit $\beta \to \infty$ (the low temperature limit) of the minimizers of $\mathcal{F}_\beta$. We let $P_{\mathbb{Z}}$ be the stationary random point process associated to the lattice $\mathbb{Z}$

$$P_{\mathbb{Z}} := \int_0^1 \delta_{x+\mathbb{Z}}\, dx, \tag{4.1.10}$$

which can also be seen as the law of the point configuration $u + \mathbb{Z}$ where u is a uniform random variable in $[0, 1]$ and where we let $x + \mathbb{Z}$ denote the point configuration $\{x + k, k \in \mathbb{Z}\}$.

Theorem 14 (Crystallization for $d = 1$)**.** *For* $d = 1$ *and in both cases* (4.1.1) *or* (4.1.3), *the random point process* $P_{\mathbb{Z}}$ *is the unique minimizer of* $\mathbb{W}^{\mathrm{elec}}$ *on* $\mathcal{P}_{s,1}(\mathcal{X})$. *Moreover if* $\{P_\beta\}_\beta$ *is a family of minimizers of* $\mathcal{F}_\beta$, *we have*

$$\lim_{\beta\to 0} P_\beta = P_{\mathbb{Z}}.$$

Theorem 14 is a *crystallization* result, proving the convergence to the one-dimensional crystal as $\beta \to \infty$. A similar result was proven in the one-dimensional logarithmic case in [Leb15c] by using the explicit expression available in the periodic setting together with an approximation by periodic point processes. The method here is similar in spirit but follows a simpler approach which works for Riesz cases as well.

4.1.3 Open questions

Let us now briefly mention some questions that are raised by, or related to our present study.

a. Minimizers of $\mathbb{W}^{\mathrm{int}}$.

Could one determine the infimum of $\mathbb{W}^{\mathrm{int}}$ (thus of $\mathbb{W}^{\mathrm{elec}}$ in the non-Coulomb cases) thanks to its explicit form? A necessary condition on the realizability of ρ_2 as a two-point correlation function of some stationary random point process of intensity one is that $T_2 := \rho_2 - 1$ satisfies

$$T_2 \geq -1, \quad \widehat{T_2} \geq -1,$$

(see [KLS07, Section 2]) where $\widehat{T_2}$ denotes the Fourier transform of T_2 (in a sense that should, in general, be precised). Thus we may start by asking whether the linear optimization problem of minimizing

$$\liminf_{R\to\infty} \int_{[-R,R]^d} T_2(v)g(v) \prod_{i=1}^{d} \left(1 - \frac{|v_i|}{R}\right) dv$$

can be solved on the convex set $\{T_2 \geq -1, \widehat{T_2} \geq -1\}$. It is unclear to us whether the symmetry of the constraints on T_2 and $\widehat{T_2}$ might be of any use (however let us observe that the two-point correlation function of $P_{\mathbb{Z}}$ is its own Fourier transform, and that the expected minimizer in $d = 2$, namely the stationary random point process associated to the triangular lattice, exhibits a self-duality of the same kind).

b. Decorrelating random point processes.

We may also investigate the problem of minimizing $\mathbb{W}^{\text{int}}$ over particular sub-classes of $\mathcal{P}_{s,1}(\mathcal{X})$. We have already mentioned the work of [BS13] where this minimization is considered over determinantal point processes in dimension $d = 1, 2$. Another interesting aspect is that of "decorrelating" random point processes. Let us say that P is a *decorrelating* random point process (or that P *decorrelates*) when $T_2(v) := \rho_2(v) - 1$ tends to 0 as $|v| \to \infty$, with some speed criterion to be fixed, or when T_2 lies in some reasonable class (e.g. L^p-spaces). It is unclear to us whether there is a minimizing sequence for $\mathbb{W}^{\text{int}}$ made of random point processes which decorrelate. A negative answer would hint at a soft kind of phase transition as β varies. Indeed as $\beta \to 0$ the minimizers of $\mathcal{F}_\beta$ converge to the law of a Poisson point process $\mathbf{\Pi}$ (according to Theorem 13), which is a typical decorrelating random point process, and as $\beta \to \infty$ they would rather leave this class.

The $1d$ Log-gas case. Such a transition may be formally observed in the one-dimensional logarithmic case. Let us recall that for $d = 1, s = 0$ there exists a "concrete" family of minimizers for $\mathcal{F}_\beta$ stemming from Random Matrix Theory (RMT), namely the Sine_β processes. They arise as the $N \to \infty$ limit of microscopic point processes observed in the so-called "Gaussian β-ensembles", which are a generalization for any β of the classical Gaussian ensembles of RMT (see [DE02] and [VV09]). It is known (see [Nak14]) that Sine_β is also the law of the limiting microscopic point process for the "Circular β-ensemble", which is another RMT model. The $N \to \infty$ limit for two-point correlation function of the Circular β-ensemble has been derived in [For93] for even values of β. Equation [For93, (11a)] should thus describe the large-x asymptotics of the two-point correlation function of Sine_β, i.e. formally we have

$$T_{2,\text{Sine}_\beta}(x) := \rho_{2,\text{Sine}_\beta}(x) - 1 \approx \sum_{k=1}^{\beta/2} f_k(x) x^{-4k^2/\beta},$$

where $f_k(x)$ is a slowly oscillating function. The leading term of $T_{2,\text{Sine}_\beta}(x)$ as $x \to \infty$ is of order $x^{-4/\beta}$, we would thus expect T_{2,Sine_β} to leave the class L^p as soon as $\beta \geq 4p$.

Decorrelating random point process of minimal energy. A negative answer would also raise the question of finding the minimizer (or a minimizing sequence) for $\mathbb{W}^{\text{int}}$ among decorrelating random point processes. It seems to us that a good candidate for a lower bound on the energy is given by the hypothetical "hardcore Poisson point process" $\mathbf{\Pi}_{\text{hc}}$ whose two-point correlation function would be $\rho_{2,\text{hc}} = 1 - \mathbf{1}_B$ where B is the ball of center 0 and unit volume in $\mathbb{R}^d$. In dimension $d = 1$ it is not hard to construct a sequence of random point processes whose energies converge to the associated energy $\int_B \log|v|$. In arbitrary dimension, it easy to see that any sub-Poissonian random point process (i.e. a random point process such that $\rho_2 \leq 1$, e.g. any determinantal point process) has a larger energy than $\mathbf{\Pi}_{\text{hc}}$.

c. Plan of the paper and ideas of proof.

In Section 4.2 we give some general definitions and notation.

In Section 4.3 we recall the definition of the *renormalized energy* in the sense of [SS15b], [RS15], [PS15], then we introduce the alternative object $\mathbb{W}^{\mathrm{int}}$ defined on the space of random point processes, and in the stationary case we give a simple expression of $\mathbb{W}^{\mathrm{int}}(P)$ in terms of the two-point correlation function of P.

In Section 4.4 we give some preliminary results. In particular we observe that while $\mathbb{W}^{\mathrm{elec}}$ is by definition computed in terms of the energy of *global* electric fields defined on the whole space $\mathbb{R}^{d+k}$, the object $\mathbb{W}^{\mathrm{int}}$ is rather a limit as $R \to \infty$ of the energy of *local* electric fields defined on hypercubes of sidelength R.

Section 4.5 is devoted to the proof of Theorem 12. The proof goes in two step: first we show that $\mathbb{W}^{\mathrm{elec}} \leq \mathbb{W}^{\mathrm{int}}$ (or $\mathbb{W}^{\mathrm{int}} + \mathcal{D}^{\log}$ in the logarithmic cases) on the space $\mathcal{P}_{s,1}(\mathcal{X})$, then conversely (in the non-Coulomb cases) for any $P \in \mathcal{P}_{s,1}(\mathcal{X})$ we prove the existence of a "recovery sequence" $\{P_N\}_N$ converging to P and such that $\lim_{N\to\infty} \mathbb{W}^{\mathrm{int}}(P_N) \leq \mathbb{W}^{\mathrm{elec}}(P)$. In both steps the key element is the *screening lemma* of [PS15] (following [SS15b], [SS15a], [RS15]) which, heuristically speaking, allows us here to construct a global electric field from a local one, and *vice versa*.

In Section 4.6 we prove Theorem 13 about the convergence to the law of the Poisson point process $\mathbf{\Pi}$ of minimizers of $\mathcal{F}_\beta$ as $\beta \to 0$. If $\mathbb{W}^{\mathrm{elec}}(\mathbf{\Pi})$ is finite we may see directly that $\mathsf{ent}[P_\beta|\mathbf{\Pi}]$ must go to zero as $\beta \to 0$, and then the *specific* Pinsker inequality implies that $P_\beta \to \mathbf{\Pi}$ as $\beta \to 0$. However in some cases the finiteness of $\mathbb{W}^{\mathrm{elec}}(\mathbf{\Pi})$ is false (e.g. $d = 1, s = 0$, see [LS15]) or yet unknown ($d = 2, s = 0$). We use the fact that $\mathbb{W}^{\mathrm{elec}} \leq \mathbb{W}^{\mathrm{int}} + \mathcal{D}^{\log}$ to construct a sequence of random point processes converging to $\mathbf{\Pi}$ in entropy sense and whose renormalized energy is finite.

In Section 4.7 we restrict ourselves to the one-dimensional cases (4.1.1) or (4.1.3) and we prove the crystallization result of Theorem 14. We start by using a convexity argument to show that $P_{\mathbb{Z}}$ is the only minimizer of $\mathbb{W}^{\mathrm{int}}$ over $\mathcal{P}_{s,1}(\mathcal{X})$. More precisely we obtain a quantitative bound below on $\mathbb{W}^{\mathrm{int}}(P) - \mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}})$ in terms of the two-point correlation function of P. This translates into a bound below for $\mathbb{W}^{\mathrm{elec}}(P) - \mathbb{W}^{\mathrm{elec}}(P_{\mathbb{Z}})$ which implies that $P_{\mathbb{Z}}$ is also the only minimizer of $\mathbb{W}^{\mathrm{elec}}$. Moreover as $\beta \to \infty$ we show that $\mathbb{W}^{\mathrm{elec}}(P_\beta)$ must go to $\mathbb{W}^{\mathrm{elec}}(P_{\mathbb{Z}})$, which in turn implies that as $\beta \to \infty$ the two-point correlation function of P_β converges to $\rho_{2,P_{\mathbb{Z}}}$ in the distributional sense. Thanks to the "rigidity" of the lattice it is not hard to deduce that in fact P_β converges to $P_{\mathbb{Z}}$ as $\beta \to \infty$.

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4.2 Definitions and notation

4.2.1 Generalities

If (X, d_X) is a metric space we let $\mathcal{P}(X)$ be the space of Borel probability measures on X and we endow $\mathcal{P}(X)$ with the distance

$$d_{\mathcal{P}(X)}(P_1, P_2) = \sup\left\{ \int F(dP_1 - dP_2) |\ F \in \mathrm{Lip}_1(X) \right\}, \tag{4.2.1}$$

where $\mathrm{Lip}_1(X)$ denotes the set of functions $F : X \to \mathbb{R}$ that are 1-Lipschitz with respect to d_X and such that $\|F\|_\infty \leq 1$. It is well-known that the distance $d_{\mathcal{P}(X)}$ metrizes the topology of weak convergence on $\mathcal{P}(X)$. If $P \in \mathcal{P}(X)$ is a probability measure and $f : X \to \mathbb{R}^d$ a measurable function, we denote by $\mathbf{E}_P[f]$ the expectation of f under P.

The ambient Euclidean space of dimension d is denoted by $\mathbb{R}^d$. We will often need to work in $\mathbb{R}^{d+k}$ where $k = 0$ (in the case of Coulomb interactions) or 1 (see Section 4.2.4).

For any $R > 0$ we denote by C_R the hypercube $[-R/2, R/2]^d \subset \mathbb{R}^d$ and by $\hat{C}_R$ the hypercube $[-R/2, R/2]^{d+k} \subset \mathbb{R}^{d+k}$. Similarly we let B_R be the ball of center 0 and radius R in $\mathbb{R}^d$ and $\hat{B}_R$ be the ball of center 0 and radius R in $\mathbb{R}^{d+k}$.

If A is a set, we let $\triangle := \{(x, x), x \in A\} \subset A \times A$ be the diagonal of A.

4.2.2 Point configurations and random point processes

a. Point configurations.

If A is a Borel set of $\mathbb{R}^d$ we denote by $\mathcal{X}(A)$ the set of locally finite point configurations in A or equivalently the set of non-negative, purely atomic Radon measures on A giving an integer mass to singletons (see [DVJ88]). We will often write $\mathcal{C}$ for $\sum_{p\in\mathcal{C}} \delta_p$. We endow the set $\mathcal{X} := \mathcal{X}(\mathbb{R}^d)$ (and the sets $\mathcal{X}(A)$ for A Borel) with the topology induced by the topology of weak convergence of Radon measure (also known as vague convergence or convergence against compactly supported continuous functions), these topologies are metrizable and we fix an arbitrary compatible distance.

The additive group $\mathbb{R}^d$ acts on $\mathcal{X}$ by translations $\{\theta_t\}_{t\in\mathbb{R}^d}$: if $\mathcal{C} = \{x_i, i \in I\} \in \mathcal{X}$ we let

$$\theta_t \cdot \mathcal{C} := \{x_i - t, i \in I\}.$$

We denote by $\mathcal{N}_R : \mathcal{X} \mapsto \mathbb{N}$ the number of points of a configuration in the hypercube C_R, and by $\mathcal{D}_R$ the discrepancy $\mathcal{D}_R = \mathcal{N}_R - R^d$.

b. Random point processes.

A random point process is a probability measure on $\mathcal{X}$. We denote by $\mathcal{P}_s(\mathcal{X})$ the set of translation-invariant (or stationary) random point processes. We endow $\mathcal{P}(\mathcal{X})$ with the topology of weak convergence of probability measures on $\mathcal{X}$. A compatible distance on $\mathcal{P}_s(\mathcal{X})$ is defined in (4.2.1).

Remark 4.2.1. *Another natural topology on $\mathcal{P}(\mathcal{X})$ is the "convergence of the finite distributions" [DVJ08, Section 11.1], also called the "convergence with respect to vague topology for the counting measure of the point process". The two notions of convergence coincide as stated in [DVJ08, Theorem 11.1.VII].*

We will use several times the operation of averaging a random point process over translations in some measurable subset. If P is a random point process in $\mathbb{R}^d$ and K a measurable subset of $\mathbb{R}^d$ with finite, non-zero Lebesgue measure, we define the average P_K^{av} of P over translations in K as the law of the random variable $u_K + \mathcal{C}$ where u_K is uniformly distributed according to the normalized Lebesgue measure on K, and $\mathcal{C}$ has law P. The sum of a vector and a point configuration is defined as the point configuration

$$x + \mathcal{C} := \{x + p, p \in \mathcal{C}\}.$$

c. Hyperuniformity.

Following [TS03] we say that a random point process P is *hyperuniform* if we have

$$\mathbf{E}_P[\mathcal{D}_R^2] = O(R^{d-1}).$$

For a stationary one-dimensional random point process P, hyperuniformity is easily seen to be equivalent to the following property: for some $r > 0$, there are P-a.s. between $k-r$ and $k+r$ points in any interval of length k.

4.2.3 Correlation functions

Let $P \in \mathcal{P}(\mathcal{X})$ be a random point process. For any $n \geq 1$ the n-point correlation function $\rho_{n,P}$ is the linear form on (a subspace of) the linear space of bounded measurable functions $\varphi_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ with compact support defined by (we abbreviate "p.d." for "pairwise distinct")

$$\rho_{n,P}(\varphi_n) = \mathbf{E}_P \sum_{x_1,\dots,x_n \in \mathcal{C} \text{ p.d.}} \varphi_n(x_1,\dots,x_n).$$

If the n-point correlation function exists as a distribution and can be identified with a measurable function, we will write $\int \rho_{n,P}\varphi_n$ instead of $\rho_{n,P}(\varphi_n)$. Heuristically speaking, the one-point correlation function ρ_1 (also called the intensity of the random point process) gives the density of the process at each point, while the two-point correlation function $\rho_2(x,y)$ gives the probability of having a point both at x and y. In this paper we will work with stationary random point processes such that $\rho_1 \equiv 1$ and we denote by $\mathcal{P}_{s,1}(\mathcal{X})$ this set.

4.2.4 Dimension extension

We recall some elements from [PS15] to which we refer for more details. Outside of the Coulomb cases the Riesz kernel g is not the convolution kernel of a local operator, but rather of a fractional Laplacian. It can be transformed into a local but inhomogeneous operator of the form $\operatorname{div}(|y|^\gamma \nabla \cdot)$ by adding one space variable $y \in \mathbb{R}$ to the space $\mathbb{R}^d$. In what follows, k will denote the dimension extension. We take $k = 0$ in all the Coulomb cases, i.e. $s = d-2$ and $d \geq 3$ or (4.1.2) and in all other cases we take $k = 1$. We use an auxiliary parameter γ defined by

$$\gamma := s - d + 2 - k$$

where the convention is to take $s = 0$ in the logarithmic cases. In particular we have $\gamma = 0$ in the logarithmic cases (4.1.1) (where $k = 1$) and (4.1.2) (where $k = 0$).

Points in the space $\mathbb{R}^d$ will be denoted by x, and points in the extended space $\mathbb{R}^{d+k}$ by $X = (x,y)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$. The interaction kernel g is naturally extended to $\mathbb{R}^{d+k}$. We will often identify $\mathbb{R}^d \times \{0\}$ and $\mathbb{R}^d$. The measure $\delta_{\mathbb{R}^d}$ is the Radon measure on $\mathbb{R}^{d+k}$ which corresponds to the Lebesgue measure on the hypersurface $\mathbb{R}^d \subset \mathbb{R}^{d+k}$.

Finally we let $c_{d,s}$ be the constant depending on d, s, such that $-\operatorname{div}(|y|^\gamma \nabla g) = c_{d,s}\delta_0$ in $\mathbb{R}^{d+k}$ (the values of $c_{d,s}$ are given in [PS15]).

4.2.5 Electric fields and random electric fields

Let $p < \frac{d+k}{s+1}$ be fixed. We think of the space $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ as the space of electric fields generated by the charged particles together with a certain uniformly charged background and we endow this space with the weak L^p topology.

a. Local electric fields.

If $\mathcal{C}$ is a finite point configuration and $R > 0$ we let

$$\Phi^{\mathrm{loc}}(\mathcal{C}) := c_{d,s}\, g * (\mathcal{C} - \mathbf{1}_{C_R}\delta_{\mathbb{R}^d}), \quad E^{\mathrm{loc}}(\mathcal{C}) := \nabla\Phi^{\mathrm{loc}} = c_{d,s}\nabla g * (\mathcal{C} - \mathbf{1}_{C_R}\delta_{\mathbb{R}^d}). \tag{4.2.2}$$

where $g*$ denotes the convolution (computed in $\mathbb{R}^{d+k}$) with the interaction kernel g. It implies that

$$-\mathrm{div}\,(|y|^\gamma E^{\mathrm{loc}}) = c_{d,s}\nabla g * (\mathcal{C} - \mathbf{1}_{C_R}\delta_{\mathbb{R}^d}). \tag{4.2.3}$$

The scalar field Φ^{loc} physically corresponds to the electrostatic potential generated by the point charges of $\mathcal{C}$ together with a background of density μ. The vector field E^{loc} can be thought of as the associated electrostatic field. It is easy to see that E^{loc} fails to be in L^2_{loc} because it blow ups like $|x|^{-(s+1)}$ near each point of $\mathcal{C}$, however E^{loc} is in $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k},\mathbb{R}^{d+k})$.

b. Electric fields.

We now introduce a special class of vector fields that correspond to electric fields generated by a system made of an infinite point configuration $\mathcal{C}$ and a negatively charged background in all $\mathbb{R}^d$. We let Elec be the class of "electric vector fields" i.e. the set of vector fields E belonging to $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k},\mathbb{R}^{d+k})$ that satisfy

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C} - \delta_{\mathbb{R}^d}) \text{ in } \mathbb{R}^{d+k} \tag{4.2.4}$$

where $\mathcal{C} \in \mathcal{X}(\mathbb{R}^d)$ is a point configuration. We say that E *is compatible with* $\mathcal{C}$ if (4.2.4) holds. If E is in Elec we let

$$\mathrm{Conf}(E) := \frac{-1}{c_{d,s}}\mathrm{div}\,(|y|^\gamma E) + \delta_{\mathbb{R}^d}$$

be the underlying point configuration $\mathcal{C}$, in other words E is compatible with $\mathcal{C}$ if and only if $\mathrm{Conf}(E) = \mathcal{C}$.

c. Truncation procedure.

The renormalization procedure of [RS15], [PS15] (inspired by the original work of [BBH94]) uses a truncation of the singularities which we now recall. We define the truncated Riesz (or Coulomb, or logarithmic) kernel as follows: for $1 > \eta > 0$ and $X \in \mathbb{R}^{d+k}$, let

$$f_\eta(X) = (g(X) - g(\eta))_+$$

with a slight abuse of notation: since g is a radial function we write $g(\eta)$ for the value of g at any point on a sphere of radius η.

If E^{loc} is a local field as in (4.2.2) we let

$$E^{\mathrm{loc}}_\eta(X) := E^{\mathrm{loc}}(X) - \sum_{p\in\mathcal{C}} \nabla f_\eta(X-p). \tag{4.2.5}$$

Similarly if E is an electric field as in (4.2.4) we let

$$E_\eta(X) := E(X) - \sum_{p\in\mathcal{C}} \nabla f_\eta(X-p). \tag{4.2.6}$$

d. Random electric process.

A probability measure on $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^{d+k},\mathbb{R}^{d+k}))$ concentrated on Elec is called a random electric process. We say that P^{elec} is stationary when it is invariant under the (push-forward by) translations $E \mapsto E(\cdot - x)$ for any $x \in \mathbb{R}^d$.

4.2.6 Specific relative entropy

Let P be a stationary random point process on $\mathbb{R}^d$. The relative specific entropy $\mathsf{ent}[P|\mathbf{\Pi}]$ of P with respect to $\mathbf{\Pi}$, the law of the Poisson point process of uniform intensity 1, is defined by

$$\mathsf{ent}[P|\mathbf{\Pi}] := \lim_{R\to\infty} \frac{1}{|C_R|}\mathrm{Ent}\left(P_{|C_R}|\mathbf{\Pi}_{|C_R}\right),$$

where $P_{|C_R}$ denotes the random point process induced in C_R, and $\mathrm{Ent}(\cdot|\cdot)$ denotes the usual relative entropy (or Kullbak-Leibler divergence) of two probability measures defined on the same probability space. We take the appropriate sign convention for the entropy so that it is non-negative: if μ, ν are two probability measures defined on the same space we let $\mathrm{Ent}\,(\mu|\nu) := \int \log \frac{d\mu}{d\nu} d\mu$ if μ is absolutely continuous with respect to ν and $+\infty$ otherwise. We have in fact by super-additivity

$$\mathsf{ent}[P|\mathbf{\Pi}] = \sup_{R\geq 1} \frac{1}{|C_R|}\mathrm{Ent}\left(P_{|C_R}|\mathbf{\Pi}_{|C_R}\right). \tag{4.2.7}$$

The functional $P \mapsto \mathsf{ent}[P|\mathbf{\Pi}]$ is affine lower semi-continous on $\mathcal{P}_{s,1}(\mathcal{X})$ and its sub-level sets are compact. We refer to [RAS09, Chap. 6] for a proof of these statements.

4.3 Definitions for the energy of a random point process

In this section we recall the derivation of a renormalized energy for random point processes from (4.1.4) then we introduce our alternative definition of a (logarithmic, Coulomb or Riesz) energy for random point processes.

4.3.1 The electric approach

a. Renormalized energy of an electric field.

We now recall the computation of the *renormalized* energy $\mathcal{W}(E)$ following [PS15] (see also [Ser15] and the references therein). For any $E \in \mathsf{Elec}$ we define

$$\mathcal{W}_\eta(E) := \limsup_{R\to\infty} \left(\frac{1}{c_{d,s}} \frac{1}{R^d} \int_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2 - g(\eta) \right),$$

where E_η is the truncated field as in (4.2.6), and we let

$$\mathcal{W}(E) := \lim_{\eta\to 0} \mathcal{W}_\eta(E).$$

For convenience we have chosen a definition of $\mathcal{W}$ which differs from a multiplicative constant $c_{d,s}$ from the one in [PS15].

b. The electric definition.

We then define

$$\mathbb{W}_\eta(\mathcal{C}) = \inf \mathcal{W}_\eta(E), \quad \mathbb{W}(\mathcal{C}) = \inf \mathcal{W}(E),$$

where both infimum are among electric fields E compatible with $\mathcal{C}$. Similarly if P is a random point process we let

$$\mathbb{W}^{\mathrm{elec}}_\eta(P) = \mathbf{E}_P\left[\mathbb{W}_\eta\right], \quad \mathbb{W}^{\mathrm{elec}}(P) = \mathbf{E}_P\left[\mathbb{W}\right]. \tag{4.3.1}$$

The following lemma was proven in [LS15]:

Lemma 4.3.1. *Let P be a stationary random point process such that $\mathbb{W}^{\mathrm{elec}}(P)$ is finite. Then there exists a stationary random electric process P^{elec} such that the push-forward of P^{elec} by* Conf *is equal to P and which satisfies*

$$\mathbf{E}_{P^{\mathrm{elec}}}[\mathcal{W}] = \mathbb{W}^{\mathrm{elec}}(P).$$

We also have the following lower semi-continuity result for the electric energy. Let us emphasize that lower-semi continuity only holds at the level of *stationary* random point processes, and not for point configurations or arbitrary random point processes.

Lemma 4.3.2. *The maps $P \mapsto \mathbb{W}^{\mathrm{elec}}_{\eta}(P)$ and $P \mapsto \mathbb{W}^{\mathrm{elec}}(P)$ are lower semi-continuous on $\mathcal{P}_{s,1}(\mathcal{X})$.*

Proof. See e.g. [PS15, Lemma 4.1.] □

This provides a definition for the energy of a random infinite point configuration with a uniform negative background. However the computation of $\mathbb{W}^{\mathrm{elec}}(P)$ or of $\mathbb{W}(\mathcal{C})$ (or even the search for an upper bound on these quantities) appears involved in general because it amounts to finding compatible electric fields for infinite point configurations. Let us mention that in the case of a periodic configuration (hence also for the stationary random point process associated to it) exact formulas are known in the cases (4.1.2) (see [SS15b]), (4.1.1) (see [SS15a]), for the higher-dimensional Coulomb case (4.1.3) with $s = d - 2$ (see [RS15]) and for Riesz gases (see [PS15]).

4.3.2 The intrinsic approach

In this section we define an energy functional $\mathbb{W}^{\mathrm{int}}$ on the space of random point processes, using only the nature of the pairwise interaction.

If A, B are two (measurable) subsets of $\mathbb{R}^d$ we define $\mathrm{Int}[A, B]$ as the interaction energy between A and B

$$\mathrm{Int}[A, B](\mathcal{C}) := \iint_{(A\times B)\setminus\triangle} g(x-y)(d\mathcal{C}(x) - dx) \otimes (d\mathcal{C}(y) - dy). \tag{4.3.2}$$

In view of (4.1.4), for any $R > 0$ we define $\mathcal{H}^{\mathrm{int}}_R : \mathcal{X}(C_R) \to \mathbb{R}$ as the interaction of C_R with itself (the diagonal being excluded) i.e. $\mathcal{H}^{\mathrm{int}}_R := \mathrm{Int}[C_R, C_R]$ or in other terms

$$\mathcal{H}^{\mathrm{int}}_R(\mathcal{C}) := \iint_{C_R^2\setminus\triangle} g(x-y)(d\mathcal{C} - dx) \otimes (d\mathcal{C} - dy).$$

Given a random point process P, a natural way of defining the energy (per unit volume) of P is the following:

Definition 4.3.3. *Let P be a random point process of intensity 1. We define its intrinsic energy $\mathbb{W}^{\mathrm{int}}$ by*

$$\mathbb{W}^{\mathrm{int}}(P) := \liminf_{R\to\infty} \frac{1}{R^d} \mathbf{E}_P\left[\mathcal{H}^{\mathrm{int}}_R(\mathcal{C})\right]. \tag{4.3.3}$$

a. Expression with correlation functions.

The energy defined by (4.3.3) can be re-written with the help of one- and two-point correlation functions of P. In order for the expression to make sense, we restrict ourselves to random point processes whose two-point correlation function exists as a Radon measure in $\mathbb{R}^d \times \mathbb{R}^d$. We will then abuse notation and consider ρ_2 as a function instead of a measure, writing $\rho_2(x, y)$ instead of $d\rho_2$.

Lemma 4.3.4. *For any random point process P of intensity 1 such that ρ_2 exists as a Radon measure, the following identity holds*

$$\mathbb{W}^{\text{int}}(P) = \liminf_{R\to\infty} \frac{1}{R^d} \iint_{C_R^2\setminus\triangle} g(x-y)(\rho_2(x,y)-1)dxdy. \tag{4.3.4}$$

Proof. First we may re-write $\mathcal{H}_R^{\text{int}}(\mathcal{C})$ as :

$$\begin{aligned}\mathcal{H}_R^{\text{int}}(\mathcal{C}) &= \iint_{C_R^2\setminus\triangle} g(x-y)(d\mathcal{C}-dx)\otimes(d\mathcal{C}-dy)\\ &= \iint_{C_R^2\setminus\triangle} g(x-y)(d\mathcal{C}\otimes d\mathcal{C}) + \iint_{C_R^2\setminus\triangle} g(x-y)(dx\otimes dy) - 2\iint_{C_R^2\setminus\triangle} g(x-y)d\mathcal{C}\otimes dy\end{aligned}$$

and by definition of the correlation functions (see Section 4.2.3) we get

$$\mathbf{E}_P\left[\mathcal{H}_R^{\text{int}}\right] = \iint_{C_R^2\setminus\triangle} g(x-y)\rho_2(x,y)dxdy + \iint_{C_R^2\setminus\triangle} g(x-y)dxdy - 2\iint_{C_R^2\setminus\triangle} g(x-y)\rho_1(x)dxdy.$$

By assumption P has intensity 1 i.e. $\rho_1\equiv 1$ and we are left with

$$\frac{1}{R^d}\mathbf{E}_P\left[\mathcal{H}_R^{\text{int}}\right] = \frac{1}{R^d}\iint_{C_R^2\setminus\triangle} g(x-y)(\rho_2(x,y)-1)dxdy,$$

which yields (4.3.4). □

b. The stationary case.

If the random point process P is stationary we may derive a somewhat simpler expression for $\mathbb{W}^{\text{int}}(P)$. In what follows $\rho_2(v)$ stands for $\rho_2(0,v)$. The change of variables $(u,v)=(x+y,x-y)$ gives

$$\begin{aligned}\iint_{([-R/2,R/2]^d)^2\setminus\triangle} g(x-y)(\rho_2(x-y)-1)dxdy &= \frac{1}{2^d}\int_{v\in[-R,R]^d\setminus\{0\}}\int_{u\in S_R(v)} g(v)(\rho_2(v)-1)dudv\\ &= \int_{v\in[-R,R]^d\setminus\{0\}} |S_R(v)|g(v)(\rho_2(v)-1)dv,\end{aligned}$$

where $S_R(v)$ denotes the set

$$S_R(v) := \{x+y : x,y\in C_R,\, x-y=v\}.$$

The Lebesgue measure $|S_R(v)|$ of $S_R(v)$ is easily computed for $v=(v_1,\dots,v_d)$

$$|S_R(v_1,\dots,v_d)| = 2^d\times\prod_{i=1}^d(R-|v_i|). \tag{4.3.5}$$

Indeed we have $S_R(v)=\{2x-v : x,y\in C_R,\, x-y=v\}$ which implies that

$$|S_R(v)| = 2\times|\{x\in C_R, x-v\in C_R\}|,$$

moreover $S_R(v)$ tensorizes i.e. if $v=(v_1,\dots,v_d)\in[-R,R]^d$ we get

$$|S_R(v_1,\dots,v_d)| = 2^d\prod_{i=1}^d|\{x_i\in[-R/2,R/2], x_i-v_i\in[-R/2,R/2]\}|$$

which leads to (4.3.5). Finally we obtain the following expression for any P in $\mathcal{P}_{s,1}(\mathcal{X})$ (such that ρ_2 is a Radon measure)

$$\boxed{\mathbb{W}^{\text{int}}(P) = \liminf_{R\to\infty}\frac{1}{R^d}\int_{[-R,R]^d\setminus\{0\}} g(v)(\rho_2(v)-1)\prod_{i=1}^d(R-|v_i|)dv.} \tag{4.3.6}$$

4.4 Preliminary results on the energy

4.4.1 Local field and local interaction

Let $R > 0$ and let $\mathcal{C}$ be a point configuration in $\mathcal{X}(C_R)$. The local electric potential (resp. field) $\Phi^{\rm loc}$ (resp. $E^{\rm loc}$) are defined in (4.2.2). For $\eta \in (0,1)$ we let also $E^{\rm loc}_\eta$ be as in (4.2.5). Let us recall that $\hat{C}_R$ denotes the hypercube $[-R/2, R/2]^{d+k}$.

Lemma 4.4.1. *The following inequalities hold*

1. *In the cases* (4.1.3)

$$\frac{1}{c_{d,s}} \int_{\mathbb{R}^{d+k}} |y|^\gamma |E^{\rm loc}_\eta|^2 - \mathcal{N}_R g(\eta) \leq \mathcal{H}^{\rm int}_R[\mathcal{C}] + \mathcal{N}_R o_\eta(1). \tag{4.4.1}$$

2. *In the logarithmic cases*

$$\frac{1}{c_{d,s}} \int_{\hat{C}_R} |y|^\gamma |E^{\rm loc}_\eta|^2 - \mathcal{N}_R g(\eta) \leq \mathcal{H}^{\rm int}_R[\mathcal{C}] + C^{\log} \mathcal{D}_R^2 \log R + \mathcal{N}_R o_\eta(1) + O(\mathcal{N}_R^2 R^{-5}) + o_R(1). \tag{4.4.2}$$

Let P be a stationary random point process of intensity 1 such that $\mathbf{E}_P[\mathcal{D}_R^2] = o(R^{2d})$. The following inequalities hold

1. *In the cases* (4.1.3)

$$\mathbf{E}_P \left[\frac{1}{c_{d,s}} \int_{\mathbb{R}^{d+k}} |y|^\gamma |E^{\rm loc}_\eta|^2 - \mathcal{N}_R g(\eta) \right] \leq \mathbf{E}_P[\mathcal{H}^{\rm int}_R] + R^d o_\eta(1). \tag{4.4.3}$$

2. *In the logarithmic cases*

$$\mathbf{E}_P \left[\frac{1}{c_{d,s}} \int_{\hat{C}_R} |y|^\gamma |E^{\rm loc}_\eta|^2 - \mathcal{N}_R g(\eta) \right] \leq \mathbf{E}_P[\mathcal{H}^{\rm int}_R] + C^{\log} \mathbf{E}_P[\mathcal{D}_R^2] \log R + R^d o_\eta(1) + o_R(1). \tag{4.4.4}$$

The terms $o_\eta(1), o_R(1)$ depends only on d, s.

Proof. The starting point is the following identity which holds for any $S > R$

$$\mathcal{H}^{\rm int}_R[\mathcal{C}] = \lim_{\eta \to 0} \left(\frac{1}{c_{d,s}} \int_{\hat{C}_S} |y|^\gamma |E^{\rm loc}_\eta|^2 - \mathcal{N}_R g(\eta) \right) + \frac{1}{c_{d,s}} \int_{\partial \hat{B}_S} |y|^\gamma \Phi^{\rm loc} E^{\rm loc} \cdot \vec{n}, \tag{4.4.5}$$

where $\vec{n}$ denotes the unit normal vector. It results from two different operations. Let us first recall that

$$\mathcal{H}^{\rm int}_R[\mathcal{C}] := \iint_{C_R^2 \setminus \triangle} g(x-y)(d\mathcal{C} - dx) \otimes (d\mathcal{C} - dy)$$

Since $E^{\rm loc} = \nabla \Phi^{\rm loc}$ satisfies (4.2.3) we may formally write

$$\mathcal{H}^{\rm int}_R[\mathcal{C}] \approx -\frac{1}{c_{d,s}} \int_{\mathbb{R}^{d+k}} \Phi^{\rm loc}(t) \mathrm{div}\, (|y|^\gamma \nabla \Phi^{\rm loc})(t).$$

Of course $\mathrm{div}\, (|y|^\gamma \nabla \Phi^{\rm loc})(t)$ is in fact supported on C_R. For any $S > R$ an integration by parts yields

$$\mathcal{H}^{\rm int}_R[\mathcal{C}] \approx \frac{1}{c_{d,s}} \int_{\hat{C}_S} |y|^\gamma |E^{\rm loc}|^2 + \frac{1}{c_{d,s}} \int_{\partial \hat{C}_S} |y|^\gamma \Phi^{\rm loc} E^{\rm loc} \cdot \vec{n}.$$

Since E^{loc} is not in L^2 the previous computation does not make sense, however it can be made rigorous in a *renormalized* fashion by truncating the interaction close to the charges at scale $\eta > 0$ and substracting a diverging term as $\eta \to 0$ (here $g(\eta)$) for each charge. We refer to [PS15] for more details.

We now turn to the boundary term in (4.4.5). A mean value argument applied to g and ∇g shows that for $S \geq 2R$ we have

$$\left|\Phi^{\mathrm{loc}}(X) - \mathcal{D}_R g(X)\right| \leq C(\mathcal{N}_R + R^d)RS^{-s-1},$$

$$|E^{\mathrm{loc}}(X) - \mathcal{D}_R \nabla g(X)| \leq C(\mathcal{N}_R + R^d)RS^{-s-2},$$

uniformly for $X \in \partial\hat{C}_S$. Indeed it is easy to see that the first derivative of $t \mapsto g(X-t)$ is bounded by CS^{-s-1} and that its second derivative is bounded by CS^{-s-2} for any t in C_R, uniformly for $X \in \partial\hat{C}_S$, and with a constant C depending only on d, s (because $S > 2R$). Moreover we have $|\nabla g(S)| \leq CS^{-s-1}$, the perimeter of $\partial\hat{C}_S$ is $O(S^{d+k-1})$ and $|y|^\gamma \leq CS^\gamma$ on $\partial\hat{C}_S$. Since $\gamma = s - d + 2 - k$ we have $S^{\gamma+d+k-1} = S^{s+1}$.

Combining the estimates above we get for any $S > 2R$

$$\Big|\int_{\partial\hat{C}_S} |y|^\gamma \Phi^{\mathrm{loc}} E^{\mathrm{loc}} \cdot \vec{n}\Big| \leq CS^{s+1}\left(\mathcal{D}_R^2 \frac{|g(S)|}{S^{s+1}} + (\mathcal{N}_R^2 + R^{2d})\left(\frac{R}{S^{2s+2}} + \frac{R|g(S)|}{S^{s+2}} + \frac{R^2}{S^{2s+3}}\right)\right)$$

with a constant C depending only on d, s.

In the cases (4.1.3) the right-hand side is $O(S^{-s})$ as $S \to \infty$ and $s > 0$. We thus obtain

$$\Big|\int_{\partial\hat{C}_S} |y|^\gamma \Phi^{\mathrm{loc}} E^{\mathrm{loc}} \cdot \vec{n}\Big| = o_{S\to\infty}(1). \tag{4.4.6}$$

In the logarithmic cases we have $s = 0$ and $g(S) = \log S$, hence if we set $S = R^4$ we get

$$\Big|\int_{\partial\hat{C}_{R^4}} |y|^\gamma \Phi^{\mathrm{loc}} E^{\mathrm{loc}} \cdot \vec{n}\Big| \leq C^{\log}\mathcal{D}_R^2 \log R + O\left(\mathcal{N}_R^2 R^{-5}\right) + o_R(1), \tag{4.4.7}$$

for a certain universal constant $C^{\log}$.

Combining (4.4.5) and (4.4.6) and letting $S \to \infty$ we obtain, in the cases (4.1.3)

$$\lim_{\eta\to 0}\left(\frac{1}{c_{d,s}}\int_{\mathbb{R}^{d+k}} |y|^\gamma |E_\eta^{\mathrm{loc}}|^2 - \mathcal{N}_R g(\eta)\right) = \mathcal{H}_R^{\mathrm{int}}[\mathcal{C}].$$

Combining (4.4.5) and (4.4.7) we obtain, in the logarithmic cases

$$\lim_{\eta\to 0}\left(\frac{1}{c_{d,s}}\int_{\hat{C}_R} |y|^\gamma |E_\eta^{\mathrm{loc}}|^2 - \mathcal{N}_R g(\eta)\right) \leq \mathcal{H}_R^{\mathrm{int}}[\mathcal{C}] + C^{\log}\mathcal{D}_R^2 \log R + O\left(\mathcal{N}_R^2 R^{-5}\right) + o_R(1),$$

where we have used the trival bound $\int_{\hat{C}_R} |y|^\gamma |E_\eta^{\mathrm{loc}}|^2 \leq \int_{\hat{C}_{R^4}} |y|^\gamma |E_\eta^{\mathrm{loc}}|^2$.

From [PS15, Lemma 2.3] we know that the limit as $\eta \to 0$ is almost monotonous, more precisely for any $\eta \in (0,1)$ and $S > 2R$ we have

$$\int_{\hat{C}_S} |y|^\gamma |E_\eta^{\mathrm{loc}}|^2 - \mathcal{N}_R g(\eta) \leq \lim_{\eta\to 0}\left(\frac{1}{c_{d,s}}\int_{\hat{C}_S} |y|^\gamma |E_\eta^{\mathrm{loc}}|^2 - \mathcal{N}_R g(\eta)\right) + \mathcal{N}_R o_\eta(1). \tag{4.4.8}$$

We thus obtain (4.4.1) and (4.4.2). Inequalities (4.4.3) and (4.4.4) follow easily by taking the expectation under P. □

4.4.2 Bound below on the interaction

The self-interaction of an electric system of point charges with a negative background in a given set K is bounded below in terms of the number of points in K (and the volume of K in the logarithmic cases).

Lemma 4.4.2. *Let K be a compact subset such that $K \subset C_R$ and let $\mathcal{C}$ be a point configuration in K. We denote by $\mathcal{N}(\mathcal{C}, K)$ the number of points in K, and we denote by $\mathrm{Int}[K, K](\mathcal{C})$ the interaction energy of K with itself as in* (4.3.2).

In the cases (4.1.3) *we have*

$$\mathrm{Int}[K, K](\mathcal{C}) \geq -C\left(\mathcal{N}(\mathcal{C}, K)\right). \tag{4.4.9}$$

In the logarithmic cases we have

$$\mathrm{Int}[K, K](\mathcal{C}) \geq -C\left(\mathcal{N}(\mathcal{C}, K) + (\mathcal{N}(\mathcal{C}, K) - |K|)^2 \log R + \mathcal{N}^2(\mathcal{C}, K) R^{-5}\right).$$

In both inequalities C is a positive constant depending on d, s.

Proof. Letting E^{loc} be the local electric field generated by $\mathcal{C}$ in K and arguing as in the proof of Lemma 4.4.1 we obtain

$$\mathrm{Int}[K, K](\mathcal{C}) = \lim_{\eta \to 0} \left(\frac{1}{c_{d,s}} \int_{\hat{C}_S} |y|^\gamma |E^{\mathrm{loc}}_\eta|^2 - \mathcal{N}_R g(\eta) \right) + \frac{1}{c_{d,s}} \int_{\partial \hat{B}_S} |y|^\gamma \Phi^{\mathrm{loc}} E^{\mathrm{loc}} \cdot \vec{n}.$$

As seen in (4.4.8) the limit as $\eta \to 0$ is almost monotonous. In the cases (4.1.3), chosing $\eta = \frac{1}{2}$ and letting $S \to \infty$ we obtain

$$\mathrm{Int}[K, K] \geq \frac{1}{c_{d,s}} \int_{\mathbb{R}^{d+k}} |y|^\gamma |E^{\mathrm{loc}}_{\frac{1}{2}}|^2 - \mathcal{N}(\mathcal{C}, K)(g(1/2) + C), \tag{4.4.10}$$

which yields (4.4.9).

In the logarithmic cases, choosing $\eta = \frac{1}{2}$ and taking $S = R^4$ we obtain

$$\mathrm{Int}[K, K] \geq \frac{1}{c_{d,s}} \int_{\hat{C}_{R^4}} |y|^\gamma |E^{\mathrm{loc}}_{\frac{1}{2}}|^2 - \mathcal{N}(\mathcal{C}, K)(g(\frac{1}{2}) + C) - \left| \int_{\partial \hat{C}_{R^4}} |y|^\gamma \Phi^{\mathrm{loc}} E^{\mathrm{loc}}_{\frac{1}{2}} \right|.$$

Controlling the boundary term as in the proof of Lemma 4.4.1 yields (4.4.10). □

4.4.3 Discrepancy estimates

The following lemma is an adaptation of the discrepancy estimates of [PS15] to show how the finiteness of $\mathbb{W}^{\mathrm{int}}(P)$ (instead of $\mathbb{W}^{\mathrm{elec}}(P)$) implies that P has a *number variance* of order $o(R^{2d})$.

Lemma 4.4.3. *Let P be a stationary random point process of intensity* 1.

In the cases (4.1.3) *if $\mathbb{W}^{\mathrm{int}}(P)$ is finite then we have*

$$\mathbf{E}_P[\mathcal{D}_R^2] = o(R^{2d}). \tag{4.4.11}$$

In the logarithmic cases if $\mathcal{D}^{\log}(P)$ is finite then we also have (4.4.11).

Proof. In the logarithmic cases, the result is straightforward since $\mathcal{D}^{\log}(P) < +\infty$ implies that $\mathbf{E}_P[\mathcal{D}_R^2] = o(R^d)$.

We now turn to the cases (4.1.3). Applying [PS15, Lemma 2.2] with $\eta = \frac{1}{2}$ we get

$$\frac{1}{R^s}\mathcal{D}_R^2 \min\left(1, \frac{\mathcal{D}_R}{R^d}\right) \leq C \int_{\mathbb{R}^{d+k}} |y|^\gamma |E^{\mathrm{loc}}_{\frac{1}{2}}|^2 + C\mathcal{N}_R.$$

Taking the expectation under P yields

$$\mathbf{E}_P\left[\frac{1}{R^s}\mathcal{D}_R^2 \min\left(1, \frac{\mathcal{D}_R}{R^d}\right)\right] \leq C\mathbf{E}_P\left[\int_{\mathbb{R}^{d+k}} |y|^\gamma |E^{\mathrm{loc}}_{\frac{1}{2}}|^2\right] + CR^d,$$

where E^{loc} denotes the local field generated by $\mathcal{C}$ in C_R and C is a constant depending on d, s. We know from Lemma 4.4.1 that

$$\int_{\mathbb{R}^{d+k}} |y|^\gamma |E^{\mathrm{loc}}_{\frac{1}{2}}|^2 - \mathcal{N}_R g(\frac{1}{2}) \leq \mathcal{H}_R^{\mathrm{int}}[\mathcal{C}] + C\mathcal{N}_R,$$

and combining the previous two estimates we obtain that

$$\mathbf{E}_P\left[\frac{1}{R^s}\mathcal{D}_R^2 \min(1, \frac{\mathcal{D}_R}{R^d})\right] \leq C\mathbf{E}_P[\mathcal{H}_R^{\mathrm{int}}] + CR^d = O(R^d).$$

Using Jensen's inequality and the fact that $s < d$ we get

$$\mathbf{E}_P[\mathcal{D}_R^2] = O(R^{\frac{4}{3}d + \frac{2}{3}s}) = o(R^{2d}),$$

which proves (4.4.11). □

4.4.4 The screening lemma

For convenience we recall the "screening lemma" following [SS15b], [SS15a], [RS15], [PS15] and [LS15]. We present it here in a simplified form which will be enough for our purposes, we refer to [LS15] for the most general statement available and to [PS15] for a proof.

The result consists in the following: given a point configuration $\mathcal{C}$ in C_R and a compatible electric field E, we wish to construct another compatible field E^{scr} such that $E^{\mathrm{scr}} \cdot \vec{n} = 0$ on the boundary of $C_R \times \mathbb{R}^k$. Indeed such *screened* fields may be pasted together in adjacent hypercubes because their normal component vanish (and in particular are equal), which will allow us to construct *global* fields defined in the whole space.

Of course this construction is not possible in general (e.g. it is easy to see that it imposes a condition on the number of points in C_R, which must match exactly the volume of C_R). However, under some conditions on E to be "screenable", by extending C_R a bit and modifiying $\mathcal{C}$ only in a thin layer of width εR we may find a new point configuration $\mathcal{C}^{\mathrm{scr}}$ and a compatible *screened* field E^{scr}, such that moreover the energy of E^{scr} is bounded in terms of the energy of E.

Lemma 4.4.4. *There exists $R_0 > 0$ depending on d, s and $\eta_0 > 0$ depending only on d such that the following holds.*

Let $0 < \varepsilon < \frac{1}{2}$ and $0 < \eta < \eta_0$ be fixed. Let C_R be a hypercube of sidelength R for some $R > 0$, and let K be the hypercube of sidelength $\lceil R \rceil$ (where $\lceil R \rceil$ denotes the smallest integer larger than R).

Assume that E is a vector field defined in $C_R \times \mathbb{R}^k$ such that

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C} - \delta_{\mathbb{R}^d}) \text{ in } C_R.$$

Let $M > 1$ such that E satisfies:

$$\frac{1}{R^d}\int_{C_R\times\mathbb{R}^k}|y|^\gamma|E_\eta|^2 \le M.$$

In the case $k = 1$ (the non-Coulomb cases) we define $e_{\varepsilon,R}$ as

$$e_{\varepsilon,R} := \frac{1}{\varepsilon^4 R^d}\int_{C_R\times(\mathbb{R}\setminus(-\varepsilon^2R,\varepsilon^2R))}|y|^\gamma|E|^2. \tag{4.4.12}$$

Under the assumption that the following inequalities are satisfied

$$R > \max\left(\frac{R_0}{\varepsilon^2}, \frac{R_0M}{\varepsilon^3}\right),$$

$$R > \begin{cases} \frac{R_0M^{1/2}}{\varepsilon^{d+3/2}} & \text{if } k=0 \\ \max(R_0M^{1/(1-\gamma)}\varepsilon^{\frac{-1-2d+\gamma}{1-\gamma}}, R_0\varepsilon^{\frac{2\gamma}{1-\gamma}}e_{\varepsilon,R}^{1/(1-\gamma)}) & \text{if } k=1, \end{cases}$$

there exists a point configuration $\mathcal{C}^{\rm scr}$ in K and a vector field $E^{\rm scr}\in L^p_{\rm loc}(\mathbb{R}^{d+k},\mathbb{R}^{d+k})$ such that

1. *The configuration $\mathcal{C}^{\rm scr}$ has exactly $|K|$ points in K.*
2. *The configurations $\mathcal{C}$ and $\mathcal{C}^{\rm scr}$ coincide on $\mathrm{Int}_\varepsilon := \{x\in C_R, \mathrm{dist}(x,\partial C_R)\ge 2\varepsilon R\}$.*
3. *We have*
$$\begin{cases} -\mathrm{div}\,(|y|^\gamma E^{\rm scr}) = c_{d,s}\,(E^{\rm scr}-\delta_{\mathbb{R}^d}) & \text{in } K\times\mathbb{R}^k \\ E^{\rm scr}\cdot\vec{n} = 0 & \text{on } \partial K\times\mathbb{R}^k, \end{cases}$$
4. *Letting $E^{\rm scr}_\eta$ be associated to $E^{\rm scr}$ as in (4.2.6) it holds*

$$\int_{K\times\mathbb{R}^k}|y|^\gamma|E^{\rm scr}_\eta|^2 \le \left(\int_{C_R\times\mathbb{R}^k}|y|^\gamma|E_\eta|^2\right)(1+C\varepsilon) + Cg(\eta)M\varepsilon R^d + Ce_{\varepsilon,R}\varepsilon R^d + o(R^{d-1}). \tag{4.4.13}$$

for some constant C depending only on s,d.

4.4.5 Minimality of the local energy

The following was proven in [LS15, Lemma 3.13.]. It expresses the fact that the local electric field $E^{\rm loc}$ has a lower energy than any "screened" field compatible with the same point configuration.

Lemma 4.4.5. *Let $R\ge 1$ be an integer and $\mathcal{C}$ be a point configuration in C_R. Let $E^{\rm loc}$ be the local electric field generated by $\mathcal{C}$ in C_R as in (4.2.2). Let $E\in L^p_{\rm loc}(\mathbb{R}^{d+k},\mathbb{R}^{d+k})$ be a vector field satisfying*

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C}-\delta_{\mathbb{R}^d}) & \text{in } C_R\times\mathbb{R}^k \\ E\cdot\vec{\nu} = 0 & \text{on } \partial C_R\times\mathbb{R}^k. \end{cases}$$

Then, for any $0<\eta<1$ we have

$$\int_{\mathbb{R}^{d+k}}|y|^\gamma|E^{\rm loc}_\eta|^2 \le \int_{C_R\times\mathbb{R}^k}|y|^\gamma|E_\eta|^2.$$

4.5 Connection of the electric and intrinsic approach

This section is devoted to the proof of Theorem 12. It goes in two steps.

1. First we establish an upper bound $\mathbb{W}^{\mathrm{elec}} \leq \mathbb{W}^{\mathrm{int}}$ (or $\mathbb{W}^{\mathrm{int}} + \mathcal{D}^{\log}$ in the logarithmic cases). The proof of this "electric-intrinsic" inequality is the purpose of Section 4.5.1.
2. Then, in the non-Coulomb cases, for any $P \in \mathcal{P}_{s,1}(\mathcal{X})$ we construct a sequence $\{P_N\}_N$ of stationary random point processes which converges to P and such that $\mathbb{W}^{\mathrm{elec}}(P)$ is bounded below by $\lim_{N\to\infty} \mathbb{W}^{\mathrm{int}}(P_N)$ (we also ensure that P_N is hyperuniform, in particular $\mathcal{D}^{\log}(P_N)$ is always zero). Moreover we have $\lim_{N\to\infty} \mathsf{ent}[P_N|\mathbf{\Pi}] = \mathsf{ent}[P|\mathbf{\Pi}]$.

   This operation is similar to the construction of a "recovery sequence" in Γ-convergence, and is proven in Section 4.5.2.

These two steps immediatly imply Theorem 12.

Re-writing of the additional term. Using the definition of the two-point correlation function (which exists by assumption as a Radon measure) we get

$$\iint_{C_R^2} (\rho_2(x,y) - 1)dxdy = \mathbf{E}_P\left[\mathcal{N}_R(\mathcal{N}_R - 1)\right] - R^{2d} = \mathbf{E}_P\left[\mathcal{N}_R^2\right] - R^{2d} - \mathbf{E}_P\left[\mathcal{N}_R\right].$$

Since P has intensity 1 we have $\mathbf{E}_P\left[\mathcal{N}_R\right] = R^d$ hence

$$\iint_{C_R^2} (\rho_2(x,y) - 1)dxdy = \mathbf{E}_P\left[\mathcal{D}_R^2\right] - R^d. \tag{4.5.1}$$

Equation (4.5.1) allows us to write the term $\mathcal{D}^{\log}$ (defined in (4.1.8)) in terms of ρ_2 equivalently as

$$\mathcal{D}^{\log}(P) = C^{\log} \limsup_{R\to\infty} \left(\frac{1}{R^d} \iint_{C_R^2} (\rho_2(x,y) - 1)dxdy + 1 \right) \log R$$
$$= C^{\log} \limsup_{R\to\infty} \frac{1}{R^d} \mathbf{E}_P\left[\mathcal{D}_R^2 \log R\right].$$

4.5.1 The electric-intrinsic inequality

Until the end of Section 4.5.1, P denotes a stationary random point process of intensity 1 on $\mathbb{R}^d$ such that $\mathbb{W}^{\mathrm{int}}$ is finite. In the logarithmic cases we assume that $\mathcal{D}^{\log}(P)$ is finite.

The following proposition is the first part of the proof of Theorem 12.

Proposition 4.5.1. *Under the above assumptions, we have in the cases* (4.1.3)

$$\mathbb{W}^{\mathrm{elec}}(P) \leq \mathbb{W}^{\mathrm{int}}(P),$$

and in the logarithmic cases $\mathbb{W}^{\mathrm{elec}}(P) \leq \mathbb{W}^{\mathrm{int}}(P) + \mathcal{D}^{\log}(P)$.

a. Screenability of the local electric fields.

Let $\eta, \varepsilon > 0$ be fixed. Let $\{R_n\}_n$ be an increasing sequence of real numbers such that $\lim_{n\to\infty} R_n = +\infty$ and

$$\lim_{n\to\infty} \frac{1}{R_n^d} \mathbf{E}_P[\mathcal{H}_{R_n}^{\mathrm{int}}] = \mathbb{W}^{\mathrm{int}}(P).$$

We start by an auxiliary lemma.

Lemma 4.5.2. *Let $E^{\mathrm{loc},n}$ denotes the local electric field generated by a point configuration in C_{R_n}. The following inequality holds*

$$P\left(\frac{1}{R_n^d}\int_{C_{R_n}\times\mathbb{R}^k}|y|^\gamma|E_\eta^{\mathrm{loc},n}|^2\le M\right)=1-O(M^{-1}),\tag{4.5.2}$$

moreover, in the case $k=1$,

$$P\left(\frac{1}{\varepsilon^4R_n^d}\int_{C_{R_n}\times(\mathbb{R}\setminus(-\varepsilon^2R_n,\varepsilon^2R_n))}|y|^\gamma|E^{\mathrm{loc},n}|^2\le\left(10R_0\varepsilon^{\frac{2\gamma}{1-\gamma}}\right)^{\gamma-1}R_n^{d-s}\right)=1-o_n(1).\tag{4.5.3}$$

Proof. The first point (4.5.2) follows directly from Markov's inequality and the assumptions on P. Indeed we have, in the cases (4.1.3) (using (4.4.3))

$$\mathbf{E}_P\left[\frac{1}{c_{d,s}}\int_{C_{R_n}\times\mathbb{R}^k}|y|^\gamma|E_\eta^{\mathrm{loc},n}|^2\right]\le\mathbf{E}_P[\mathcal{H}^{\mathrm{int}}_{R_n}]+O(R_n^d),$$

and in the logarithmic cases (using (4.4.4))

$$\mathbf{E}_P\left[\frac{1}{c_{d,s}}\int_{C_{R_n}\times\mathbb{R}^k}|y|^\gamma|E_\eta^{\mathrm{loc},n}|^2\right]\le\mathbf{E}_P[\mathcal{H}^{\mathrm{int}}_{R_n}]+C^{\log}\mathbf{E}_P[\mathcal{D}^2_{R_n}]\log R_n+O(R_n^d),$$

where the terms $O(R_n^d)$ depend on η.

To prove the second point (4.5.3) let us fix $X=(x,y)$ with $x\in C_{R_n}$ and $y\ge\varepsilon^2R_n$. We may estimate $E^{\mathrm{loc},n}(X)$ as follows: let $R_0>0$ and let us divide R_n into $O(R_n^d/R_0^d)$ hypercubes $\{\bar C_i\}_{i\in I}$ of sidelength $\in(\frac12R_0,\frac32R_0)$. For any $i\in I$ we have by a mean value argument

$$\left|\int_{\bar C_i}\nabla g(X-t)(d\mathcal{C}(t)-dt)\right|\le C|\mathcal{D}_i||y|^{-s-1}+CR_0(\mathcal{N}_i+R_0^d)|y|^{-s-2},$$

where $\mathcal{D}_i$ (resp. $\mathcal{N}_i$) denotes the discrepancy (resp. the number of points) in $\bar C_i$. Summing over $i\in I$ we get

$$|E^{\mathrm{loc},n}(X)|\le C\sum_{i\in I}|\mathcal{D}_i||y|^{-s-1}+CR_0(\mathcal{N}_{R_n}+R_n^d)|y|^{-s-2}.$$

We may thus write, using the stationarity of P

$$\mathbf{E}_P[|E^{\mathrm{loc},n}(X)|^2]\le C\frac{R_n^{2d}}{R_0^{2d}}\mathbf{E}_P[\mathcal{D}^2_{R_0}]|y|^{-2s-2}+CR_0(\mathbf{E}[\mathcal{N}^2_{R_n}]+R_n^{2d})|y|^{-2s-4}.$$

Using Lemma 4.4.11 or the fact that $\mathcal{D}^{\log}(P)$ is finite we have $\mathbf{E}[\mathcal{N}^2_{R_n}]=R_n^{2d}+O(R_n^{2d})$. Finally, integrating over $C_R\times\mathbb{R}\setminus(-\varepsilon^2R_n,\varepsilon^2R_n))$ we obtain

$$\mathbf{E}_P\left[\frac{1}{\varepsilon^4R_n^d}\int_{C_{R_n}\times(\mathbb{R}\setminus(-\varepsilon^2R_n,\varepsilon^2R_n))}|y|^\gamma|E^{\mathrm{loc},n}|^2\right]$$
$$\le\frac{1}{\varepsilon^4}\left(C\mathbf{E}_P[\mathcal{D}^2_{R_0}]\frac{R_n^{2d}}{R_0^{2d}}\frac{1}{(\varepsilon^2R_n)^{d+s}}+CR_0R_n^{2d}\frac{1}{(\varepsilon^2R_n)^{d+s+2}}\right).$$

Using the fact that $\frac{1}{R_0^{2d}}\mathbf{E}_P[\mathcal{D}^2_{R_0}]=o(1)$ as $R_0\to\infty$ (see Lemma 4.4.3), we obtain

$$\mathbf{E}_P\left[\frac{1}{\varepsilon^4R_n^d}\int_{C_{R_n}\times(\mathbb{R}\setminus(-\varepsilon^2R_n,\varepsilon^2R_n))}|y|^\gamma|E^{\mathrm{loc},n}|^2\right]=o(R_n^{d-s}).$$

The bound (4.5.3) follows by Markov's inequality. □

b. Proof of Proposition 4.5.1.

We now turn to the proof of Proposition 4.5.1.

Proof. **Screening the local electric fields.** When $\eta, \varepsilon > 0$ and $M > 0$ are fixed we denote by $\mathcal{S}^{M,\varepsilon}_{R_n,\eta}$ the set of point configurations in C_{R_n} such that $E^{\mathrm{loc},n}$ satisfies (4.5.2) and, in the case $k = 1$, (4.5.3). For any $\mathcal{C}$ in $\mathcal{S}^{M,\varepsilon}_{R_n,\eta}$ the conclusions of Lemma 4.4.4 apply to $E^{\mathrm{loc},n}$. We may thus find a point configuration $\mathcal{C}^{\mathrm{scr}}$ in $K_n := C_{\lceil R_n \rceil}$ and a compatible field E^{scr} such that

1. The point configurations $\mathcal{C}$ and $\mathcal{C}^{\mathrm{scr}}$ coincide on a large subset of C_{R_n}, namely $\{x \in C_{R_n}, \mathrm{dist}(x, \partial C_{R_n} \geq 2\varepsilon R_n\}$.
2. The vector field E^{scr} is screened i.e. $E^{\mathrm{scr}} \cdot \vec{n} = 0$ on $\partial K_n \times \mathbb{R}^k$.
3. The energy of E^{scr} is bounded in terms of that of $E^{\mathrm{loc},n}$ as in (4.4.13)

$$\int_{K_n \times \mathbb{R}^k} |y|^\gamma |E^{\mathrm{scr}}_\eta|^2 \leq \left(\int_{C_{R_n} \times \mathbb{R}^k} |y|^\gamma |E^{\mathrm{loc},n}_\eta|^2 \right) (1 + C\varepsilon) + Cg(\eta) M \varepsilon R_n^d + C e_{\varepsilon,R} \varepsilon R_n^d + o(R_n^{d-1}), \quad (4.5.4)$$

where $e_{\varepsilon,R}$ is defined in (4.4.12).

Constructing a global electric field. Any such point configuration (resp. vector field) may be extended periodically in the whole space $\mathbb{R}^d$. The main point is that since E^{scr} is screened we can paste together several copies of E^{scr} periodically without creating divergence at the boundary of two tiles. Let $\mathcal{C}^{\mathrm{per}}$ (resp. E^{per}) be the resulting periodic point configuration (resp. vector field)

We let P_n be the conditional expectation of P knowing $\mathcal{S}^{M,\varepsilon}_{R_n,\eta}$ and we let P_n^{per} be the push-forward of P_n by the map $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{per}}$ defined above. Finally we let P_n^{av} be the average of P_n^{per} over translations in K_n. Taking the expectation of (4.5.4) under P we see that (with $\mathbb{W}^{\mathrm{elec}}_\eta$ as defined in (4.3.1))

$$\begin{aligned}
\mathbb{W}^{\mathrm{elec}}_\eta(P_n^{\mathrm{av}}) &\leq \frac{1}{R_n^d} \mathbf{E}_{P_n} \left[\frac{1}{c_{d,s}} \int_{K_n \times \mathbb{R}^k} |y|^\gamma |E^{\mathrm{scr}}_\eta|^2 \right] - g(\eta) \\
&\leq \frac{1}{R_n^d} \mathbf{E}_{P_n} \left[\frac{1}{c_{d,s}} \int_{C_{R_n} \times \mathbb{R}^k} |y|^\gamma |E^{\mathrm{loc},n}_\eta|^2 \right] (1 + C\varepsilon) - g(\eta) + Cg(\eta) M\varepsilon \\
&\qquad + C\varepsilon \mathbf{E}_{P_n}[e_{\varepsilon,R_n}] + o(R_n^{-1}).
\end{aligned}$$

From Lemma 4.5.2 we see that as $M \to \infty$, $n \to \infty$ the random point process P_n converges to P. In particular we may bound the expectations under P_n in the right-hand side by the expectation under P at a small cost

$$\mathbf{E}_{P_n} \left[\int_{C_{R_n} \times \mathbb{R}^k} |y|^\gamma |E^{\mathrm{loc},n}_\eta|^2 \right] \leq \mathbf{E}_P \left[\int_{C_{R_n} \times \mathbb{R}^k} |y|^\gamma |E^{\mathrm{loc},n}_\eta|^2 \right] (1 + o(1)),$$

$$\mathbf{E}_{P_n}[e_{\varepsilon,R_n}] \leq \mathbf{E}_P[e_{\varepsilon,R}](1 + o(1))$$

where both terms are $o(1)$ as $M \to \infty, n \to \infty$ (keeping ε, η fixed). By definition of $e_{\varepsilon,R}$ and a mean value argument we see that up to changing ε into 2ε we may assume that

$$\mathbf{E}_P[e_{\varepsilon,R_n}] = \mathbf{E}_P \left[\frac{1}{\varepsilon^4 R_n^d} \int_{C_{R_n} \times (\mathbb{R} \setminus (-\varepsilon^2 R_n, \varepsilon^2 R_n))} |y|^\gamma |E^{\mathrm{loc},n}|^2 \right] \leq \frac{1}{\varepsilon^6 R_n} M.$$

We thus get

$$\mathbb{W}_\eta^{\mathrm{elec}}(P_n^{\mathrm{av}}) \leq \frac{1}{R_n}\mathbf{E}_P\left[\frac{1}{c_{d,s}}\int_{C_{R_n}\times\mathbb{R}^k}|y|^\gamma|E_\eta^{\mathrm{loc},n}|^2\right](1+o(1))(1+C\varepsilon) - g(\eta) + Cg(\eta)M\varepsilon + C\varepsilon^{-5}R_n^{-1}.$$

The convergence of P_n to P implies the convergence of P_n^{av} to P as $M, n \to \infty$ and $\varepsilon \to 0$. This might be seen as follows: the topology on $\mathcal{P}(\mathcal{X})$ is such that for any fixed $\delta > 0$, if two random point processes coincide (or are close to each other) in C_{R_0} for R_0 large enough, then they are δ-close. In particular P_n and P are very close to each other in C_{R_n} because on the one hand the vast majority of point configurations under P are screenable and on the other hand the screening procedure does not modify the points in a large interior part of C_{R_n}. Heuristically speaking, if R_n is larger than some R_0 then P_n and P should be δ-close. Now when averaging the random point process P_n over translations in C_{R_n} there is a nonzero proportion of $z \in C_n$ which are such that the translation by z of the thin layer of C_{R_n} in which the points have been modified ends up intersecting C_{R_0}, thus P_n^{av} look *less* like P in C_0. However when $R_n \gg R_0$ this proportion is of order ε, hence P_n^{av} still converges to P when taking M, n large and ε small.

Conclusion. Taking M large, letting $n \to \infty$ and using (4.4.3) or (4.4.4) we obtain, by lower semi-continuity of $\mathbb{W}_\eta^{\mathrm{elec}}$ over $\mathcal{P}_{s,1}(\mathcal{X})$, (see Lemma 4.3.2) that

$$\mathbb{W}_\eta^{\mathrm{elec}}(P) \leq \mathbb{W}^{\mathrm{int}}(P)(1+C\varepsilon) + O(\varepsilon) + o_\eta(1),$$

plus an additional $\mathcal{D}^{\log}(P)$ term in the logarithmic cases. Sending $\varepsilon \to 0$ and $\eta \to 0$ we finally obtain that $\mathbb{W}^{\mathrm{elec}}(P) \leq \mathbb{W}^{\mathrm{int}}(P)$ or, in the logarithmic cases $\mathbb{W}^{\mathrm{elec}}(P) \leq \mathbb{W}^{\mathrm{int}}(P) + \mathcal{D}^{\log}(P)$, which concludes the proof of Proposition 4.5.1. □

4.5.2 Construction of a recovery sequence

In this section P denotes a stationary random point process of intensity 1 such that $\mathbb{W}^{\mathrm{elec}}(P)$ is finite. Moreover we assume that we are in one of the non-Coulomb cases, i.e. $d = 1$ or $d \geq 2$ and $s > d - 2$.

The following result forms the second step in the proof of Theorem 12.

Proposition 4.5.3. *There exists a sequence $\{P_N\}_N$ of hyperuniform random point processes in $\mathcal{P}_{s,1}(\mathcal{X})$ such that*

$$\lim_{N\to\infty} P_N = P, \quad \lim_{N\to\infty} \mathsf{ent}[P_N|\mathbf{\Pi}] = \mathsf{ent}[P|\mathbf{\Pi}],$$

and satisfying

$$\lim_{N\to\infty} \mathbb{W}^{\mathrm{int}}(P_N) = \mathbb{W}^{\mathrm{elec}}(P). \tag{4.5.5}$$

Let us observe that since the random point processes are hyperuniform, in the logarithmic cases they satisfy $\mathcal{D}^{\log}(P_N) = 0$ for any N.

The proof of Proposition 4.5.3 goes in two steps.

1. First we construct an auxiliary sequence of random point processes which converges to P and such that almost every point configuration is finite and "screened" i.e. there exists an associated *screened* electric field. This is done in Lemma 4.5.4.
2. Next, we extend this random point process in the whole space and make it stationary, before bounding its interaction energy in terms of $\mathbb{W}^{\mathrm{elec}}(P)$.

a. An auxiliary sequence

Lemma 4.5.4. *There exists a sequence $\{P_N^{(1)}\}_N$ of random point processes in $\mathcal{P}(\mathcal{X}(C_N))$ such that*

0. *The sequence $\{P_N^{(1)}\}_N$ converges to P as $N \to \infty$. More precisely, there exists a sequence $\{L_N\}_N$ such that $L_N = N(1 - o(1))$ and the respective restrictions of P_N^{mod} and P to C_{L_N} are arbitrarily close as $N \to \infty$.*

1. *For $P_N^{(1)}$-a.e. point configuration $\mathcal{C}^{(1)}$ there exists a screened electric field $E^{(1)}$ satisfying*

$$-\mathrm{div}\,(|y|^\gamma E^{(1)}) = c_{d,s}(\mathcal{C}^{(1)} - \delta_{\mathbb{R}^d}) \text{ in } C_N \times \mathbb{R}^k, \tag{4.5.6}$$

$$E^{(1)} \cdot \vec{n} = 0 \text{ on } \partial C_N \times \mathbb{R}^k. \tag{4.5.7}$$

In particular the point configurations have $P_N^{(1)}$-a.s. N^d points in C_N. We also have

$$\min_{p \in \mathcal{C}^{(1)}} \mathrm{dist}(p, \partial C_N) \geq \eta_0, \tag{4.5.8}$$

for some $\eta_0 > 0$ depending only on d, s.

2. *The following estimate holds*

$$\limsup_{N\to\infty} \lim_{\eta \to 0} \mathbf{E}_{P_N^{(1)}} \left[\frac{1}{c_{d,s}} \frac{1}{N^d} \int_{C_N \times \mathbb{R}^k} |y|^\gamma |E_\eta^{(1)}|^2 - g(\eta) \right] \leq \mathbb{W}^{\mathrm{elec}}(P). \tag{4.5.9}$$

3. *The relative entropies of $P_N^{(1)}$ and $P_{|C_N}$ with respect to $\mathbf{\Pi}_{|C_N}$ are close*

$$\mathrm{Ent}[P_N^{(1)} | \mathbf{\Pi}_{|C_N}] = \mathrm{Ent}[P_{|C_N} | \mathbf{\Pi}_{|C_N}] + o(N^d). \tag{4.5.10}$$

Proof. This follows from the analysis of [LS15], and we sketch here the main steps.

Let P^{elec} be a stationary electric process associated to P as in Lemma 4.3.1. For fixed $R, M, \varepsilon, \eta > 0$ we say that an electric field E is in $\mathcal{S}_{R,\eta}^{M,\varepsilon}$ (or is *screenable*) if its energy is controlled as follows

$$\frac{1}{R^d} \int_{C_R \times \mathbb{R}^d} |y|^\gamma |E_\eta|^2 \leq M \text{ and, if } k = 1, \frac{1}{\varepsilon^4 R^d} \int_{C_R \times \mathbb{R}^d \setminus (-\varepsilon^2 R, \varepsilon^2 R)} |y|^\gamma |E_\eta|^2 \leq 1.$$

Under the assumption that $\mathbb{W}^{\mathrm{elec}}(P)$ is finite, then the probability $P^{\mathrm{elec}}(\mathcal{S}_{R,\eta}^{M,\varepsilon})$ tends to 1 as $M, R \to \infty$ for any $\varepsilon, \eta > 0$ fixed. This is proven in [LS15, Lemma 5.10] and is similar in spirit to Lemma 4.5.2.

If R is an integer much larger than M and E is in $\mathcal{S}_{R,\eta}^{M,\varepsilon}$, the screening procedure as in Lemma 4.4.4 (see also [PS15, Proposition 6.1] and [LS15, Proposition 5.2]) applies. In particular we may change the underlying point configuration in a thin layer of size $\leq \varepsilon R$ close to the boundary of C_R and obtain a new *screened* point configuration $\mathcal{C}^{\mathrm{scr}}$ in $\mathcal{X}(C_R)$ as well as a compatible *screened* electric field E^{scr} which satisfies (4.5.7). It also ensures that (4.5.8) holds. The screening procedure is described in [LS15, Section 5.1]. The energy of E^{scr} is bounded in terms of the energy of E as in (4.4.13).

The next step is to regularize the point configurations $\mathcal{C}^{\mathrm{scr}}$ by separating the pair of points which are close from each other. This regularization procedure is described in [LS15, Section 5.2], and another electric field E^{mod} can be associated to the regularized point configurations, with a good energy bound. The main benefit of this procedure is to control the difference between $\frac{1}{R^d} \int_{C_R \times \mathbb{R}^k} |y|^\gamma |E_\eta^{\mathrm{mod}}|^2 - g(\eta)$ and its limit as $\eta \to 0$. In general the limit may be much larger because of the contribution of pairs of points which are very close, at distance $\ll \eta$, and which are not "seen" when truncating at scale $\eta > 0$.

We let ε, η tend to 0 and M tend to infinity (depending on N) and we pick $R = N$ large enough. We let $P_N^{(1)}$ be the associated random point process in C_N. Most of the point configurations (or electric fields) are "screenable", the screening procedure only modifies the configuration in a thin boundary layer of C_N, and the regularization moves only a *small* fraction of the points by a *small* distance. This ensures that $\mathbb{P}_N^{(1)}$ converges to P as $N \to \infty$.

The estimates on the energy of the screened-then-regularized electric fields are such that (4.5.9) holds (see [LS15, Section 6.3.4.]) with $E^{(1)} := E^{\mathrm{mod}}$.

Concerning the entropy, letting the new/modified points of the configurations move in small balls allow us to recover a small, nonzero volume in phase space without affecting the energy, since only a small fraction of the points have been deleted/created/modified, it gives (4.5.10) (see [LS15, Section 6.3.5] for a precise analysis of the volume loss). □

b. Proof of Proposition 4.5.3

Proof. **Step 1.** *Construction of the random point process.* Let $\{P_N^{(1)}\}_N$ be as in Lemma 4.5.4 and let us extend $P_N^{(1)}$ in the whole space $\mathbb{R}^d$ as follows. Let $\{\bar{C}_i\}_{i\in I}$ be a tiling of $\mathbb{R}^d$ by a family of hypercubes of sidelength N and let x_i be the center of x_i (we may impose that one of the x_i's is 0). Let $\{P_{N,i}^{(1)}\}_{i\in I}$ be the laws of independent point processes distributed as $P_N^{(1)}$.

To any family $\{\mathcal{C}^{(i)}\}_{i\in I}$ of point configurations in C_N we may associate the point configuration

$$\mathcal{C} := \sum_{i\in I} \theta_{x_i} \cdot \mathcal{C}^{(i)}$$

which amounts to "paste" the point configuration $\mathcal{C}^{(i)}$ in the hypercube $\bar{C}_i$.

For any $i \in I$, let $E^{(i)}$ be an electric field which is compatible with $\mathcal{C}^{(i)}$ as in (4.5.6) and screened as in (4.5.7). By the latter condition, the normal component of each $E^{(i)}$ vanishes on the boundary of $C_N \times \mathbb{R}^k$, thus we may paste together such fields along the boundaries and their energy is additive. In particular the electric field E defined by

$$E(x) := \sum_{i\in I} E^{(i)}(x - x_i)$$

is compatible with $\mathcal{C}$ and moreover we have

$$\int_A |E_\eta|^2 = \sum_{i\in I} \int_A |E_\eta^{(i)}|^2$$

for any measurable subset $A \subset \mathbb{R}^{d+k}$ and any $\eta > 0$. Let us also observe that, by construction, the normal component of E vanishes on the boundary of $C_{mN} \times \mathbb{R}^k$ for any $m \geq 1$ (in fact it is easy to see that it vanishes on any path included in $N\mathbb{Z}^d \times \mathbb{R}^k$).

We let $P_N^{(2)}$ be the random point process obtained by pasting $P_{N,i}^{(1)}$ in $\bar{C}_i$ for $i \in I$, or in other terms the push-forward of the product measure of the $\{P_{N,i}^{(1)}\}_{i\in I}$ by the map

$$\{\mathcal{C}^{(i)}\}_{i\in I} \mapsto \mathcal{C} := \sum_{i\in I} \theta_{x_i} \cdot \mathcal{C}^{(i)}.$$

For any $z \in C_N$, we let $P_{N,z}^{(2)} = \theta_z \cdot P_N^{(2)}$ be the push-forward of $P_N^{(2)}$ by the translation by a vector z. We also define $P_N^{(3)}$ as the uniform average of $P_{N,z}^{(2)}$ for $z \in C_N$. It is not hard to check that $P_N^{(3)}$ is a hyperuniform stationary random point process which converges to P as $N \to \infty$.

Step 2. *Estimates on the energy.* We fix $z \in C_N$ and $m \geq 1$, and we use the subscript z to denote a translation by z, e.g. $C_{mN,z}$ denotes the hypercube C_{mN} translated by z.

By construction, to $P^{(2)}_{N,z}$-almost every point configuration in $C_{(m+1)N,z}$ we may associate an electric field E whose normal component vanishes on the boundary of $C_{(m+1)N,z} \times \mathbb{R}^k$.

Let us recall that $\mathrm{Int}[A,B](\mathcal{C})$ denotes the interaction energy between the sets A and B (cf. (4.3.2)). By minimality of the local energy (cf. Lemma 4.4.5) we have for $P^{(2)}_{N,z}$-almost every $\mathcal{C}$

$$\mathrm{Int}[C_{(m+1)N,z}, C_{(m+1)N,z}](\mathcal{C}) \leq \lim_{\eta\to 0}\left(\frac{1}{c_{d,s}}\int_{C_{(m+1)N,z}\times\mathbb{R}^k} |E_\eta|^2 - N^d g(\eta)\right).$$

Using (4.5.9) we see that the right-hand side is bounded in terms of $\mathbb{W}^{\mathrm{elec}}(P)$ as $N\to\infty$, more precisely we obtain

$$\limsup_{N\to\infty}\limsup_{m\to\infty}\frac{1}{(mN)^d}\mathbf{E}_{P^{(2)}_{N,z}}\left[\mathrm{Int}[C_{(m+1)N,z}, C_{(m+1)N,z}]\right] \leq \mathbb{W}^{\mathrm{elec}}(P), \tag{4.5.11}$$

and both limits (as $m\to\infty$ and $N\to\infty$) are uniform for $z\in C_N$.

This gives an asymptotic upper bound on the expectation of $\mathrm{Int}[C_{(m+1)N,z}, C_{(m+1)N,z}]$ under $P^{(2)}_{N,z}$, however the relevant quantity to control in order to get (4.5.5) is rather the expectation of $\mathrm{Int}[C_{mN}, C_{mN}]$. We thus need to bound the difference $\mathrm{Int}[C_{(m+1)N,z}, C_{(m+1)N,z}] - \mathrm{Int}[C_{mN}, C_{mN}]$.

Let us write $\mathrm{Int}[C_{(m+1)N,z}, C_{(m+1)N,z}]$ as

$$\begin{aligned}\mathrm{Int}[C_{(m+1)N,z}, C_{(m+1)N,z}] = \mathrm{Int}[C_{mN}, C_{mN}] + 2\mathrm{Int}[C_{mN}, C_{(m+1)N,z}\backslash C_{mN}]\\ + \mathrm{Int}[C_{(m+1)N,z}\backslash C_{mN}, C_{(m+1)N,z}\backslash C_{mN}].\end{aligned} \tag{4.5.12}$$

We may bound the last term in the right-hand side of (4.5.12) using Lemma 4.4.2 with (for the notations of the lemma) $K = C_{(m+1)N,z}\backslash C_{mN}$ and $R = (m+1)N$. In $C_{(m+1)N,z}\backslash C_{mN}$ there are $O(m^{d-1})$ points, and the discrepancy between the number of points and the volume is also such that $(\mathcal{N}(\mathcal{C},K) - |K|)^2 = O(m^{d-1})$. Applying (4.4.9) in the cases (4.1.3) and (4.4.10) in the logarithmic case we obtain

$$\mathrm{Int}[C_{(m+1)N,z}\backslash C_{mN}, C_{(m+1)N,z}\backslash C_{mN}] \geq o(m^d) \text{ as } m\to\infty. \tag{4.5.13}$$

We are left to estimate the interaction term $\mathrm{Int}[C_{mN}, C_{(m+1)N,z}\backslash C_{mN}]$. We may split it as

$$\begin{aligned}\mathrm{Int}[C_{mN}, C_{(m+1)N,z}\backslash C_{mN}] = \mathrm{Int}[C_{(m-1)N,z}, C_{(m+1)N,z}\backslash C_{mN}]\\ + \mathrm{Int}[C_{mN}\backslash C_{(m-1)N,z}, C_{(m+1)N,z}\backslash C_{mN}],\end{aligned} \tag{4.5.14}$$

and we will prove

$$\mathbf{E}_{P^{(2)}_{N,z}}\left[\mathrm{Int}[C_{mN}, C_{(m+1)N,z}\backslash C_{mN}]\right] = o(m^d). \tag{4.5.15}$$

Proof of (4.5.15) First we claim that

$$\mathbf{E}_{P^{(2)}_{N,z}}\left[\mathrm{Int}[C_{(m-1)N,z}, C_{(m+1)N,z}\backslash C_{mN}]\right] = o(m^d). \tag{4.5.16}$$

To prove (4.5.16) let us write

$$\mathrm{Int}\left[C_{(m-1)N,z}, C_{(m+1)N,z}\backslash C_{mN}\right] = -\frac{1}{c_{d,s}}\int_{C_{(m-1)N,z}} \Phi^a \mathrm{div}\,(|y|^\gamma E^b),$$

where Φ^a is the local electric potential generated by the system of charges in $C_{(m+1)N,z}\backslash C_{mN}$ i.e.

$$\Phi^a(x) := \int_{C_{(m+1)N,z}\backslash C_{mN}} g(x-t)(d\mathcal{C}(t) - dt)$$

and E^b is the screened electric field associated to the system of charges in $C_{(m-1)N,z}$. Using (4.5.8) we may also write

$$\mathrm{Int}\left[C_{(m-1)N,z}, C_{(m+1)N,z}\backslash C_{mN}\right] = -\frac{1}{c_{d,s}}\int_{C_{(m-1)N,z}} \Phi^a_{\eta_0}\mathrm{div}\,(|y|^\gamma E^b_{\eta_0}),$$

because the minimal distance between a point charge in $C_{(m-1)N,z}$ and one in $C_{(m+1)N,z}\backslash C_{mN}$ is $\geq \eta_0$. An integration by parts and Cauchy-Schwarz's inequality yield (observing also that $C_{(m-1)N,z} \subset C_{(m+2)N}$)

$$\begin{aligned}\mathrm{Int}&\left[C_{(m-1)N,z}, C_{(m+1)N,z}\backslash C_{mN}\right]\\ &\geq -C\left(\int_{C_{(m+2)N}\times\mathbb{R}^k} |y|^\gamma |E^a_{\eta_0}|^2\right)^{1/2}\left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |E^b_{\eta_0}|^2\right)^{1/2}, \quad (4.5.17)\end{aligned}$$

for some constant C depending on d, s. Using the definition of $P^{(2)}_{N,z}$ we have

$$\mathbf{E}_{P^{(2)}_{N,z}}\left[\int_{\mathbb{R}^{d+k}} |E^b_{\eta_0}|^2\right] \leq m^d \mathbf{E}_{P^{(2)}_N}\left[\int_{C_N\times\mathbb{R}^k} |E^{(1)}_{\eta_0}|^2\right],$$

where $E^{(1)}$ is as in (4.5.6). Using (4.5.9) we obtain that

$$\limsup_{N\to\infty, m\to\infty} \mathbf{E}_{P^{(2)}_{N,z}}\left[\frac{1}{(mN)^d}\int_{\mathbb{R}^{d+k}} |E^b_{\eta_0}|^2\right] \leq (\mathbb{W}^{\mathrm{elec}}(P) + C)$$

with C depending only on d, s. In particular this yields

$$\mathbf{E}_{P^{(2)}_{N,z}}\left[\int_{\mathbb{R}^{d+k}} |E^b_{\eta_0}|^2\right] = O(m^d).$$

Using Jensen's inequality we may thus bound the second term in the right-hand side of (4.5.17) as

$$\mathbf{E}\left[\left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |E^b_{\eta_0}|^2\right)^{1/2}\right] = O(m^{d/2}). \quad (4.5.18)$$

It remains to control the the first term in the right-hand side of (4.5.17). We claim that

$$\mathbf{E}_{P^{(2)}_{N,z}}\left[\int_{C_{(m+2)N}\times\mathbb{R}^k} |y|^\gamma |E^a_{\eta_0}|^2\right] = o(m^d). \quad (4.5.19)$$

Since E^a is the local field generated by the configuration in $C_{(m+1)N,z}\backslash C_{mN}$ we may, by almost monotonicty as in Lemma 4.4.1 compare $\int_{C_{(m+2)N}\times\mathbb{R}^k} |y|^\gamma |E^a_{\eta_0}|^2$ with the interaction energy

$$\mathrm{Int}[C_{(m+1)N,z}\backslash C_{mN}, C_{(m+1)N,z}\backslash C_{mN}],$$

up to a boundary term in the logarithmic case (which is bounded as usual) and a term of order $O(m^{d-1})g(\eta_0) = o(m^d)$. We now prove that

$$\mathbf{E}_{P^{(2)}_{N,z}}\left[\mathrm{Int}[C_{(m+1)N,z}\backslash C_{mN}, C_{(m+1)N,z}\backslash C_{mN}]\right] = o(m^d). \quad (4.5.20)$$

Let us recall that $\{\bar{C}_i\}_{i\in I}$ denotes a tiling of $\mathbb{R}^d$ by a family of hypercubes of sidelength N. We let $J_{N,z}$ be the set of indices

$$J_{N,z} := \{j \in I, (\bar{C}_{j,z} \cap C_{(m+1)N,z} \backslash C_{mN}) \neq \emptyset\}.$$

and it is clear that the cardinal of $J_{N,z}$ is $O(m^{d-1})$. We may then write the interaction of $C_{(m+1)N,z}\backslash C_{mN}$ with itself as

$$\begin{aligned}
&\mathrm{Int}\left[C_{(m+1)N,z}\backslash C_{mN}, C_{(m+1)N,z}\backslash C_{mN}\right] \\
&= -\sum_{j_1\neq j_2\in J_{N,z}} \int_{(\bar{C}_{j_1,z}\cap C_{(m+1)N,z}\backslash C_{mN})\times(\bar{C}_{j_2,z}\cap C_{(m+1)N,z}\backslash C_{mN})} g(x-y)(d\mathcal{C}(x)-dx)(d\mathcal{C}(y)-dy), \\
&\qquad + \sum_{j\in J_{N,z}} \mathrm{Int}[\bar{C}_{j,z}\cap C_{(m+1)N,z}\backslash C_{mN}), \bar{C}_{j,z}\cap C_{(m+1)N,z}\backslash C_{mN}].
\end{aligned}$$

The previous identity is nothing but writing the interaction of a collection of (possibly truncated) hypercubes with itself as the sum of hypercubes-hypercubes interactions plus the sum of self-interactions.

Since there are $O(m^{d-1})$ elements in $J_{N,z}$ we have

$$\mathbf{E}_{P^{(2)}_{N,z}}\left[\sum_{j\in J_{N,z}} \mathrm{Int}[\bar{C}_{j,z}\cap C_{(m+1)N,z}\backslash C_{mN}), \bar{C}_{j,z}\cap C_{(m+1)N,z}\backslash C_{mN}]\right] = O(m^{d-1}).$$

It remains to bound the sum of interactions between two disjoint hypercubes. We have, if the two hypercubes are not adjacent (since there are only $O(m^{d-1})$ pairs of adjacent hypercubes in the sum, all giving a contribution of order $O(1)$, we may neglect these terms),

$$\begin{aligned}
&\int_{(\bar{C}_{j_1,z}\cap C_{(m+1)N,z}\backslash C_{mN})\times(\bar{C}_{j_2,z}\cap C_{(m+1)N,z}\backslash C_{mN})} g(x-y)(d\mathcal{C}(x)-dx)(d\mathcal{C}(y)-dy) \\
&\qquad\qquad \geq -Cg(\mathrm{dist}(\bar{C}_{j_1},\bar{C}_{j_2})),
\end{aligned}$$

with a constant C depending on N and d, s. For any fixed $j \in J_{N,z}$ we have (the sum is implicitely restricted to non-adjacent hypercubes)

$$\sum_{j'\neq j\in J_{N,z}} g(\mathrm{dist}(\bar{C}_j,\bar{C}_{j'})) \leq C\int_1^{mN} r^{-s}r^{d-2}dr \leq C(m^{d-1-s}+1),$$

with a constant C depending on N and on d, s. The element of volume is only r^{d-2} because we are summing terms on the *boundary* of C_{mN}. We may then estimate the sum of pairwise hypercube interactions as

$$\sum_{j_1\neq j_2\in J_{N,z}} g(\mathrm{dist}(\bar{C}_{j_1},\bar{C}_{j_2})) \leq O(m^{d-1})(O(m^{d-1-s})+O(1)) = o(m^d),$$

because we are restricted to the non-Coulomb cases for which $s > d-2$. It proves (4.5.20) hence also (4.5.19). Combining (4.5.19) and (4.5.18) we obtain (4.5.16).

It remains to control the last term in the right-hand side of (4.5.14), which is another boundary-boundary interaction and is bounded as in (4.5.20).

Conclusion for the energy. Finally, combining (4.5.13) and (4.5.15) we obtain that

$$\mathbf{E}_{P^{(2)}_{N,z}}\left[\mathrm{Int}[C_{(m+1)N,z}, C_{(m+1)N,z}] - \mathrm{Int}[C_{mN}, C_{mN}]\right] \geq o(m^d),$$

with a $o(m^d)$ depending on N and d, s, but uniform in $z \in C_N$. Combining this estimate with (4.5.11) we thus conclude that

$$\limsup_{N\to\infty} \limsup_{m\to\infty} \mathbf{E}_{P_N^{(3)}} [\mathrm{Int}[C_{mN}, C_{mN}]] \leq \mathbb{W}^{\mathrm{elec}}(P).$$

Step 3. *Entropy and conclusion.* The convergence of the entropy follows easily from (4.5.10) and the definition of the relative specific entropy, indeed we have

$$\mathsf{ent}[P_N^{(3)}|\mathbf{\Pi}] = \frac{1}{N^d}\mathrm{Ent}[P_N|\mathbf{\Pi}_{C_N}] + o(1).$$

To summarize, we have shown that the sequence $\{P_N^{(3)}\}_N$ satisfies the requirements of Proposition 4.5.3, which concludes the proof. □

4.6 High-temperature limit

In this section we apply the results of Section 4.5 to study the limit as $\beta \to 0$ (the high-temperature limit) of the minimizers of $\mathcal{F}_\beta$. We prove their convergence to the law of the Poisson point process in all cases (4.1.1), (4.1.2), (4.1.3) as stated in Theorem 13.

4.6.1 Specific Pinsker inequality

The well-known Pinsker inequality gives an upper-bound on the total variation distance between probability measures in terms of their Kullback-Leibler divergence:

$$|P - Q|_{\mathrm{TV}} \leq \sqrt{\frac{1}{2}\mathrm{Ent}(P||Q)}$$

where $|P - Q|_{\mathrm{TV}}$ is the total variation defined by

$$|P - Q|_{\mathrm{TV}} := \sup\{P(A) - Q(A), A \text{ measurable}\}.$$

Combining the Pinsker inequality with the property (4.2.7) of the specific relative entropy we get for any stationary random point process P the following specific (infinite-volume) Pinsker inequality:

$$\sup_{N\geq 1} \frac{|P_{|C_N} - \mathbf{\Pi}_{|C_N}|_{\mathrm{TV}}}{|C_N|^{\frac{1}{2}}} \leq \sqrt{\frac{1}{2}\mathsf{ent}[P|\mathbf{\Pi}]}.$$

Since the total variation convergence implies weak convergence of probability measures, it is clear that the convergence in specific relative entropy sense implies the weak convergence of random point processes i.e. if a sequence of stationary random point processes $\{P_k\}_k$ satisfies

$$\lim_{k\to\infty} \mathsf{ent}[P_k|\mathbf{\Pi}] = 0$$

then the sequence $\{P_k\}_k$ converges to $\mathbf{\Pi}$.

4.6.2 Finite energy approximation the Poisson point process

Since the two-point correlation function of the Poisson point process satisfies $\rho_{2,\mathbf{\Pi}} \equiv 1$ we clearly have $\mathbb{W}^{\mathrm{int}}(\mathbf{\Pi}) = 0$ in all cases (4.1.1), (4.1.2) and (4.1.3). In the case (4.1.3) it thus follows that $\mathbb{W}^{\mathrm{elec}}(\mathbf{\Pi})$ is finite, according to the electric-intrinsic inequality of Section 4.5.1. For the one-dimensional Log-gas it has been proven in [LS15] that $\mathbb{W}^{\mathrm{elec}}(\mathbf{\Pi}) = +\infty$, and the answer is

unknown in the two-dimensional Log-gas case (the result of Section 4.5.1 is not enough because $\mathcal{D}^{\log}(\mathbf{\Pi})$ is infinite). However we may always construct random point processes which converge in entropy sense to $\mathbf{\Pi}$ and whose renormalized energies are finite.

Lemma 4.6.1. *There exists a sequence $\{\pi_k\}_k$ of stationary random point processes in $\mathcal{P}_{s,1}(\mathcal{X})$ satisfying*

1. *$\mathbb{W}^{\mathrm{elec}}(\pi_k)$ is finite for all k.*
2. *$\lim_{k\to\infty} \mathsf{ent}[\pi_k|\mathbf{\Pi}] = 0$.*

Proof. For any $k \geq 1$, let $\{C_k^i\}_{i\in I}$ be a tiling of $\mathbb{R}^d$ by a countable family of disjoint copies of the hypercube C_k, and let $\{\mathbf{B}_k^i\}_i$ be the law of a family of independent Bernoulli point processes with k^d points in C_k^i. We let π_k^t be the random point process consisting of the union of all $\mathbf{B}_k^i$ for $i \in I$. Finally we define π_k by averaging π_k^t over a "fundamental domain" i.e. we let

$$\pi_k := u_k + \pi_k^t,$$

where u_k is a uniform random variable in C_k (if $\mathcal{C}$ is a point configuration we let $x + \mathcal{C}$ denote the point configuration $\{x + p, p \in \mathcal{C}\}$, cf. also (4.1.10)).

The random point processes π_k defined this way are clearly stationary and of intensity 1. The two-point correlation function of π_k^t is easy to compute:

$$\rho_2^t(x,y) = \begin{cases} 1 - \frac{1}{k^d} \text{ if } x \text{ and } y \text{ belong to the same hypercube } C_i^k, \\ 1 \text{ otherwise.} \end{cases} \tag{4.6.1}$$

The two-point correlation function of π_k could be deduced from (4.6.1) by averaging $\rho_2^t(x,y)$ over translations of both coordinates by a vector in C_k. Let us simply observe that $\rho_2(x,y) - 1$ is bounded (because ρ_2^t is) and has compact support (e.g. $\rho_2(x,y) = 1$ as soon as $|x-y| \geq \sqrt{d}k$) which implies (using the expression (4.3.6)) that $\mathbb{W}^{\mathrm{int}}(\pi_k)$ is finite. Moreover, observing that

$$\int_{C_R^2} (\rho_2^t(x,y) - 1) = -1$$

we also get that

$$\int_{C_R^2} (\rho_2(x,y) - 1) = -1 + O(R^{d-1})$$

which implies in view of (4.5.1) that $\mathbf{E}_{\pi_k}[\mathcal{D}_R^2] = O(R^{d-1})$, hence $\mathcal{D}^{\log}(\pi_k)$ is zero. Using the electric-intrinsic inequality we conclude that $\mathbb{W}^{\mathrm{elec}}(\pi_k)$ is finite for all $k \geq 1$ which proves the first point.

We are left to prove the second point of the lemma. Let $\mathcal{E}_k^t$ be the measurable subset of point configurations which have exactly k^d points in each hypercube C_k^i, and let $\mathcal{E}_k$ be the subset obtained by averaging $\mathcal{E}_k^t$ over C_k, more precisely we let

$$\mathcal{E}_k := \bigcup_{x\in C_k, \mathcal{C}\in\mathcal{E}_k^t} x + \mathcal{C}$$

where the sum $x + \mathcal{C}$ is defined as above. By definition π_k coincides with the law of the Poisson point process conditioned to the event $\mathcal{E}_k$. For any $R > 0$ we may thus estimate the relative entropy

$$\mathrm{Ent}\left[(\pi_k)_{C_R}|\mathbf{\Pi}_{|C_R}\right] = -\log \mathbf{\Pi}_{|C_R}(\mathcal{E}_k). \tag{4.6.2}$$

Since $\mathcal{E}_k^t \subset \mathcal{E}_k$ we may bound below $\mathbf{\Pi}_{|C_R}(\mathcal{E}_k)$ by $\mathbf{\Pi}_{|C_R}(\mathcal{E}_k^t)$ which is easier to compute, indeed we only have to estimate the probability that

$$N_{R,k} = \left\lceil \frac{R}{k} \right\rceil^d \approx \frac{R^d}{k^d}$$

disjoint hypercubes of sidelength k receive exactly k^d points, and we can bound below $\mathbf{\Pi}_{|C_R}(\mathcal{E}_k^t)$ by

$$\mathbf{\Pi}_{|C_R}(\mathcal{E}_k^t) \geq \left(e^{-k^d} \frac{(k^d)^{k^d}}{(k^d)!} \right)^{N_{R,k}}.$$

An elementary estimate using Stirling's formula shows that

$$-\log \mathbf{\Pi}_{|C_R}(\mathcal{E}_k^t) \leq C \frac{R^d}{k^d} \tag{4.6.3}$$

with a universal constant C. We deduce from (4.6.2) and (4.6.3) that

$$\frac{1}{R^d} \mathrm{Ent}\left[(\pi_k)_{C_R} | \mathbf{\Pi}_{|C_R} \right] = O(k^{-d})$$

hence by definition of $\mathsf{ent}[\cdot|\mathbf{\Pi}]$ we also have $\mathsf{ent}[\pi_k|\mathbf{\Pi}] = O(k^{-d})$, which proves the second point of the lemma. □

4.6.3 Proof of Theorem 13

From Lemma 4.6.1 the proof of Theorem 13 is straightforward.

Proof. For any $\beta > 0$, let P_β be a minimizer of $\mathcal{F}_\beta$. In particular we have

$$\beta \mathbb{W}^{\mathrm{elec}}(P_\beta) + \mathsf{ent}[P_\beta|\mathbf{\Pi}] \leq \beta \mathbb{W}^{\mathrm{elec}}(\pi_k) + \mathsf{ent}[\pi_k|\mathbf{\Pi}]$$

for any $k \geq 1$, where $\{\pi_k\}_k$ is the sequence of random point processes constructed in Lemma 4.6.1. Since $\mathbb{W}^{\mathrm{elec}}$ is bounded below by some constant depending only on d, s we have

$$\sup_{\mathcal{F}_\beta(P_\beta) = \min \mathcal{F}_\beta} \mathsf{ent}[P_\beta|\mathbf{\Pi}] \leq \mathsf{ent}[\pi_k|\mathbf{\Pi}] + \beta \left(\mathbb{W}^{\mathrm{elec}}(\pi_k) - \min \mathbb{W}^{\mathrm{elec}} \right).$$

Since $\mathsf{ent}[\pi_k|\mathbf{\Pi}] = o_k(1)$ and $\mathbb{W}^{\mathrm{elec}}(\pi_k)$ is always finite, we get (4.1.9) by considering k large enough and β small enough (depending on k). The fact that convergence in entropy sense implies weak convergence was observed in Section 4.6.1. □

Since Sine_β was proven to be a minimizer of $\mathcal{F}_\beta$ for the one-dimensional Log-gas, we get Corollary 4.1.2 as an immediate consequence of Theorem 13. This convergence result was recently established in [AD14] by analysing the family of coupled diffusion processes defining the point processes Sine_β. Here we rely only on the fact that the Sine_β process minimizes the free energy functional $\mathcal{F}_\beta$.

4.7 Low temperature limit in one dimension

In this section we prove Theorem 14 i.e. we use the link between $\mathbb{W}^{\mathrm{elec}}$ and $\mathbb{W}^{\mathrm{int}}$ to give a minimization result on the energy in the one-dimensional case. As can be expected the minimizer of $\mathbb{W}^{\mathrm{elec}}$ is attained by a "crystalline state" which in dimension 1 corresponds simply

to the lattice $\mathbb{Z}$. In the remaining of this section we deal with the cases (4.1.1) or (4.1.3) with $d = 1$ and $0 < s < 1$.

In [SS15a] (in the one-dimensional logarithmic case) the minimality of $\mathbb{W}(\mathbb{Z})$ among the energies of periodic configurations was proven using an explicit formula valid in the periodic setting, together with the convexity of the interaction kernel. An argument of approximation by periodic configurations was then used to prove that $\mathbb{Z}$ is a global minimizer of the energy (however, it is not unique). In [Leb15c] we turned this convexity argument into a quantitative estimate in order to bound below the difference $\mathbb{W}^{\mathrm{elec}}(P) - \mathbb{W}^{\mathrm{elec}}(P_{\mathbb{Z}})$ in terms of the two-point correlation function of P, first in the periodic case, then in the general stationary case using the same kind of approximation. It was enough to prove that $P_{\mathbb{Z}}$ is the unique minimizer of $\mathbb{W}^{\mathrm{elec}}$ among stationary point processes in the case $d = 1, s = 0$. It also yields the fact that if $\mathbb{W}^{\mathrm{elec}}(P_n) \to \mathbb{W}^{\mathrm{elec}}(P_{\mathbb{Z}})$ then the two-point correlation function of P_n converges to that of $P_{\mathbb{Z}}$. In the following we prove the same result in all one-dimensional cases, first at the level of *hyperuniform* random point processes (which include periodic point processes) then in the general stationary case using the approximation argument of Proposition 4.5.3. We also observe that convergence of the two-point correlation functions to that of $P_{\mathbb{Z}}$ in fact implies weak convergence of the random point processes.

4.7.1 The k-th neighbor correlation functions

In the one-dimensional case the two-point correlation function of a stationary random point process admits a decomposition as the sum of the k-th neighbor correlation functions.

Let P be in $\mathcal{P}_{s,1}(\mathcal{X})$ such that the two-point correlation ρ_2 exists as a Radon measure in $\mathbb{R} \times \mathbb{R}$. For any $k \geq 1$ we define the k-th neighbor correlation function $\rho_{2,k}$ by duality, letting for any $\varphi \in C_c(\mathbb{R} \times \mathbb{R})$

$$\int \varphi \rho_{2,k} := \frac{1}{2} \mathbf{E}_P \left[\sum_{x,y \in \mathcal{C}, y \ k\text{-th neighbor of } x} \varphi(x,y) + \varphi(y,x) \right]. \tag{4.7.1}$$

In (4.7.1) if x, y belong to a point configuration $\mathcal{C}$ we say that y is the k-th neighbor of x if $x < y$ and $\mathcal{C}([x,y]) = k+1$. We will abbreviate "k-th neighbor of" by *k.n.o.* in the formulas. Since P is stationary we may see $\rho_{2,k}$ as a measure on $\mathbb{R}$ by letting $\rho_{2,k}(x) := \rho_{2,k}(0,x)$ (in the rest of this section we will use the same notation for both interpretations of $\rho_{2,k}$).

Lemma 4.7.1. *For any $k \geq 1$, for any compactly supported, measurable, even function $\varphi : \mathbb{R} \mapsto \mathbb{R}$, we have*

$$\mathbf{E}_P \left[\sum_{x,y \in \mathcal{C} \cap [-L/2, L/2], y \ k.n.o. \ x} \varphi(x-y) \right] = \int_0^L \varphi(x) \rho_{2,k}(x)(1 - x/L). \tag{4.7.2}$$

Proof. We use the definition (4.7.1) together with a change of variable $(x,y) \mapsto (x-y, x+y)$ as in the re-writing of $\mathbb{W}^{\mathrm{int}}$ in Section 4.3.2. □

4.7.2 Minimization: the hyperuniform case

Lemma 4.7.2. *The unique minimizer of $\mathbb{W}^{\mathrm{int}}$ among random point processes in $\mathcal{P}_{\mathrm{hyp}}(\mathcal{X})$ is the random point process $P_{\mathbb{Z}}$ defined in* (4.1.10)*. Moreover for any such P we have*

$$\mathbb{W}^{\mathrm{int}}(P) - \mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}}) \geq c \sum_{k=1}^{+\infty} \int_0^{+\infty} \min\left(\frac{(x-k)^2}{k^{s+2}}, 1 \right) \rho_{2,k}(x), \tag{4.7.3}$$

with a constant c depending only on s.

Proof. Since there are a.s. r points in any interval of length r the k-th neighbor correlation function of P is supported in $[k-r, k+r]$. We may thus write, for any $R > 0$

$$\frac{1}{R}\int_{-R}^{R} g(x)(\rho_2(x)-1)(R-|x|)dx = \frac{2}{R}\int_0^R g(x)\left(\sum_{k=1}^{R+r}\rho_{2,k}-1\right)(R-x)dx.$$

Let $\psi_R : x \mapsto \frac{2}{R}g(x)(R-x)$. By definition of $\mathbb{W}^{\mathrm{int}}$ (see (4.3.6)) we get

$$\mathbb{W}^{\mathrm{int}}(P) = \liminf_{R\to\infty}\int_0^R \psi_R(x)\left(\sum_{k=1}^{R+r}\rho_{2,k}-1\right)dx, \tag{4.7.4}$$

and for $P_{\mathbb{Z}}$ it is easy to see that the lim inf is actually a lim and we have

$$\mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}}) = \lim_{R\to\infty}\int_0^R \psi_R(x)\left(\sum_{k=1}^{\lfloor R\rfloor}\delta_k-1\right)dx. \tag{4.7.5}$$

Substracting (4.7.5) in (4.7.4) we get

$$\mathbb{W}^{\mathrm{int}}(P)-\mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}}) = \liminf_{R\to\infty}\int_0^R \psi_R(x)\left(\sum_{k=1}^{R+r}\rho_{2,k}-\sum_{k=1}^{\lfloor R\rfloor}\delta_k\right).$$

We may re-write the previous expression as

$$\mathbb{W}^{\mathrm{int}}(P)-\mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}}) = \liminf_{R\to\infty}\int_0^{+\infty} \psi_R(x)\left(\sum_{k=1}^{\lfloor R-r\rfloor}(\rho_{2,k}-\delta_k)\right)+E_{R,r} \tag{4.7.6}$$

where the error term $E_{R,r}$ is bounded using the fact that $\rho_{2,k}$ is supported on $[k-r,k+r]$ and that $|\psi_R|$ is decreasing on $[R-2r,R]$ (for R large enough).

$$|E_{R,r}| \le \int_0^R \psi_R(x)\left(\sum_{k=\lfloor R-r\rfloor}^{R+r}\rho_{2,k}+\sum_{k=\lfloor R-r\rfloor}^{\lfloor R\rfloor}\delta_k\right) \le Cr|\psi(R-2r)| = o_R(1). \tag{4.7.7}$$

Let us now observe that ψ_R is a convex function, more precisely for $x \in (0,+\infty)$ we have $\psi_R''(x) \ge \frac{c}{x^{s+2}}$ for some positive constant c depending on s. Moreover for all $k \ge 1$, since P is of intensity 1 we have $\int \rho_{2,k} = 1$ ($\rho_{2,k}$ is the probability law of the k-th neighbor) and since P is periodic the expectation $\int x\rho_{2,k}(x)$ is finite and thus equal to k (the k-th neighbor is in average at distance k). Combining this observation with the convexity estimate we may write

$$\int_0^{+\infty}\psi_R(x)(\rho_{2,k}-\delta_k) \ge \int_0^{+\infty}\frac{(x-k)^2}{\max(x,k)^{s+2}}\rho_{2,k}(x) \ge c\int_0^{+\infty}\min\left(\frac{(x-k)^2}{k^{s+2}},1\right)\rho_{2,k}(x),$$

with c depending only on s. Inserting this bound in (4.7.6) and using (4.7.7) we get

$$\mathbb{W}^{\mathrm{int}}(P)-\mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}}) \ge \liminf_{R\to\infty}\sum_{k=1}^{\lfloor R-r\rfloor}\int_0^{+\infty}\min\left(\frac{(x-k)^2}{k^{s+2}},1\right)\rho_{2,k}(x)+o_R(1)$$

Hence finally by taking the limit $R\to\infty$ we obtain that

$$\mathbb{W}^{\mathrm{int}}(P)-\mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}}) \ge \sum_{k=1}^{+\infty}\int_0^{+\infty}\min\left(\frac{(x-k)^2}{k^{s+2}},1\right)\rho_{2,k}(x),$$

which proves (4.7.3). This lower bound implies that $P_{\mathbb{Z}}$ is a minimizer of $\mathbb{W}^{\mathrm{int}}$ among hyperuniform random point processes and also that it is unique. Indeed $\mathbb{W}^{\mathrm{int}}(P) = \mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}})$ implies (by (4.7.3)) that $\rho_{2,k} = \delta_k$ for all $k \ge 1$, hence the two-point correlation function of P coincides with the one of $P_{\mathbb{Z}}$, which is enough to conclude that $P = P_{\mathbb{Z}}$ (see e.g. [Leb15c]). □

4.7.3 Proof of Theorem 14

We may now give the proof of our minimization result for $\mathbb{W}^{\mathrm{elec}}$. First let us observe that $\mathbb{W}^{\mathrm{elec}}(P_{\mathbb{Z}}) = \mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}})$. The inequality $\leq$ is true by Proposition 4.5.1. Moreover Proposition 4.5.3 ensures that there exists a sequence of *hyperuniform* random point processes such that $\limsup_{N\to\infty} \mathbb{W}^{\mathrm{int}}(P_N) \leq \mathbb{W}^{\mathrm{elec}}(P_{\mathbb{Z}})$. By Lemma 4.7.2 above we know that $\mathbb{W}^{\mathrm{int}}(P_N) \geq \mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}})$, hence in fact $\mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}}) \leq \mathbb{W}^{\mathrm{elec}}(P_{\mathbb{Z}})$ and equality holds.

Proof. **Step 1.** *Minimization of* $\mathbb{W}^{\mathrm{elec}}$. Let P be a minimizer of $\mathbb{W}^{\mathrm{elec}}$ on $\mathcal{P}_{s,1}(\mathcal{X})$. From Proposition 4.5.3 we get a sequence $\{P_n\}_n$ of hyperuniform random point processes converging to P and such that $\{\mathbb{W}^{\mathrm{int}}(P_n)\}_n$ converges to $\mathbb{W}^{\mathrm{elec}}(P)$. For any $k, n \geq 1$ let $\rho_{2,k}^{(n)}$ denote the k-th neighbour correlation function of P_n. In the hyperuniform case, (4.7.3) implies that for any $M > 0$ and $k \geq 1$ we have

$$\mathbb{W}^{\mathrm{int}}(P_n) - \mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}}) \geq \int_0^{2M} \min\left(\frac{(x-k)^2}{k^{s+2}}, 1\right) \rho_{2,k}^{(n)}(x), \tag{4.7.8}$$

and the right-hand side is bounded below by (see (4.7.2))

$$\int_0^{2M} \min\left(\frac{(x-k)^2}{k^{s+2}}, 1\right) \rho_{2,k}^{(n)}(x) \left(1 - \frac{x}{2M}\right) = \mathbf{E}_{P_n}\left[\sum_{x,y\in\mathcal{C}\cap[-M,M],y \text{ k.n.o. } x} \varphi_k(x-y)\right]$$

where we let $\varphi_k(x) := \min\left(\frac{(x-k)^2}{k^{s+2}}, 1\right)$. Since $\{P_n\}_n$ converges to P we have

$$\begin{aligned}\mathbf{E}_{P_n}\left[\sum_{\mathcal{C}\cap[-M,M],y \text{ k.n.o. } x} \varphi_k(x-y)\right] &= \mathbf{E}_P\left[\sum_{\mathcal{C}\cap[-M,M],y \text{ k.n.o. } x} \varphi_k(x-y)\right] + o_n(1)\\ &= \int_0^{2M} \varphi_k(x)\rho_{2,k}(x)\left(1 - \frac{x}{2M}\right) + o_n(1) \geq \frac{1}{2}\int_0^M \varphi_k\rho_{2,k}(x) + o_n(1). \end{aligned} \tag{4.7.9}$$

Combining (4.7.8) and (4.7.9) we see that

$$\int_0^M \min\left(\frac{(x-k)^2}{k^{s+2}}, 1\right) \rho_{2,k}(x) \leq 2\left(\mathbb{W}^{\mathrm{int}}(P_n) - \mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}})\right) + o_n(1),$$

but as $n \to \infty$ we have $\mathbb{W}^{\mathrm{int}}(P_n) \to \mathbb{W}^{\mathrm{elec}}(P) \leq \mathbb{W}^{\mathrm{elec}}(P_{\mathbb{Z}}) = \mathbb{W}^{\mathrm{int}}(P_{\mathbb{Z}})$. It implies that

$$\int_0^M \min\left(\frac{(x-k)^2}{k^{s+2}}, 1\right) \rho_{2,k}(x) = 0$$

for all $M > 0$ and $k \geq 1$. Finally we get that $\rho_{2,k} = \delta_k$ for all $k \geq 1$ and we conclude as in the proof of Lemma 4.7.2 that $P = P_{\mathbb{Z}}$, which ensures that $P_{\mathbb{Z}}$ is the unique minimizer of $\mathbb{W}^{\mathrm{elec}}$ on $\mathcal{P}_{s,1}(\mathcal{X})$.

If P is not a minimizer the same argument shows that

$$\mathbb{W}^{\mathrm{elec}}(P) - \mathbb{W}^{\mathrm{elec}}(P_{\mathbb{Z}}) \geq \sum_{k=1}^{+\infty} \int_0^{+\infty} \min\left(\frac{(x-k)^2}{k^{s+2}}, 1\right) \rho_{2,k}(x) \tag{4.7.10}$$

as in the hyperuniform setting.

Step 2. *Energy of minimizers of $\mathcal{F}_\beta$ tends to $\mathbb{W}^{\mathrm{elec}}(P_\mathbb{Z})$.* On the other hand we claim that if $\{P_\beta\}_{\beta>0}$ is a family of minimizers of $\mathcal{F}_\beta$ then we must have

$$\lim_{\beta\to 0} \mathbb{W}^{\mathrm{elec}}(P_\beta) = \mathbb{W}^{\mathrm{elec}}(P_\mathbb{Z}). \tag{4.7.11}$$

To prove (4.7.11) we cannot directly evaluate $\mathcal{F}_\beta$ over $P_\mathbb{Z}$ and compare with P_β because $\mathsf{ent}[P_\mathbb{Z}|\mathbf{\Pi}]$ is infinite. However we may argue as in Section 4.6.2 and show that there exists a sequence $\{\pi_k\}_k$ of stationary random point processes in $\mathcal{P}_{s,1}(\mathcal{X})$ satisfying

1. $\mathsf{ent}[\pi_k|\mathbf{\Pi}]$ is finite for all k.
2. $\lim_{k\to\infty} \mathbb{W}^{\mathrm{elec}}[\pi_k] = \mathbb{W}^{\mathrm{elec}}(P_\mathbb{Z})$.

Such a sequence can be constructed by chosing a "vibrating" approximation of $P_\mathbb{Z}$. For any $k \geq 1$ we let $\{V_{k,m}\}_{m\in\mathbb{Z}}$ be a countable family of i.i.d. random variables distributed uniformly in $[-\frac{1}{k}, \frac{1}{k}]$, then we let π_k^t be the random point process

$$\pi_k^t := \sum_{m\in\mathbb{Z}} \delta_{m+V_{k,m}}$$

and finally we define π_k by averaging π_k^t over $[0,1]$. It is easy to check that π_k is a stationary random point process of intensity 1. In fact π_k may equivalently be defined as a renewal process with increments distributed as $1 + V_{k,2} - V_{k,1}$. The specific relative entropy of π_k coincides with its "entropy rate", it is finite (see [DVJ08, Section 14.8.]) and blows up as $k \to \infty$ like the entropy of $V_{k,2} - V_{k,1}$. Concerning the energy, we have the bound

$$\mathbb{W}^{\mathrm{elec}}(P_\mathbb{Z}) \leq \mathbb{W}^{\mathrm{elec}}(\pi_k) \leq \mathbb{W}^{\mathrm{int}}(\pi_k)$$

and the fact that $\mathbb{W}^{\mathrm{int}}(\pi_k)$ converges to $\mathbb{W}^{\mathrm{int}}(P_\mathbb{Z}) = \mathbb{W}^{\mathrm{elec}}(P_\mathbb{Z})$ can be checked directly with the help of the formula defining $\mathbb{W}^{\mathrm{int}}$. Indeed the two-point correlation function of π_k may be written as

$$\rho_{2,\pi_k} = \sum_{m\in\mathbb{Z}} \psi_k(m+\cdot)$$

where ψ_k is a triangular "hat function" of width $\frac{1}{2k}$ and integral 1. For any $m \in \mathbb{Z}$ and $R > 0$, a mean value argument shows that

$$\left| \int \psi_k(m+\cdot) \log|x| (1 - \frac{|x|}{R}) - \log|m| (1 - \frac{m}{R}) \right| \leq \frac{C}{k^2} \left(\frac{1}{m^2} + \frac{1}{mR} \right).$$

Consequently, we get

$$\limsup_{R\to\infty} \left| \int_{[-R,R]} (\rho_{2,\pi_k} - \rho_{2,\mathbb{Z}}) (1 - \frac{|x|}{R}) \right| = O\left(\frac{1}{k^2} \right),$$

and we obtain that $\lim_{k\to\infty} \mathbb{W}^{\mathrm{int}}(\pi_k) = \mathbb{W}^{\mathrm{int}}(P_\mathbb{Z})$.

With the help of the sequence $\{\pi_k\}_k$ we obtain (4.7.11) by arguing as in Section 4.6.3, indeed since $\mathsf{ent}[P_\beta|\mathbf{\Pi}]$ is always non-negative we have

$$\mathbb{W}^{\mathrm{elec}}(P_\beta) \leq \mathbb{W}^{\mathrm{elec}}(\pi_k) + \frac{1}{\beta}\mathsf{ent}[\pi_k|\mathbf{\Pi}] \leq \mathbb{W}^{\mathrm{elec}}(P_\mathbb{Z}) + \frac{1}{\beta}\mathsf{ent}[\pi_k|\mathbf{\Pi}] + o_k(1)$$

and (4.7.11) follows by chosing k, β large enough.

Step 3. *Convergence of the two-point function of minimizers of $\mathcal{F}_\beta$.* Combining (4.7.10) and (4.7.11) we see that if $\rho_{2,k}^{(\beta)}$ denotes the k-th neighbor correlation function of P_β, we have for any

$$x \sum_{k=1}^{+\infty} \int_0^{+\infty} \min\left(\frac{(x-k)^2}{k^{s+2}}, 1\right) \rho_{2,k}^{(\beta)}(x) \to 0$$

as $\beta \to \infty$. Arguing as in the proof of [Leb15c, Lemma 2.3] we deduce that $\rho_2^{(\beta)}$ converges to $\sum_{k\in\mathbb{Z}^*} \delta_k$ in the distributional sense as $\beta \to \infty$. Let us observe that

$$\rho_{2,\mathbb{Z}} := \sum_{k\in\mathbb{Z}^*} \delta_k$$

is the two-point correlation function of $P_\mathbb{Z}$.

Step 4. *Weak convergence of the minimizers of $\mathcal{F}_\beta$.* It is not hard to see that this convergence implies in fact the weak convergence of P_β to $P_\mathbb{Z}$ as $\beta \to \infty$. For any $\frac{1}{2} > \varepsilon > 0$ let χ_ε be a smooth non-negative function which is equal to 1 on the set $\cup_{k\in\mathbb{Z}}[k-1+\varepsilon, k-\varepsilon]$ and to 0 on $\mathbb{Z}$. For any $T > 0$ we let $\tau_{T,\varepsilon}$ be a non-negative continuous function such that $\tau_{T,\varepsilon} \equiv 1$ on $[-T+\varepsilon, T-\varepsilon]$ and 0 outside $[-T, T]$. We let $\varphi_{\varepsilon,T}$ be the continuous, compactly supported map

$$\varphi_{\varepsilon,T}(x,y) := \chi_\varepsilon(x-y)\tau_{T,\varepsilon}(x)\tau_{T,\varepsilon}(y).$$

Let $A_{T,\varepsilon}$ be the event "there is no pair (x,y) of points of the configuration in $[-T+\varepsilon, T-\varepsilon]$ such that $|x-y| \in \cup_{k\in\mathbb{Z}}[k-1+\varepsilon, k-\varepsilon]$". Since $\int \varphi_{\varepsilon,T}\rho_{2,\mathbb{Z}} = 0$ and since $\rho_2^{(\beta)}$ converges to $\rho_{2,\mathbb{Z}}$ as $\beta \to \infty$ we have

$$P_\beta\left(A_{T,\varepsilon}\right) \longrightarrow 1 \tag{4.7.12}$$

as $\beta \to \infty$. In other words, with probability tending to 1 as $\beta \to \infty$, a configuration under P_β locally looks like a (translated) subset of $\mathbb{Z}$ in which all the points have been displaced at a distance at most ε.

The variance under P_β of the number of points in $[-T, T]$ is bounded as $\beta \to \infty$, because it is controlled by the energy $\mathbb{W}^{\mathrm{elec}}(P_\beta)$, which itself converges (to $\mathbb{W}^{\mathrm{elec}}(P_\mathbb{Z})$). This follows from the discrepancy estimates (see e.g. [Leb15c, Lemma 2.1] or [LS15, Lemma 3.10]). In particular we have uniform integrability under P_β of the number of points in $[-T, T]$ as $\beta \to \infty$. In particular, conditioning P_β to $A_{T,\varepsilon}$, we have an average of $2T + o(1)$ points in $[-T, T]$ as $\beta \to \infty$, because (4.7.12) holds.

Finally we deduce that for any $\varepsilon > 0$, with probability tending to 1 as $\beta \to \infty$, a configuration under P_β locally looks like a translate of $\mathbb{Z}$ in which all the points have been displaced at a distance at most ε. This implies the convergence of P_β to $P_\mathbb{Z}$ as $\beta \to \infty$. □

Chapitre 5

Minimiseurs du log-gas uni-dimensionnel

Ce chapitre est constitué de l'article "A uniqueness result for minimizers of the 1D log-gas renormalized energy" [Leb15c], paru dans le *Journal of Functional Analysis*. Les résultats sont pour l'essentiel subsumés par ceux du chapitre précédent.

Contents

5.1 Introduction and statement of the results

5.1.1 Introduction

The N-particle Log-gas Hamiltonian w_N is defined on $\mathbb{R}^N$ by:

$$w_N(x_1, \dots, x_N) = -\sum_{i \neq j} \log |x_i - x_j| + N \sum_{i=1}^{N} V(x_i). \tag{5.1.1}$$

where V is a confining potential satisfying some growth conditions to be given later. The factor N before the second term in (5.1.1) corresponds to a mean-field scaling where both terms in the Hamiltonian (the interaction term $-\sum_{i \neq j} \log |x_i - x_j|$ and the potential term $N \sum_{i=1}^{N} V(x_i)$) have the same order of magnitude N^2.

While w_N has an obvious physical interpretation as the energy of N charged particles $x_1, \dots, x_N$ living on the real line, interacting pairwise through a potential $g(x, y) = -\log |x - y|$ (which is the Coulomb potential in two-dimensional physics) and subject to an external field

V, the Hamiltonian (5.1.1) also appears in random matrix theory as an exponential weight in the law of the eigenvalues of random Hermitian matrices (for a survey see [For10]). Minimizers of w_N are also called (N-points) "weighted Fekete sets" and arise in interpolation, cf. [ST97]. The "renormalized energy" W of Sandier-Serfaty (introduced in [SS12], see also [SS15b] for the two-dimensional case, [RS15] for an alternative approach that allows to handle the higher dimensional case as well, and [Ser15] for a pedagogical survey) is a way to make sense of the Hamiltonian w_N in the $N \to \infty$ limit, by deriving an energy functional which allows to consider the energy of infinite point configurations, and which is the second-order Γ-limit of w_N.

It is proven in [SS15a] that W is minimal at $\mathbb{Z}$ among infinite point configurations of density one, however this minimizer is not unique: in particular it was observed that local defects in the lattice, by the mean of arbitrary perturbations of $\mathbb{Z}$ on a compact set, form non-lattice configurations with the same minimal energy. In this paper we prove that the local defects essentially account for all the ground state degeneracy, by showing that there is no *translation-invariant* probability measure on point configurations having minimal energy in expectation, but the one associated to $\mathbb{Z}$ by averaging $\mathbb{Z}$ over translations in $[0, 1]$. This uniqueness result is obtained as consequence of our main theorem, which gives a lower bound on the (mean) renormalized energy of a stationary point process in terms of the difference between its two-point correlation function and that of the stationary process associated to the one-dimensional lattice $\mathbb{Z}$.

5.1.2 Definition and properties of the renormalized energy

In this section, and in all the paper, we follow mainly the definitions and notation from [SS15a]. We have chosen to work here with the definition of the "renormalized energy" as given in [SS15a] or [SS15b]. An alternative definition has been given in [RS15] and extended for more general interactions in [PS15], for which our results would apply readily in the case of a logarithmic interaction. For more general interactions (when the logarithmic kernel is replaced by a two-body interaction of the type $g(x, y) = \frac{1}{|x-y|^s}$ with $0 < s < 1$) the lattice $\mathbb{Z}$ is again expected to minimize the renormalized energy by a simple convexity argument (see [PS15, Proposition 1.5.]), and the type of quantitative bounds that we show can probably be extended to these interactions as well.

Let us start by recalling the definition of the renormalized energy. The renormalized energy (in the spirit of the "renormalized energy" of [BBH94]) of an infinite configuration of points can be understood as a way of computing the electrostatic energy of those points, seen as interacting charged particles of charge $+1$, together with an infinite negatively charged uniform background. In 1D, the renormalized energy is obtained by "embedding" the real line into the plane and computing the renormalized energy in the plane according to its two-dimensional definition of [SS15b]. In particular, the pairwise interaction $g(x, y) = -\log|x - y|$ is not the Coulomb electrostatic interaction of one-dimensional physics, but a restriction on $\mathbb{R} \subset \mathbb{R}^2$ of the Coulomb two-dimensional interaction, hence the term "Log-gases".

In what follows, $\mathbb{R}$ will denote the set of real numbers but also the real line of the plane $\mathbb{R}^2$ i.e. points of the form $(x, 0) \in \mathbb{R}^2$. For the sake of clarity, we will denote points in $\mathbb{R}$ by the letter x and points in the plane by $z = (x, y)$. We denote by $\delta_{\mathbb{R}}$ the measure of length on $\mathbb{R}$ seen as embedded in $\mathbb{R}^2$, that is

$$\int_{\mathbb{R}^2} \varphi \delta_{\mathbb{R}} = \int_{\mathbb{R}} \varphi(x, 0)\, dx$$

for any smooth compactly supported test function φ in $\mathbb{R}^2$.

The "admissible classes" $\mathcal{A}_m$ correspond to the electric fields generated by infinite configurations on the real line together with a background of uniform density m:

Definition 5.1.1. *Let $m \geq 0$. Let E be a gradient vector field in $\mathbb{R}^2$. We say E belongs to the admissible class $\mathcal{A}_m$ if*

$$\text{div } E = 2\pi(\nu - m\delta_{\mathbb{R}}) \quad \text{in } \mathbb{R}^2 \tag{5.1.2}$$

where ν has the form $\nu = \sum_{x\in\Lambda} \delta_x$ for some discrete set $\Lambda \subset \mathbb{R} \subset \mathbb{R}^2$ (where δ_x denotes the Dirac mass at x), and $\frac{\nu([-R,R])}{R}$ is bounded by a constant independent of $R > 1$.

We now turn to the central definition:

Definition 5.1.2 (Renormalized energy)**.** *Let m be a nonnegative number. For any bounded function χ and any E satisfying* (5.1.2) *we let*

$$W(E,\chi) = \lim_{\eta\to 0} \left(\frac{1}{2} \int_{\mathbb{R}^2 \setminus \cup_{p\in\Lambda} B(p,\eta)} \chi |E|^2 + \pi \log \eta \sum_{p\in\Lambda} \chi(p) \right)$$

and the renormalized energy W is defined by

$$W(E) = \limsup_{R\to\infty} \frac{W(E,\chi_R)}{R}, \tag{5.1.3}$$

where $\{\chi_R\}_{R>0}$ is a family of cut-off functions satisfying

$$|\nabla \chi_R| \leq C, \quad Supp(\chi_R) \subset [-R/2, R/2] \times \mathbb{R}, \quad \chi_R(z) = 1 \text{ if } |x| < R/2 - 1,$$

for some C independent of R.

The various admissible classes $\mathcal{A}_m$ ($m \in \mathbb{R}^+$) are related to each other by the following scaling relation: if E belongs to $\mathcal{A}_m$ then $E' := \frac{1}{m} E(\cdot/m)$ belongs to $\mathcal{A}_1$ and

$$W(E) = m \left(W(E') - \pi \log m \right). \tag{5.1.4}$$

Moreover, it is easy to see that the point configurations associated to E and E' coincide up to an homothety.

For reasons related to the physical interpretation of the Hamiltonian w_N, the gradient vector field E is sometimes called the "electric field" associated to a configuration (seen as charged point particles). Starting from a discrete set of points $\Lambda \subset \mathbb{R} \subset \mathbb{R}^2$, there might be several gradient vector fields E satisfying (5.1.2) with $\nu = \sum_{x\in\Lambda} \delta_x$: if E is any such field (let us note that, due to the infinite setting, there might not be any) we can simply add to E the gradient of any harmonic function on $\mathbb{R}^2$. In the two-dimensional case this is indeed an issue, but for one-dimensional Log-gases the following lemma shows that there is in fact a natural choice of the electric vector field E:

Lemma 5.1.3. *[SS15a, Lemma 1.7.] Let $E \in \mathcal{A}_m$ be such that $W(E) < +\infty$. Then any other E' satisfying* div $E' =$ div E *and $W(E') < +\infty$, is such that $E' = E$. In other words, W only depends on the points.*

By simple considerations similar to [SS15b, Section 1.2] this makes W a measurable function of the point configuration Λ and with an abuse of notation we will write $W(\Lambda)$ as well as $W(E)$, where E is the only admissible vector field of finite energy associated to Λ (when it exists). We will frequently use the following map to get from an electric field $E \in \mathcal{A}_1$ to its underlying point configuration:

$$E \mapsto \frac{1}{2\pi} \text{div } E + \delta_{\mathbb{R}}.$$

It is not difficult to show (for a proof see [SS15b]) that an admissible gradient vector field is in $L^q_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ for $q < 2$. We endow the admissible classes $\mathcal{A}_m$ with the Borel σ-algebra inherited from $L^q_{loc}(\mathbb{R}^2, \mathbb{R}^2)$ for some $q < 2$.

Definition 5.1.4. *Let $m > 0$. If P is a translation-invariant probability measure on $\mathcal{A}_m$, we define*

$$\overline{W}(P) = \int W(E)dP(E).$$

We say that such a probability measure is translation-invariant (or stationary) when P is invariant by (the push-forward of) the maps $E \mapsto E(\cdot - \lambda)$ for any $\lambda \in \mathbb{R}$.

Finally, when X is a measurable space, $\mathcal{P}(X)$ will denote the space of probability measures on X. If $P \in \mathcal{P}(X)$ is a probability measure and $f : X \to \mathbb{R}$ a measurable function, we denote by $\mathbf{E}_P[f]$ the expectation of f under P.

5.1.3 Periodic case and minimization

When the configuration is assumed to have some periodicity, there is an explicit formula for W in terms of the points. The following lemma is proven in [BS13, Section 2.5.] (here we can reduce to the class $\mathcal{A}_1$ by scaling, as seen above in (5.1.4)).

Lemma 5.1.5. *In the case $m = 1$ and when the set of points Λ is periodic with respect to some lattice $N\mathbb{Z}$, then it can be viewed as a set of N points $a_1, \dots, a_N$ over the torus $\mathbb{T}_N := \mathbb{R}/(N\mathbb{Z})$. In this case, by Lemma 5.1.3 there exists a unique E satisfying* (5.1.2) *and for which $W(E) < +\infty$. It is periodic and equal to $E_{\{a_i\}} = \nabla H$, where H is the solution on $\mathbb{T}_N \times \mathbb{R}$ to $-\Delta H = 2\pi(\sum_i \delta_{a_i} - \delta_{\mathbb{R}})$, and we have the explicit formula:*

$$W(E_{\{a_i\}}) = -\frac{\pi}{N} \sum_{i \neq j} \log \left| 2 \sin \frac{\pi(a_i - a_j)}{N} \right| - \pi \log \frac{2\pi}{N}.$$

Henceforth we will denote by $W(\mathbb{Z})$ the energy of the periodic electric field associated to $\mathbb{Z}$ as above. In this periodic, one-dimensional setting, $\mathbb{Z}$ is shown to be the (unique) minimizer of W by a simple convexity argument. The key point of our proof is to make this argument quantitative in order to get a lower bound on $W(E_{\{a_i\}})$ in terms of the local defects with respect to the lattice configuration (this is Lemma 5.1.9).

A general argument of approximating any gradient vector field E of finite energy by periodic electric fields implies a minimization result for W on $\mathcal{A}_m$, without any periodicity assumption. It is proven in [SS15a, Theorem 2] that:

Theorem (crystallization in 1D)**.** $\min_{\mathcal{A}_m} W = -\pi m \log(2\pi m)$ *and this minimum is achieved by the perfect lattice i.e. $\Lambda = \frac{1}{m}\mathbb{Z}$.*

Let us emphasize that as a consequence of the definition of W as a limit (5.1.3) over large intervals W does not feel compact perturbations of the points (as long as the configuration stays simple i.e. two points of Λ are always distinct) hence no uniqueness of the minimizer can be expected at the level of point configurations.

5.1.4 Point processes and correlation functions

In this paragraph we give some definitions about point processes (for a complete presentation see [DVJ88]).

Definition 5.1.6 (Point processes)**.**

— *Let $\mathcal{X}$ be the set of locally finite, simple point configurations on $\mathbb{R}$. If $B \subset \mathbb{R}$ is a Borel set, we let $N_B : \mathcal{X} \mapsto \mathbb{N}$ be the map giving the number of points of a configuration that lie in B. The set $\mathcal{X}$ is endowed with the initial σ-algebra associated to the maps $\{N_B, B \text{ Borel}\}$.*

— *A point process is a probability measure on $\mathcal{X}$. The additive group $\mathbb{R}$ acts on $\mathcal{X}$ by translations $\{\theta_t\}_{t\in\mathbb{R}}$: if $\mathcal{C} = \{x_i, i \in I\} \in \mathcal{X}$ we let $\theta_t \cdot \mathcal{C} = \{x_i - t, i \in I\}$. It also acts on the set $\mathcal{P}(\mathcal{X})$ of point processes in the natural way, by pushing-forward $P \in \mathcal{P}(\mathcal{X})$ by the map $\mathcal{C} \mapsto \theta_t \cdot \mathcal{C}$ for each $t \in \mathbb{R}$.*
— *A point process is said to be translation-invariant (or stationary) when it is invariant by the action of $\mathbb{R}$.*
— *If $\Lambda \in \mathcal{X}$ is a periodic configuration of points on $\mathbb{R}$ with $\theta_L \cdot \Lambda = \Lambda$, we may associate to $\mathcal{C}$ the following stationary point process:*

$$P_\Lambda := \frac{1}{L}\int_0^L \delta_{\theta_t\cdot\Lambda} dt.$$

In particular, we will use the stationary processes associated to $\mathbb{Z}$ and its dilations $\frac{1}{m}\mathbb{Z}$ (for $m > 0$), which we denote by $P_\mathbb{Z}$, $P_{\frac{1}{m}\mathbb{Z}}$. We may abuse notation, relying on Lemma 5.1.3, and use the same notation for the stationary "electric" probability measure (concentrated on $\mathcal{A}_m$ and of finite energy) corresponding to $\frac{1}{m}\mathbb{Z}$.

Definition 5.1.7 (Correlation functions)**.** *Let $P_\Lambda \in \mathcal{P}(\mathcal{X})$ be a point process. For $k \geq 1$ the k-point correlation function ρ_{k,P_Λ} is a linear form on the vector space of measurable functions $\varphi_k : \mathbb{R}^k \longrightarrow \mathbb{R}$, defined by:*

$$\rho_{k,P_\Lambda}(\varphi_k) = \mathbf{E}_{P_\Lambda}\left(\sum_{x_1,\dots,x_k\in\mathcal{C}|x_i,x_j \text{ pairwise distinct}} \varphi_k(x_1,\dots,x_k)\right).$$

Strictly speaking, it is only defined on the subspace of functions φ_k such that the map $\mathcal{C} \mapsto \sum_{x_1,\dots,x_k\in\mathcal{C}|x_i,x_j \text{ pairwise distinct}} \varphi_k(x_1,\dots,x_k)$ is integrable against dP_Λ.

When the k-point correlation function exists as a distribution and can be identified with a measurable function, we will write $\int \rho_k\varphi_k$ instead of $\rho_k(\varphi_k)$. Heuristically, ρ_1 (also called the intensity of the point process) gives the density of the process at each point, while $\rho_2(x, y)$ gives the probability of having a point both at x and y.

Remark 5.1.8. *If P is a translation-invariant (stationary) probability measure concentrated on the class of admissible electric fields $\mathcal{A}_m$, the push-forward P_Λ of P by $E \mapsto \frac{1}{2\pi}\mathrm{div}\ E + m\delta_\mathbb{R}$ is a stationary point process. Let us assume that $\overline{W}(P)$ is finite. Then:*
— *The one-point correlation function may be identified as (testing against) the function $\rho_{1,P_\Lambda} \equiv m$.*
— *The two-point correlation ρ_{2,P_Λ} is well-defined as a Radon measure on $\mathbb{R}^2$.*

Physically speaking, this is because there must be approximately m points per unit volume in order to compensate (without overwhelming) the background charge $-m\delta_\mathbb{R}$, so that the configuration is globally neutral (non-neutrality would generate too much energy). We will give a proof of Remark 5.1.8 in Section 5.2.1.

Henceforth, if φ is a compactly supported, continous function on $\mathbb{R}^2$ and $\mathcal{C}$ a point configuration, we will denote by $\langle\varphi, \mathcal{C}\rangle$ the quantity

$$\langle\varphi, \mathcal{C}\rangle := \sum_{x\neq y|x,y\in\mathcal{C}} \varphi(x, y)$$

which is always a well-defined number since φ is compactly supported and $\mathcal{C} \in \mathcal{X}$ is locally finite.

5.1.5 Statement of the results

We start by stating a quantitative version of the convexity argument on which the minimality of $W(\mathbb{Z})$ as in [SS15a, Proposition 4.3.] relies:

Lemma 5.1.9. *Let $a_1 < \cdots < a_N$ be any points in $[0, N]$ and $E_{\{a_i\}}$ be the corresponding periodic vector field as in Lemma 5.1.5. Let $u_{p,i} = a_{i+p} - a_i$ with the convention $a_{N+l} = a_l + N$ and let $b_{p,i} = u_{p,i} - p$. Then, for some universal constant C:*

$$W(E_{\{a_i\}}) - W(\mathbb{Z}) \geq C \sum_{p=1}^{N/2} \frac{1}{N} \sum_{i=1}^{N} \min\left(\frac{b_{p,i}^2}{p^2}, 1\right). \tag{5.1.5}$$

The proof is given in section 5.2.2. The quantity $b_{p,i}$ in the right-hand side of (5.1.5) measures a local defect with respect to the lattice: the spacing error $u_{p,i} - p$ between two p-th neighbours (in $\mathbb{Z}$ two p-th neighbours are always at distance p).

We then state our main theorem and its consequence for the minimization problem. The theorem gives for any translation-invariant probability measure $P \in \mathcal{P}(\mathcal{A}_1)$ of finite energy a lower bound on $\overline{W}(P) - W(\mathbb{Z})$ in terms of the two-point correlation functions of the stationary point process associated to P.

Theorem 15. *Let P be a stationary probability measure concentrated on $\mathcal{A}_1$ such that $\overline{W}(P)$ is finite, let P_Λ be the push-forward of P by the map (5.1.2) and ρ_{2,P_Λ} be the two-point correlation function of P_Λ. Then for any $T > 1$ and any function $\varphi \in C^1([-T, T]^2)$ we have*

$$\left| \int (\rho_{2,P_\Lambda} - \rho_{2,P_{\mathbb{Z}}})\varphi \right| \leq C_\varphi (\overline{W}(P) + C)^{\frac{1}{2}} \left(\overline{W}(P) - W(\mathbb{Z})\right)^{\frac{1}{2}}$$

with C_φ depending only on $||\varphi||_\infty, ||\nabla\varphi||_\infty$ and T, and C a universal constant.

The proof is given in Section 5.2.4. The constant C_φ appearing in Theorem 15 can be made explicit and is seen to be of the type:

$$C_\varphi = C(||\varphi||_\infty + ||\nabla\varphi||_\infty)T^\alpha$$

for some power α which could be tracked down if necessary, and where C is a universal constant.

An easy consequence of Theorem 15 is the following uniqueness result:

Corollary 5.1.10. *For $m > 0$, the only minimizer of $\overline{W}$ on the set of stationary probability measures concentrated on $\mathcal{A}_m$ is $P_{\frac{1}{m}\mathbb{Z}}$.*

Theorem 15 also allows to track down the crystallization of the statistical mechanics model via the convergence (in distributional sense) of the two-point correlation functions of to $\rho_{2,\mathbb{Z}}$ when the inverse temperature $\beta \to +\infty$, as stated in [SS15a, Corollary 1.14.]. This was one of the main motivations for this paper. More precisely, it is shown in [SS15a, Corollary 1.14.] that any limit point of the canonical Gibbs measure (after averaging over translations) associated to the one-dimensional Log-gas Hamiltonian 5.1.1 has its two-point correlation function close to $\rho_{2,\mathbb{Z}}$ with a bound becoming sharp as $\beta \to \infty$.

5.1.6 Connection with the Log-gas Hamiltonian

Let us now briefly give a motivation for studying the renormalized energy and its minimization (for a thorough study we refer again to [SS15b], [SS15a], [RS15] and to the survey [Ser15]).

Starting again from w_N, when the potential V in the definition (5.1.1) is lower semi-continuous and satisfies the growth assumption $V(x) - 2\log|x| \to +\infty$ (when $|x| \to +\infty$),

it is known that for any sequence $\{\vec{x_N}\}_N$ of minimizers of w_N, the empirical measures $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ converge weakly as $N \to +\infty$ to some measure μ_0 on $\mathbb{R}$, called the "equilibrium measure". We assume that the equilibirum measure μ_0 has a compact support Σ which is a finite union of compact intervals, and is absolutely continous with respect to the Lebesgue measure with a density $m_0 \in L^\infty(\mathbb{R})$. The result is in fact much stronger, since the convergence of the empirical measures to μ_0 holds not only for minimizers but almost surely under the Gibbs measure associated to w_N at any finite temperature (see [BAG97], [HP00]). The renormalized energy appears as the Hamiltonian w_N at second order: there is an exact splitting formula

$$w_N(x_1, \dots, x_N) = N^2\mathcal{F}(\mu_0) - N\log N + NF_N(\nu)$$

where $\mathcal{F}(\mu_0)$ is a "first-order" potential energy associated to the equilibrium measure, and F_N is a function of the finite point configuration $\sum_{i=1}^N \delta_{x_i}$. To any such finite point configuration we associate a probability measure P_{ν_N} on $\Sigma \times \mathcal{A}$ obtained by averaging over $x \in \Sigma$ the electric field E_N associated to the finite configuration $\nu'_N = \sum_{i=1}^N \delta_{Nx_i}$ (for finite configurations such a field always exists), translated by Nx:

$$P_{\nu_N} := \fint_\Sigma \delta_{(x, E_N(Nx+\cdot))}.$$

Let us emphasize that in this setting the scaling $x_i \mapsto x'_i = Nx_i$ is necessary: since we know that the empirical measures $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ typically converge to a compactly supported measure μ_0 on $\mathbb{R}$ it is relevant to scale the distances by N so that the spacing between two consecutive points becomes of order 1. If $\{\nu_N\}_N$ is a sequence of finite point configurations such that $\{F_N(\nu_N)\}_N$ is bounded then it is proven in [SS15a, Theorem 3.] that up to extraction the sequence $\{P_{\nu_N}\}_N$ converges to some admissible probability measure P (the definition of "admissible" is given below) and $\liminf_{N\to+\infty} F_N(\nu_N) \geq \widetilde{W}(P)$. More precisely, the sequence of functionals $\{F_N\}_N$ (for each N we can see F_N as a function of probability measures $P \in \mathcal{P}(\Sigma \times \mathcal{A})$ which is infinite outside the image of the map $\nu_N \mapsto P_{\nu_N}$) has $\widetilde{W}$ for Γ-limit (see [Bra02]), which implies that the minimizers of F_N (hence of w_N) converge to minimizers of $\widetilde{W}$. This reduces the second-order study of the Hamiltonian w_N in the limit $N \to \infty$ to the study of $\widetilde{W}$ on admissible probabilities.

Definition 5.1.11 (Admissible probabilities)**.** *We say $P \in \mathcal{P}(\Sigma \times \mathcal{A})$ is* admissible if
- The first marginal of P is the normalized Lebesgue measure on Σ.
- It holds for P-a.e. (x, E) that $E \in \mathcal{A}_{m_0(x)}$.
- P is invariant by the maps $(x, E) \mapsto (x, E(\lambda+\cdot))$ for all $\lambda \in \mathbb{R}$ (this is a weaker assumption than the "$T_{\lambda(x)}$-invariance" of [SS15a] but it is sufficient for our purpose).

When P is admissible, we denote by $\widetilde{W}$ the expectation of W under P:

$$\widetilde{W}(P) = \frac{|\Sigma|}{\pi}\int W(E)\, dP(x, E).$$

For Log-gases, Theorem 15 implies the following uniqueness result:

Corollary 5.1.12. *The equilibrium measure μ_0 and its density m_0 being fixed, the only minimizer of $\widetilde{W}$ on the set of admissible probability measures is given by*

$$P_0 = \frac{dx_{|\Sigma}}{|\Sigma|} \otimes P_{\frac{1}{m_0(x)}\mathbb{Z}}. \tag{5.1.6}$$

In (5.1.6) we make a slight abuse of notation, it is not strictly speaking a tensor product since there is a dependency on x in $P_{\frac{1}{m_0(x)}\mathbb{Z}}$. The probability measure $P_0 \in \mathcal{P}(\Sigma \times \mathcal{A})$ is rigorously

defined by duality : if $F \in C^0(\Sigma \times \mathcal{A})$ is a test function we let

$$\int F dP_0 := \int_\Sigma \frac{dx}{|\Sigma|} \int F(x,\cdot) dP_{\frac{1}{m_0(x)}\mathbb{Z}}.$$

Let us note that $P_{\frac{1}{m}\mathbb{Z}}$ is not well-defined for $m = 0$, however we may assume that the set $\{x \in \Sigma, m_0(x) = 0\}$ has zero Lebesgue measure. Corollary 5.1.12 as well as Corollary 5.1.10 are proven in Section 5.2.5.

The minimization problem is of physical relevance, indeed minimizers of an Hamiltonian describe the behaviour of the physical system at zero temperature. It is believed (see e.g. [CK07, Conjecture 9.4.]) that for a wide class of interactions the minimizing infinite configurations are lattices. The one-dimensional crystallization is somewhat easier to prove (see e.g. [EL62] for positive results concerning the Coulomb interaction in $1d$, [BL75], [Kun74] for one-dimensional long-range order, [ALS10] for zeroes of orthogonal polynomials), but the higher-dimensional cases are largely open (see [The06] for a result in $d = 2$ and [FT14] for recent progress in $d = 3$).

5.2 Proof of the results

5.2.1 Preliminary bounds on the density of points

Since we are dealing with two-point correlation functions, we will often need to bound the variance (for some point process) of the number of points that lie in some fixed interval. For this purpose we use a deviation estimate of [SS15a] which gives a lower bound on the renormalized energy in terms of the local non-neutrality of a point configuration (together with the uniform background). We summarize the consequences for correlation functions in the following lemma:

Lemma 5.2.1. *If $E \in \mathcal{A}_1$ is an electric field, we denote by $\mathcal{N}(E,T)$ the number of points of $\frac{1}{2\pi}\mathrm{div}\ E + \delta_{\mathbb{R}}$ that lie in the inverval $[-T,T]$. Let P be a stationary probability measure concentrated on $\mathcal{A}_1$.*

— *We have, for any $T > 0$,*

$$\mathbf{E}_P\left[\mathcal{N}(E,T)\right] = 2T.$$

In particular, the one-point correlation function (the intensity) of the point process associated to P may be identified as (testing against) the function $\rho_{1,P_\Lambda} \equiv 1$. This, together with the scaling relation (5.1.4)*, proves the first claim of Remark 5.1.8.*

— *The following bound holds:*

$$\int \mathcal{N}(E,T)^2 dP(E) \leq C_T\left(C + \overline{W}(P)\right).$$

where C_T depends only on T and C is universal.

The proof of such discrepancy estimates (especially the second point) involves technical details of [SS15a] that we will quote when needed. It might be interesting to note that when working with the alternative definition of a renormalized energy given in [RS15], [PS15], the proof of similar estimates is more natural.

Proof. In what follows C is a universal constant, which may vary from line to line. Let E be in $\mathcal{A}_1$ and $T \geq 1$ and let $\Lambda(E) := \frac{1}{2\pi}\mathrm{div}\ E + \delta_{\mathbb{R}}$. Denoting by $D(E,T)$ the quantity $D(E,T) = \mathcal{N}(E,T) - 2T$ (D measures the "discrepancy" between the expected number of points and the actual one, hence is a measure of local non-neutrality), [SS15a, Proposition 4.6.] reads:

$$\int_{[-2T,2T]} dg \geq -CT + cD(E,T)^2 \min\left(1, \frac{|D(E,T)|}{T}\right)$$

where the density g is defined in [SS15a, Proposition 2.1.] (we quote below the results that we need) and $c > 0$ is universal. From [SS15a, Proposition 2.1.i)], we know that g is bounded below by $-C$, hence if χ_T is a smooth cut-off function satisfying $\chi_T \equiv 1$ on $[-2T, 2T]$ and $\chi_T \equiv 0$ outside $[-2(T+1), 2(T+1)]$ with $\|\nabla\chi_T\|_\infty \leq 1$, which we extend by $\bar{\chi}_T(x, y) = \chi_T(x)$ on $\mathbb{R}^2$, we have:

$$\int \bar{\chi}_T dg \geq -CT + cD(E,T)^2 \min\left(1, \frac{|D(E,T)|}{T}\right).$$

We also know from [SS15a, Proposition 2.1.iii)] that for such a function χ_T the following bound holds:

$$\left|W(E,\chi_T) - \int \bar{\chi}_T \, dg\right| \leq Cn(\log n + 1)\|\nabla\chi\|_\infty,$$

where n is a boundary term bounded by the number of points of the configuration $\Lambda(E)$ in $[-3T, -2T] \cup [2T, 3T]$. Let ψ be the map $\psi : x \mapsto x^2 \min(1, \frac{|x|}{T})$. Combining (5.2.1) and (5.2.1) we easily get

$$\psi(D(E,T)) \leq C\left(C + T + W(E,\chi_T) + n(\log n + 1)\right).$$

Taking the expectation under P yields:

$$\int \psi(D(E,T))dP(E) \leq C\left(C + T + \int \left(W(E,\chi_T) + n\log n + n\right) dP(E)\right) \tag{5.2.1}$$

with $n = n(E)$ bounded by the number of points of $\Lambda(E)$ in $[-3T, -2T] \cup [2T, 3T]$. Since P is stationary the average number (under P) of points of $\Lambda(E)$ in any interval of length T is the same. Hence if we write $\mathcal{N}_+(E) = \max(\mathcal{N}(E,T), 2)$ we have by stationarity (and using the Cauchy-Schwarz inequality):

$$\int n\log n dP \leq 4\left(\int \mathcal{N}_+^2 dP\right)^{\frac{1}{2}}\left(\int (\log\mathcal{N}_+)^2 dP\right)^{\frac{1}{2}}. \tag{5.2.2}$$

Obviously the right-hand side of (5.2.2) also bounds the term $\int n(E)dP(E)$. For any $0 < \alpha \leq \frac{1}{2}$, we may find C_α large enough such that

$$\left(\int \mathcal{N}_+^2 dP\right)^{\frac{1}{2}}\left(\int (\log\mathcal{N}_+)^2 dP\right)^{\frac{1}{2}} \leq C_\alpha\left(\int \mathcal{N}_+^2 dP\right)^{\frac{1}{2}+\alpha}.$$

Together with (5.2.1) and (5.2.2) we thus obtain:

$$\int \psi(D(E,T))dP(E) \leq C\left(C + T^2 + \int W(E,\chi_T)dP(E) + C_\alpha\left(\int \mathcal{N}_+^2 dP\right)^{\frac{1}{2}+\alpha}\right). \tag{5.2.3}$$

From Definition 5.1.2 it is easy to see that $W(E,\cdot)$ is additive in the second variable i.e. $W(E,\chi_1+\chi_2) = W(E,\chi_1) + W(E,\chi_2)$. Using the stationarity of P we get for any $R > 0$:

$$\int W(E,\chi_T)dP(E) = \frac{1}{2R}\int_{-R}^{R}\int W(E,\chi_T(x+\cdot))dP(E)dx = \frac{1}{2R}\int W(E,\mathbf{1}_{[-R,R]} * \chi_T)dP(E).$$

Let us observe that the family of functions $\{\chi'_R\}_R$ defined as

$$\chi'_R := \frac{\mathbf{1}_{[-R,R]} * \chi_T}{\int \chi_T}$$

satisfy the conditions of Definition 5.1.2 so that sending R to $+\infty$ we get: $\int W(E,\chi'_R)dP(E) = \overline{W}(P)$. This in turn implies that

$$\int W(E,\chi_T)dP(E) = \overline{W}(P)\left(\int \chi_T\right) \leq CT(\overline{W}(P)+C). \tag{5.2.4}$$

Moreover since $D(E,T) = \mathcal{N}(E,T) - 2T$ an elementary computation shows that for any $0 < \alpha \leq \frac{1}{2}$ we have:

$$CC_\alpha\left(\int \mathcal{N}_+^2 dP\right)^{\frac{1}{2}+\alpha} \leq \frac{1}{2}\int D^2(E,T)dP + C'_\alpha T^{1+\alpha} \tag{5.2.5}$$

where C, C_α are the constants in (5.2.3) and C'_α depends only on α. Combining (5.2.3), (5.2.4) and (5.2.5) we get for any $0 < \alpha \leq \frac{1}{2}$:

$$\int \psi(D(E,T))dP(E) \leq CT(\overline{W}(P)+C) + C'_\alpha T^{1+2\alpha} \tag{5.2.6}$$

with a constant C'_α depending only on α and C universal. Equation (5.2.6) implies (by distinguishing the events $D \leq T$ and $D \geq T$) that $\int D(E,T)^2 dP(E) = o(T^2)$. The following is then a consequence :

— We have $\int D(E,T)dP(E) = o(T)$ hence $\int(\mathcal{N}(E,T) - 2T)dP(E) = o(T)$ but by stationarity we have $\int \mathcal{N}(E,T)dP(E) = T\int \mathcal{N}(E,1)dP(E)$ so that in fact

$$\int \mathcal{N}(E,T)dP(E) = 2T$$

for all $T > 0$, which proves the first claim of the lemma.

— Since $\mathcal{N}(E,T)^2 \leq 4T^2 + 2\int D(E,T)^2$ we also get a bound on the mean square number of points $\int \mathcal{N}(E,T)^2 dP(E)$ as in the second claim of the lemma.

□

We now use Lemma 5.2.1 to show that two-point correlation functions exist as Radon measures for point processes of finite renormalized energy.

Lemma 5.2.2. *Let $T \geq 1$, let $\varphi \in C^0([-T,T]^2)$ and let P be a stationary probability measure on $\mathcal{A}_1$ such that $\overline{W}(P)$ is finite. Let also P_Λ be the push-forward of P by the map $E \mapsto \frac{1}{2\pi}\mathrm{div}\, E + \delta_{\mathbb{R}}$. The following bound holds:*

$$\left|\int \langle \varphi, \mathcal{C}\rangle \, dP_\Lambda(\mathcal{C})\right| \leq C_\varphi(\overline{W}(P)+C)$$

with C_φ depending only on $||\varphi||_\infty$ and T, and C a universal constant.

Proof. For $x \in \mathbb{R}$ and $E \in \mathcal{A}_1$, let us denote by $\mathcal{N}(E,T)$ the number of points of $\frac{1}{2\pi}\mathrm{div}\, E + \delta_{\mathbb{R}}$ lying in $[-T,T]$. Since φ is supported on $[-T,T]^2$ the following bound is obvious by definition

$$|\langle \varphi, \mathcal{C}\rangle| \leq \mathcal{N}(E,T)^2 ||\varphi||_\infty.$$

Integrating against dP_Λ, we get

$$\left|\int \langle \varphi, \mathcal{C}\rangle \, dP_\Lambda(\mathcal{C})\right| \leq ||\varphi||_\infty \int \mathcal{N}(E,T)^2 dP(E). \tag{5.2.7}$$

By Lemma 5.2.1 we know that the right-hand side of (5.2.7) is bounded by $C_\varphi(C + \overline{W}(P))$ which concludes the proof. □

Lemma 5.2.2 has the following implication: if P is a stationary probability measure concentrated on $\mathcal{A}_1$ such that $\overline{W}(P)$ is finite, then the push-forward of P by $E \mapsto \frac{1}{2\pi}\mathrm{div}\, E + \delta_{\mathbb{R}}$ admits a two-point correlation function in distributional sense. Indeed, the linear form $\varphi \mapsto \int \langle \varphi, E\rangle \, dP(E)$ is shown to be bounded by $O(||\varphi||_\infty)$ uniformly for test functions in $C^0([-T,T]^2)$, for all T. This proves the second claim of Remark 5.1.8 (the case of probability measures concentrated on $\mathcal{A}_m$ reduces to the former case by scaling as in (5.1.4)).

5.2.2 Energy lower bound near the ground state for periodic configurations

We prove a quantitative version of the minimization of W on periodic configurations, as stated in Lemma 5.1.9.

Proof. Let $u_{p,i} = a_{i+p} - a_i$, with the convention $a_{N+l} = a_l + N$, and let $b_{p,i} = u_{p,i} - p$. We know from (5.1.5) that

$$W(E_{\{a_i\}}) - W(\mathbb{Z}) = \frac{2\pi}{N}\sum_{p=1}^{N/2}\left(\log\left|2\sin\frac{p\pi}{N}\right| - \frac{1}{N}\sum_{i=1}^{N}\log\left|2\sin\frac{\pi u_{p,i}}{N}\right|\right).$$

Using a Taylor expansion of the function $F : x \mapsto \log|2\sin x|$, we get for each p, i:

$$F\left(\frac{\pi u_{p,i}}{N}\right) = F\left(\frac{1}{N}\sum_{i=1}^{N}\frac{\pi u_{p,i}}{N}\right) + F'\left(\frac{1}{N}\sum_{i=1}^{N}\frac{\pi u_{p,i}}{N}\right)\left(\frac{\pi u_{p,i}}{N} - \frac{1}{N}\sum_{i=1}^{N}\frac{\pi u_{p,i}}{N}\right) + \frac{1}{2}F''(x_{p,i})\left(\frac{\pi u_{p,i}}{N} - \frac{1}{N}\sum_{i=1}^{N}\frac{\pi u_{p,i}}{N}\right)^2$$

for a certain $x_{p,i}$ with

$$|x_{p,i}| \le \max\left(\frac{\pi|u_{p,i}|}{N}, \frac{1}{N}\left|\sum_{i=1}^{N}\frac{\pi u_{p,i}}{N}\right|\right).$$

Observing that $\sum_{i=1}^{N} u_{p,i} = pN$, we have $|x_{p,i}| \le \frac{p\pi}{N} + \frac{|b_{p,i}|\pi}{N}$ and

$$\frac{1}{x_{p,i}^2}\left(\frac{\pi u_{p,i}}{N} - \frac{\pi p}{N}\right)^2 = \frac{\pi^2 b_{p,i}^2}{N^2 x_{p,i}^2} \ge \frac{1}{2}\frac{\pi^2 b_{p,i}^2}{(p\pi)^2 + (b_{p,i}\pi)^2} \ge \frac{1}{6}\min\left(\frac{b_{p,i}^2}{p^2}, 1\right). \tag{5.2.8}$$

The last inequality in (5.2.8) is obtained by observing that $\frac{1}{2}\frac{x^2}{p^2+x^2} \ge \frac{1}{6}\min(\frac{x^2}{p^2}, 1)$ on $\mathbb{R}$. Summing the Taylor expansions (5.2.2) for $i = 1 \dots N$ gives, for any $p \le N/2$:

$$\log\left|2\sin\frac{p\pi}{N}\right| - \frac{1}{N}\sum_{i=1}^{N}\log\left|2\sin\frac{\pi u_{p,i}}{N}\right| = \sum_{i=1}^{N}\frac{1}{2}F''(x_{p,i})\left(\frac{\pi u_{p,i}}{N} - \frac{\pi p}{N}\right)^2.$$

An explicit computation shows that, for any $x \in \mathbb{R}$, $F''(x) = \frac{1}{\sin^2 x} \ge \max\left(1, \frac{1}{x^2}\right)$, so by combining (5.2.8) and (5.2.2) we get

$$\log\left|2\sin\frac{p\pi}{N}\right| - \frac{1}{N}\sum_{i=1}^{N}\log\left|2\sin\frac{\pi u_{p,i}}{N}\right| \ge \sum_{i=1}^{N}\frac{1}{2}\max\left(\frac{\pi^2 b_{p,i}^2}{N^2}, \frac{1}{6}\min\left(\frac{b_{p,i}^2}{p^2}, 1\right)\right).$$

Finally, inserting the previous inequality for $1 \le p \le N/2$ into (5.2.2) gives

$$W(E_{\{a_i\}}) - W(\mathbb{Z}) \ge \sum_{p=1}^{N/2}\frac{1}{2}\max\left(\frac{\pi^2 b_{p,i}^2}{N^2}, \frac{1}{6}\min\left(\frac{b_{p,i}^2}{p^2}, 1\right)\right)$$

which yields the inequality

$$W(E_{\{a_i\}}) - W(\mathbb{Z}) \geq C \sum_{p=1}^{N/2} \frac{1}{N} \sum_{i=1}^{N} \min\left(\frac{b_{p,i}^2}{p^2}, 1\right)$$

for some universal constant C. □

5.2.3 Consequences for correlation functions

We now recast Lemma 5.1.9 in the context of stationary point processes associated to periodic point configurations. The following lemma proves in particular Theorem 1 for *periodic* stationary point processes.

Lemma 5.2.3. *For any $N \geq 1$, let $a_1 < \cdots < a_N$ be any points in $[0, N]$ and $E_{\{a_i\}}$ be the corresponding periodic vector field. Let Λ be the corresponding infinite periodic configuration in $\mathbb{R}$, and P_Λ be the stationary point process associated to Λ, defined in (5.1.6) by averaging translated copies of Λ over $[0, N]$. Assume that $W(E_{\{a_i\}})$ is finite. The following bound holds:*

$$\left| \int (\rho_{2,P_{\mathbb{Z}}} - \rho_{2,P_\Lambda})\, \varphi \right| \leq C_\varphi (C + W(E_{\{a_i\}}))^{\frac{1}{2}} \left(W(E_{\{a_i\}}) - W(\mathbb{Z})\right)^{\frac{1}{2}}$$

for any $\varphi \in C^1([-T, T]^2)$ with C_φ depending only on $||\varphi||_\infty, ||\nabla\varphi||_\infty$ and $T \geq 1$, and C universal.

Proof. Let $\varphi \in C_c^1([-T, T]^2)$ (without loss of generality we assume $T \geq 1$). Since $W(E_{\{a_i\}})$ is finite we know by Remark 5.1.8 that ρ_{2,P_Λ} exists as a Radon measure, we will abuse notation and write $\int \rho_{2,\Lambda}\varphi$ for $\rho_2(\varphi)$. By Definition 5.1.7, we have:

$$\int \rho_{2,\Lambda}\varphi = \mathbf{E}_{P_\Lambda}\left[\langle \varphi, \cdot \rangle\right].$$

Let $u_{p,i} = a_{i+p} - a_i$, with the convention $a_{N+l} = a_l + N$, and let us write the expectation $\mathbf{E}_{P_\Lambda}\left[\langle \varphi, \cdot \rangle\right]$ as:

$$\begin{aligned}
\mathbf{E}_{P_\Lambda}\left[\langle \varphi, \cdot \rangle\right] &= \int \Big(\sum_{x \neq y | x,y \in \mathcal{C}} \varphi(x,y) \Big) dP_\Lambda(\mathcal{C}) = \frac{1}{N} \int_{a_1}^{a_1+N} \Big(\sum_{x \neq y | x,y \in \theta_t \cdot \Lambda} \varphi(x,y) \Big) dt \\
&= \frac{1}{N} \sum_{i=1}^{N} \int_{a_i}^{a_{i+1}} \Big(\sum_{k \in \mathbb{Z}} \sum_{p=1}^{+\infty} (\varphi(a_{i+k} - t, a_{i+k} - t + u_{p,i+k}) + \varphi(a_{i+k} - t + u_{p,i+k}, a_{i+k} - t)) \Big) dt.
\end{aligned} \tag{5.2.9}$$

The first equality in (5.2.9) is simply an explicitation of the measure P_Λ as an average of Λ on translations in any interval of length N as in (5.1.6), and the second equality amounts to writing the sum of φ over couples of distinct points by taking a_i as the "origin" of Λ on each interval $[a_i, a_{i+1}]$:

$$\begin{aligned}
\sum_{x \neq y | x,y \in \theta_t \cdot \Lambda} \varphi(x,y) &= \sum_{x < y | x,y \in \theta_t \cdot \Lambda} \varphi(x,y) + \varphi(y,x) = \sum_{x < y | x,y \in \Lambda} \varphi(x - t, y - t) + \varphi(y - t, x - t) \\
&= \sum_{k \in \mathbb{Z}} \sum_{p=1}^{+\infty} \varphi(a_k - t, a_{k+p} - t) + \varphi(a_{k+p} - t, a_k - t) \\
&= \sum_{k \in \mathbb{Z}} \sum_{p=1}^{+\infty} \varphi(a_{i+k} - t, a_{i+k+p} - t) + \varphi(a_{i+k+p} - t, a_{i+k} - t)
\end{aligned}$$

and using the fact that, by definition, $a_{i+k+p} = a_{i+k} + u_{p,i+k}$.

For $i = 1 \dots N$, let $c_i = 1$ when $|u_{1,i} - 1| \leq 1$ and $c_i = \lfloor u_{1,i} \rfloor$ otherwise, so that $|u_{1,i} - c_i| \leq \min(|b_{1,i}|, 1)$. Let us recall that the numbers $b_{p,i}$ are defined as $b_{p,i} = u_{p,i} - p$. The following bounds are easily seen

$$\sum_{i=1}^{N} |u_{1,i} - c_i| \leq \sum_{i=1}^{N} \min(|b_{1,i}|, 1) \text{ and } \left| N - \sum_{i=1}^{N} c_i \right| \leq \sum_{i=1}^{N} \min(|b_{1,i}|, 1),$$

the second inequality following from the first one by observing that since $\sum_{i=1} u_{1,i} = N$ we have $N - \sum_{i=1}^{N} c_i = \sum_{i=1}^{N} (u_{1,i} - c_i)$. We may now write that, by 1-periodicity of $\mathbb{Z}$, and the fact that c_i $(i = 1 \dots N)$ is an integer:

$$\int_0^1 \delta_{\theta_t \cdot \mathbb{Z}} dt = \frac{1}{N} \sum_{i=1}^{N} \int_0^1 \left(\int_0^{u_{1,i}} \delta_{\theta_t \cdot \mathbb{Z}} dt + \int_{u_{1,i}}^{c_i} \delta_{\theta_t \cdot \mathbb{Z}} dt \right) + \frac{1}{N} \int_0^{N - \sum_{i=1}^{N} c_i} \delta_{\theta_t \cdot \mathbb{Z}} dt. \tag{5.2.10}$$

The decomposition of (5.2.10) is meant to adapt the average of $\mathbb{Z}$ over translations in $[0, N]$ to the decomposition as a sum over translations in $[a_i, a_{i+1}]$ used in (5.2.9), at the cost of an error term which feels the spacing irregularities in Λ. We may haved use the same way of writing $\mathbf{E}_{P_{\mathbb{Z}}}[\langle \varphi, \cdot \rangle]$ as in (5.2.9), i.e.

$$\mathbf{E}_{P_{\mathbb{Z}}}[\langle \varphi, \cdot \rangle] = \frac{1}{N} \sum_{i=1}^{N} \int_{a_i}^{a_{i+1}} \Big(\sum_{k \in \mathbb{Z}} \sum_{p=1}^{+\infty} (\varphi(i + k - t, i + k - t + p) + \varphi(i + k - t + p, i + k - t)) \Big) dt \tag{5.2.11}$$

however expressions (5.2.11) and (5.2.9) are not easy to compare. Using (5.2.10) when testing against φ yields, by making a change of variables $t \mapsto t + a_i$ on each interval $[0, u_{1,i}]$:

$$\int \rho_{2,P_{\mathbb{Z}}} \varphi = \frac{1}{N} \sum_{i=1}^{N} \int_{a_i}^{a_{i+1}} \left(\sum_{k \in \mathbb{Z}} \sum_{p=1}^{+\infty} \varphi(a_i + k - t, a_i + k - t + p) + \varphi(a_i + k - t + p, a_i + k - t) \right) dt$$
$$+ \frac{1}{N} \sum_{i=1}^{N} \int_{u_{1,i}}^{c_i} \langle \varphi, \theta_t \cdot \mathbb{Z} \rangle \, dt + \frac{1}{N} \int_0^{N - \sum_{i=1}^{N} c_i} \langle \varphi, \theta_t \cdot \mathbb{Z} \rangle \, dt,$$

Since φ is compactly supported on $[-T, T]^2$, the terms $\langle \varphi, \theta_t \cdot \mathbb{Z} \rangle$ are bounded uniformly on $t \in \mathbb{R}$ by $(2T + 1)^2 ||\varphi||_\infty$, because there is at most $(2T + 1)^2$ couples of distinct points of $\mathbb{Z}$ in any interval of length $2T$. Since we may bound the lengths of the intervals $|u_{1,i} - c_i|$ $(i = 1 \dots N)$ and $|N - \sum_{i=1}^{N} c_i|$ according to (5.2.3), we get

$$\int \rho_{2,P_{\mathbb{Z}}} \varphi = \frac{1}{N} \sum_{i=1}^{N} \int_{a_i}^{a_{i+1}} \Big(\sum_{k \in \mathbb{Z}} \sum_{p=1}^{+\infty} \varphi(a_i - t + k, a_i - t + k + p)$$
$$+ \varphi(a_i - t + k + p, a_i - t + k) \Big) dt + \frac{1}{N} \sum_{i=1}^{N} \min(|b_i|, 1) \, O(||\varphi||_\infty)$$

where the terms $O(||\varphi||_\infty)$ are bounded by $(2T + 1)^2 ||\varphi||_\infty$. In the rest of the proof, we denote by C_φ a constant, which may vary from line to line, depending only on φ via $||\varphi||_\infty$ and $||\nabla \varphi||_\infty$ and T.

Let us recall that $a_{i+k} = a_i + u_{k,i}$. A first order expansion of φ yields, for any t, i, k, p such that a_{i+k} and a_{i+k+p} lie in $[-T + t, T + t]$,

$$|\varphi(a_{i+k} - t, a_{i+k} - t + u_{i,k+p}) - \varphi(a_i - t + k, a_i - t + k + p)|$$
$$\leq \min(||\nabla \varphi||_\infty |b_{k,i}|, 2||\varphi||_\infty) + \min(||\nabla \varphi||_\infty |b_{k+p,i}|, 2||\varphi||_\infty)$$
$$\leq C_\varphi \left(\min(|b_{k,i}|, 1) + \min(|b_{k+p,i}|, 1) \right). \tag{5.2.12}$$

Let us emphasize that the constant C_φ appearing in (5.2.12) is of the type $C\times(||\varphi||_\infty+||\nabla\varphi||_\infty)$.

We may now compare (5.2.9) and the main term of (5.2.3) by summing the expansions (5.2.12):

$$\left|\frac{1}{N}\sum_{i=1}^{N}\int_{a_i}^{a_{i+1}}\Big(\sum_{k\in\mathbb{Z}}\sum_{p=1}^{+\infty}\varphi(a_{i+k}-t,a_{i+k}-t+u_{p,i+k})+\varphi(a_{i+k}-t+u_{p,i+k},a_{i+k}-t)\right.$$
$$\left.-\varphi(a_i-t+k,a_i-t+k+p)-\varphi(a_i-t+k+p,a_i-t+k)\Big)dt\right|$$
$$\leq C_\varphi\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{\infty}m_{k,i}\min(|b_{k,i}|,1)$$

where the numbers $m_{k,i}$ are given by

$$m_{k,i}=\int_{a_1}^{a_{N+1}}\mathbf{1}_{a_i\in[-T+t,T+t]}\mathbf{1}_{a_{i+k}\in[-T+t,T+t]}dt.$$

Indeed, the first-order expansions (5.2.12) allow us to bound every term in the left-hand side of (5.2.3) by a sum of four terms of the type $\min(|b_{k,i}|,1)$. For $t\in[a_1,a_{N+1}]$ the term $\min(|b_{k,i}|,1)$ appears only if a_i-t and $a_{i+k}-t$ lie in $[-T,T]$, which gives the expression for $m_{k,i}$. Moreover since there is at most N points of Λ in any interval of length N, if $N\geq 2T$ we have $m_{k,i}=0$ for all $k>N$. The assumption $N\geq 2T$ is not restrictive since we may always consider a N-periodic configuration as rN-periodic for any integer r.

It is easy to see that $m_{k,i}\leq 2T$ hence we may bound $\sum_{i=1}^{N}\sum_{k=1}^{N}m_{k,i}^2$ in the following way :

$$\sum_{i=1}^{N}\sum_{k=1}^{N}m_{k,i}^2\leq 2T\sum_{i=1}^{N}\sum_{k=1}^{N}m_{k,i}=2T\int_{a_1}^{a_{N+1}}\left(\sum_{i=1}^{N}\sum_{k=1}^{N}\mathbf{1}_{a_i\in[-T+t,T+t]}\mathbf{1}_{a_{i+k}\in[-T+t,T+t]}\right)dt$$
$$\leq 2T\int_{a_1}^{a_{N+1}}\mathcal{N}^2(\theta_t\cdot\Lambda,T)$$

where $\mathcal{N}(\Lambda,T)$ denotes the number of points of Λ in $[-T,T]$. By definition of P_Λ we may re-write the last term as

$$\int_{a_1}^{a_{N+1}}\mathcal{N}^2(\theta_t\cdot\Lambda,T)=N\int\mathcal{N}^2(\mathcal{C},T)dP_\Lambda(\mathcal{C}).$$

By Lemma 5.2.1 we know that

$$\int\mathcal{N}^2(\mathcal{C},T)dP_\Lambda(\mathcal{C})\leq C_T(C+\overline{W}(P_\Lambda))=C_T(C+W(E_{\{a_i\}})),$$

so that we finally get:

$$\sum_{i=1}^{N}\sum_{k=1}^{N}m_{k,i}^2\leq NC_T(C+W(E_{\{a_i\}})). \tag{5.2.13}$$

Combining (5.2.3) and (5.2.3) we obtain:

$$\left|\int(\rho_{2,P_\mathbb{Z}}-\rho_{2,P_\Lambda})\varphi\right|\leq\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N}m_{k,i}\min(|b_{k,i}|,1)+\frac{1}{N}\sum_{i=1}^{N}\min\left(|b_{1,i}|,1\right)O(||\varphi||_\infty). \tag{5.2.14}$$

If $m_{k,i}$ is nonzero it means that $u_{k,i}\leq 2T$ (since a_i and a_{i+k} lie in some common interval of length $2T$) hence the spacing error $|b_{k,i}|$ is larger than $k-2T$. Consequently, if $m_{k,i}$ is nonzero for $k\geq 3T$, we have $\frac{b_{k,i}}{k}\geq\frac{1}{3}$ so that for any i,k, since $T\geq 1$:

$$m_{k,i}\min(|b_{k,i}|,1)\leq 3Tm_{k,i}\min\left(\frac{|b_{k,i}|}{k},1\right). \tag{5.2.15}$$

Combining (5.2.14) and (5.2.15) gives:

$$\left|\int (\rho_{2,P_{\mathbb{Z}}} - \rho_{2,P_\Lambda})\varphi\right| \leq \frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N} 3T m_{k,i} \min\left(\frac{|b_{k,i}|}{k}, 1\right) + \frac{1}{N}\sum_{i=1}^{N} \min\left(|b_{1,i}|, 1\right) O(||\varphi||_\infty). \tag{5.2.16}$$

The second sum in (5.2.16) is controlled by Lemma 5.1.9 as follows:

$$\frac{1}{N}\sum_{i=1}^{N} \min\left(|b_{1,i}|, 1\right) \leq \left(\frac{1}{N}\sum_{i=1}^{N} \min\left(|b_{1,i}^2|, 1\right)\right)^{\frac{1}{2}} \leq C\left(W(E_{\{a_i\}}) - W(\mathbb{Z})\right)^{\frac{1}{2}}$$

hence we have

$$\left|\int \rho_{2,P_{\mathbb{Z}}} - \rho_{2,P_\Lambda}\varphi\right| \leq C_\varphi \frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N} m_{k,i} \min\left(\frac{|b_{k,i}|}{k}, 1\right) + C\left(W(E_{\{a_i\}}) - W(\mathbb{Z})\right)^{\frac{1}{2}}.$$

Using the Cauchy-Schwarz inequality in (5.2.3), the bound (5.2.13) for the sum of the $m_{k,i}^2$ and the bound (5.1.5) of Lemma 5.1.9 we get:

$$\begin{aligned}\left|\int (\rho_{2,P_{\mathbb{Z}}} - \rho_{2,P_\Lambda})\,\varphi\right| &\leq C_\varphi \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N} m_{k,i}^2\right)^{\frac{1}{2}} \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N} \min\left(\frac{|b_{k,i}|^2}{k^2}, 1\right)\right)^{\frac{1}{2}} \\ &\quad + C\left(W(E_{\{a_i\}} - W(\mathbb{Z})\right)^{\frac{1}{2}} \leq C_\varphi \left(C + W(E_{\{a_i\}})\right)^{\frac{1}{2}} \left(W(E_{\{a_i\}}) - W(\mathbb{Z})\right)^{\frac{1}{2}}\end{aligned}$$

which concludes the proof of the lemma. Let us note that altough the bound of Lemma 5.1.9 only controls $\sum_{i=1}^{N}\sum_{k=1}^{N/2} \min\left(\frac{|b_{k,i}|^2}{k^2}, 1\right)$ we may easily bound $\sum_{i=1}^{N}\sum_{k=1}^{N} \min\left(\frac{|b_{k,i}|^2}{k^2}, 1\right)$ as well by periodicity. □

An inspection of the proof shows that C_φ may be taken as $C_\varphi = C \times (||\varphi||_\infty + ||\nabla\varphi||_\infty)T^\alpha$ for some $\alpha > 0$ and C universal.

A reason for the "natural" role of two-point correlation function is that the renormalized energy itself derives from an Hamiltonian with two-body interaction. This is what allows the energy gap $W(E_{\{a_i\}}) - W(\mathbb{Z})$ to be bounded below by quantities which are "pairwise" in nature as in Lemma 5.1.9. The arguments of Lemma 5.2.3 could be applied to higher correlation functions if we knew how to bound the higher moments of $\mathcal{N}(\mathcal{C}, T)$ uniformly for point processes which are close to the ground state. It could be expected that for $\varepsilon > 0$ small enough, the family of point processes whose renormalized energy is $\leq W(\mathbb{Z}) + \varepsilon$ is such that a uniform bound on $\mathbf{E}_P\left[\mathcal{N}(\mathcal{C}, T)^n\right]$ holds for any n (one can think e.g. of harmonic deformations of the perfect lattice). It would imply a control of the crystallization for any higher correlation function, in the neighborhood of the ground state. Unfortunately we do not know if such a result is true.

5.2.4 Extension to the non-periodic case

Let us now turn to the proof of the main result, Theorem 15.

Proof. In the following we denote by C_φ a constant, which may vary from line to line, depending only on φ via $||\varphi||_\infty$ and $||\nabla\varphi||_\infty$ and T and of the type $C \times (||\varphi||_\infty + ||\nabla\varphi||_\infty)T^\alpha$.

Since $\overline{W}(P)$ is finite, let us recall that by Remark 5.1.8 the two-point correlation function of P_Λ exists at least in distributional sense.

- *Step 1: Choosing a large set where the controls are uniform.* A straightforward adaptation of [SS15a, Lemma 3.6.] (the only modification is that we are dealing with probability measures

on the electric fields only, with no dependance on Σ) ensures that for any $\varepsilon > 0$, we may find a subset $G_\varepsilon \subset \mathcal{A}_1$ such that G_ε has almost full P-measure, and on which we have a uniform control for the relevant quantities. Precisely, the lemma ensures that:

1. $P({G_\varepsilon}^c) < \varepsilon$
2. The convergence (5.1.3) in the definition of the renormalized energy is uniform with respect to $E \in G_\varepsilon$.
3. Writing $\operatorname{div} E = 2\pi(\nu_E - 1)$, both $W(E)$ and $\nu_E(I_R)/R$ are bounded uniformly with respect to $E \in G_\varepsilon$ and $R > 1$.
4. Uniformly with respect to $E \in G_\varepsilon$ we have

$$\lim_{y_0 \to +\infty} \lim_{R \to +\infty} \fint_{I_R} \int_{|y|>y_0} |E|^2 = 0.$$

This is a technical assumption needed for the "screening" construction of Step 2.

Moreover, we may assume (this is Equation (5.3) in [SS15a, Lemma 3.6.v)]) that G_ε is almost translation-invariant in that for any $E \in G_\varepsilon$, $E(\lambda + \cdot) \in G_\varepsilon$ for all $\lambda \in \mathbb{R}$ except for a set of bounded Lebesgue measure (the set depends on E but its measure is bounded uniformly on G_ε). Note that, strictly speaking, it is not precised in [SS15a, Lemma 3.6.] that one may choose G_ε both of almost full P-measure and almost translation-invariant, however it is a consequence of Equation (3.6.) in [SS15a, Lemma 3.6.v)]), and is written as Equation (7.6) in [SS15b, Lemma 7.6] (which handles the purely 2D case, but from which [SS15a, Lemma 3.6.] is essentially deduced).

For $\varepsilon < 1$, let P_ε be the probability measure induced by P on G_ε, let $P_{\Lambda,\varepsilon}$ be the push-forward of P_ε by the map $E \mapsto \frac{1}{2\pi}\operatorname{div} E + \delta_{\mathbb{R}}$ and let $\rho_{2,P_{\Lambda,\varepsilon}}$ be the two-point correlation function of $P_{\Lambda,\varepsilon}$. In the rest of the proof we make the following abuse of notation: we denote by $\mathbf{1}_{G_\varepsilon}$ both the characteristic function of G_ε and its push-forward by the map $E \mapsto \frac{1}{2\pi}\operatorname{div} E + \delta_{\mathbb{R}}$. We claim that

$$\left| \int (\rho_{2,P_\Lambda} - \rho_{2,P_{\Lambda,\varepsilon}})\varphi \right| = o_{\varepsilon \to 0}(1).$$

Indeed, we know that $\int \rho_{2,P_\Lambda}|\varphi| = \mathbf{E}_{P_\Lambda}\left[\langle |\varphi|, \cdot \rangle\right]$ is finite (see Lemma 5.2.2), and that $P(G_\varepsilon^c) < \varepsilon$. By uniform continuity of the integral, if ε is small enough, then $|\mathbf{E}_{P_\Lambda}\left[\langle \varphi, \cdot \rangle\right] - \mathbf{E}_{P_\Lambda}\left[\mathbf{1}_{G_\varepsilon}\langle \varphi, \cdot \rangle\right]|$ is arbitrarily small. This proves the claim, because we also have, by definition of $P_{\Lambda,\varepsilon}$:

$$\int \rho_{2,P_{\Lambda,\varepsilon}}\varphi = \mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\mathbf{1}_{G_\varepsilon}\langle \varphi, \cdot \rangle\right] \frac{1}{P(G_\varepsilon)}.$$

- Step 2: Obtaining periodic fields by screening. We now construct, for R large enough and for each E in G_ε, a periodic field $\bar{E}_R$ of period R, which approximates E, and we use these fields to approximate P_ε by an average of stationary measures on $\mathcal{A}_1$ associated to periodic electric fields.

To this aim, we apply [SS15a, Proposition 3.1.]. This screening result allows us to truncate E outside of a large interval, to approximate E on this interval by some field which is "screened" so that we may paste identical copies of it in order to get a periodic electric field on $\mathbb{R}$, whilst letting E unchanged in some large interval. For $R > 0$ we let $I_R = [-R/2, R/2]$.

Let $\alpha > 0$. We get from [SS15a, Proposition 3.1.] that there exists $R_0 > 0$ (depending on ε and α) such that for every integer $R \geq R_0$, for every $E \in G_\varepsilon$, there exists a vector field $E_R \in L^q_{loc}(I_R \times \mathbb{R}, \mathbb{R}^2)$ (for $q < 2$) satisfying:

i) $E_R \cdot \vec{\nu} = 0$ on $\partial I_R \times \mathbb{R}$, where $\vec{\nu}$ denotes the outer unit normal.

ii) There is a discrete subset $\Lambda \subset I_R$ such that

$$\text{div } E_R = 2\pi \left(\sum_{p\in\Lambda} \delta_p - \delta_{\mathbb{R}} \right) \quad \text{in } I_R \times \mathbb{R}.$$

iii) $E_R(z) = E(z)$ for $x \in [-R/2 + \alpha R, R/2 - \alpha R]$.

iv)

$$\frac{W(E_R, \mathbf{1}_{I_R})}{R} \leq W(E) + \alpha.$$

The "screened" property is expressed by i), the point iii) shows that E is unchanged on a large interval and iv) gives an upper bound on the new energy.

For any integer $R \geq R_0$, we extend the electric fields E_R periodically, and make them gradients. This amounts to first pasting together identical copies of E_R to make it periodic of period R (the point i) allows us to make such a construction), and then considering the L^2-projection of the constructed field onto the space of gradient vector fields, which, together with point ii) guarantees that we end up in the class $\mathcal{A}_1$. It is proved that the projection can only decrease the energy, so that iv) is conserved. Moreover, projecting onto gradients leave the divergence of E_R unchanged, so that iii) becomes:

$$\text{div } \bar{E}_R(z) = \text{div } E(z) \text{ for } x \in [-R/2 + \alpha R, R/2 - \alpha R].$$

Details are given in the proof of [SS15a, Proposition 4.1.], and we only state the conclusions: we get, for each $E \in G_\varepsilon$, and any $R \geq R_0$ (let us emphasize that R_0 depends on ε and α) an electric field $\bar{E}_R$ which is R-periodic, which coincides with E on $[-R/2 + \alpha R, R/2 - \alpha R]$, and such that

$$\frac{W(\bar{E}_R, \mathbf{1}_{I_R})}{R} \leq W(E) + \alpha.$$

- Step 3: Approximate stationary processes. For each $E \in G_\varepsilon$, and any $R \geq R_0$, we now consider the stationary probability measure $⨍_{-R/2}^{R/2} \delta_{\theta_t \cdot \bar{E}_R} dt$ on $\mathcal{A}_1$ associated to $\bar{E}_R$ by averaging $\bar{E}_R$ over translations in $[-R/2, R/2]$, and we define P_ε^R as the pushforward of the probability measure P_ε by the map

$$E \mapsto ⨍_{-R/2}^{R/2} \delta_{\theta_t \cdot \bar{E}_R} dt$$

(let us note that this map is only defined on G_ε, but P_ε itself is concentrated on G_ε). The process P_ε^R is stationary as an average of stationary probability measures, we denote by $P_{\Lambda,\varepsilon}^R$ its push-forward by the map $E \mapsto \frac{1}{2\pi}\text{div } E + \delta_{\mathbb{R}}$ and we let $\rho_{2,P_{\Lambda,\varepsilon}}^R$ be the two-point correlation function of $P_{\Lambda,\varepsilon}^R$. We now claim that

$$\left| \int (\rho_{2,P_{\Lambda,\varepsilon}}^R - \rho_{2,P_{\Lambda,\varepsilon}}) \varphi \right| \leq o_{R\to\infty}(1) + \alpha C_\varphi (\bar{W}(P) + C).$$

Indeed, by definition we have $\mathbf{E}_{P_{\Lambda,\varepsilon}} [\langle \varphi, \cdot \rangle] = \int \rho_{2,P_{\Lambda,\varepsilon}} \varphi$, and

$$\begin{aligned} \int \rho_{2,P_{\Lambda,\varepsilon}}^R \varphi = \mathbf{E}_{P_{\Lambda,\varepsilon}^R} \left[⨍_{-R/2}^{R/2} \langle \varphi, \theta_t \cdot \rangle \, dt \right] &= \frac{1}{R} \int_{-R/2+2\alpha R}^{R/2-2\alpha R} \mathbf{E}_{P_{\Lambda,\varepsilon}^R} [\langle \varphi, \theta_t \cdot \rangle] \, dt \\ &+ \frac{1}{R} \int_{-R/2}^{-R/2+2\alpha R} \mathbf{E}_{P_{\Lambda,\varepsilon}^R} [\langle \varphi, \theta_t \cdot \rangle] \, dt + \frac{1}{R} \int_{R/2-2\alpha R}^{R/2} \mathbf{E}_{P_{\Lambda,\varepsilon}^R} [\langle \varphi, \theta_t \cdot \rangle] \, dt. \end{aligned}$$

Since φ is compactly supported and since div $\bar{E}_R$ coincides with div E on the interval $[-R/2+\alpha R, R/2-\alpha R]$, if R is large enough (depending on φ) we have P_ε-a.s. that div $\bar{E}_R(\cdot - t) =$ div $E(\cdot - t)$ for $t \in [-R/2+2\alpha R, R/2-2\alpha R]$ (i.e. the screening and periodization have not affected the point configuration on a large interval). It means that for R large enough, we may express the first integrand in the right-hand side of (5.2.4) as

$$\mathbf{E}_{P^R_{\Lambda,\varepsilon}}\left[\langle\varphi,\theta_t\cdot\rangle\right] = \mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\theta_t\cdot\rangle\right] = \frac{1}{P(G_\varepsilon)}\mathbf{E}_{P_\Lambda}\left[\mathbf{1}_{G_\varepsilon}\langle\varphi,\theta_t\cdot\rangle\right].$$

The probability measure P is, by assumption, translation-invariant hence so is P_Λ, so that for any $t \in \mathbb{R}$ we have

$$\mathbf{E}_{P_\Lambda}\left[\mathbf{1}_{G_\varepsilon}\langle\varphi,\theta_t\cdot\rangle\right] = \mathbf{E}_{P_\Lambda}\left[\mathbf{1}_{G_\varepsilon}(\theta_{-t}\cdot)\langle\varphi,\cdot\rangle\right],$$

which in turn gives

$$\mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\theta_t\cdot\rangle\right] = \frac{1}{P(G_\varepsilon)}\mathbf{E}_{P_\Lambda}\left[\mathbf{1}_{G_\varepsilon}\langle\varphi,\theta_t\cdot\rangle\right] = \frac{1}{P(G_\varepsilon)}\mathbf{E}_{P_\Lambda}\left[\mathbf{1}_{G_\varepsilon}(\theta_{-t}\cdot)\langle\varphi,\cdot\rangle\right].$$

We now claim to control the default of invariance of $P_{\Lambda,\varepsilon}$ under translations in the following way:

$$\left|\frac{1}{R}\int_{-R/2+2\alpha R}^{R/2-2\alpha R} dt\,\mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\theta_t\cdot\rangle\right] - \frac{R-4\alpha R}{R}\mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\cdot\rangle\right]\right| = o_{R\to\infty}(1)$$

with a $o_{R\to\infty}(1)$ depending on φ, ε, P.

Indeed, we have, for $t \in [-R/2+2\alpha R, R/2-2\alpha R]$:

$$\mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\theta_t\cdot\rangle\right] - \mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\cdot\rangle\right] = \frac{1}{P(G_\varepsilon)}\int \langle\varphi,\mathcal{C}\rangle\left(\mathbf{1}_{G_\varepsilon}(\theta_{-t}\cdot\mathcal{C}) - \mathbf{1}_{G_\varepsilon}(\mathcal{C})\right)dP_\Lambda(\mathcal{C}).$$

Integrating (5.2.4) between $[-R/2+2\alpha R, R/2-2\alpha R]$ yields:

$$\begin{aligned}
&\frac{1}{R}\int_{-R/2+2\alpha R}^{R/2-2\alpha R} dt\,\mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\theta_t\cdot\rangle\right] - \frac{R-4\alpha R}{R}\mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\cdot\rangle\right] \\
&\qquad = \frac{1}{R}\int_{-R/2+2\alpha R}^{R/2-2\alpha R} dt\,\frac{1}{P(G_\varepsilon)}\int \langle\varphi,\mathcal{C}\rangle\left(\mathbf{1}_{G_\varepsilon}(\theta_{-t}\cdot\mathcal{C}) - \mathbf{1}_{G_\varepsilon}(\mathcal{C})\right)dP_\Lambda(\mathcal{C}) \\
&\qquad\qquad = \frac{1}{RP(G_\varepsilon)}\int dP_\Lambda(\mathcal{C})\,\langle\varphi,\mathcal{C}\rangle\int_{-R/2+2\alpha R}^{R/2-2\alpha R} dt\left(\mathbf{1}_{G_\varepsilon}(\theta_{-t}\cdot\mathcal{C}) - \mathbf{1}_{G_\varepsilon}(\mathcal{C})\right).
\end{aligned}$$

We know that for $E \in G_\varepsilon$, there is a set $\Gamma(E)$ such that $|\Gamma(E)| \le C_\varepsilon$ (for some constant depending only on G_ε) and if $\lambda \notin \Gamma(E)$ then $E(\cdot - \lambda) \in G_\varepsilon$. This property is clearly pushed forward at the level of the point configurations. This yields the following bound

$$\left|\int_{-R/2+2\alpha R}^{R/2-2\alpha R} dt\left(\mathbf{1}_{G_\varepsilon}(\theta_{-t}\cdot\mathcal{C}) - \mathbf{1}_{G_\varepsilon}(\mathcal{C})\right)\right| \le C_\varepsilon$$

and since $\int dP_\Lambda(\mathcal{C})\,\langle|\varphi|,\mathcal{C}\rangle$ is finite (again, by Lemma 5.2.2) we get

$$\left|\frac{1}{R}\int_{-R/2+2\alpha R}^{R/2-2\alpha R} dt\,\mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\theta_t\cdot\rangle\right] - \frac{R-4\alpha R}{R}\mathbf{E}_{P_{\Lambda,\varepsilon}}\left[\langle\varphi,\cdot\rangle\right]\right| \le \frac{C_{\varepsilon,\varphi,P}}{R}$$

with a constant depending on ε, φ, P, which proves (5.2.4).

We are now left to bound the two error terms in (5.2.4), for which we have, applying Lemma 5.2.2 in the last inequality:

$$\left|\frac{1}{R}\int_{R/2-2\alpha R}^{R/2}\mathbf{E}_{P^R_{\Lambda,\varepsilon}}\left[\langle\varphi,\theta_t\cdot\rangle\right]\right|dt \leq 2\alpha\sup_{t\in\mathbb{R}}\mathbf{E}_{P^R_{\Lambda,\varepsilon}}\left[\langle|\varphi|,\theta_t\cdot\rangle\right] \leq \alpha C_\varphi(\overline{W}(P)+C).$$

The other term $\frac{1}{R}\int_{R/2}^{R/2-2\alpha R}\mathbf{E}_{P^R_{\Lambda,\varepsilon}}\left[\langle\varphi,\theta_t\cdot\rangle\right]dt$ is bounded the same way, moreover with the same application of Lemma 5.2.2 we get

$$\left|4\alpha\mathbf{E}_{P^R_{\Lambda,\varepsilon}}\left[\langle\varphi,\cdot\rangle\right]\right| \leq \alpha C_\varphi(\overline{W}(P)+C).$$

Combining (5.2.4) with the estimates (5.2.4), (5.2.4), (5.2.4), we have

$$\left|\int(\rho^R_{2,P_{\Lambda,\varepsilon}}-\rho_{2,P_{\Lambda,\varepsilon}})\varphi\right| \leq \frac{C_{\varepsilon,\varphi,P}}{R}+\alpha C_\varphi(\overline{W}(P)+C)$$

which proves the claim (5.2.4).

- *Step 4: Using the result of the periodic case.* We may now come back to the proof of Theorem 15. Let us fix $\eta>0$, and take $\alpha=\frac{\eta}{C_\varphi(\overline{W}(P)+C)}$, where C_φ and C are the constant in (5.2.4). Then for R large enough (depending on α and G_ε) we have

$$\left|\int(\rho^R_{2,P_{\Lambda,\varepsilon}}-\rho_{2,P_{\Lambda,\varepsilon}})\varphi\right| \leq \eta+\frac{C_{\varphi,\varepsilon,P}}{R}.$$

Let us now apply Lemma 5.2.3 for the periodic case. For each E (under P_ε), and for any $R>0$, we consider the stationary measure $\fint_{-R/2}^{R/2}\delta_{\theta\cdot\bar{E}_R}dt$ whose energy is finite P_ε-a.s., and we denote by $\rho_{2,E,R}$ the two-point correlation function of its push-forward by the map (5.1.2). From Proposition 15 we get

$$\left|\int_{\mathbb{R}^2}(\rho_{2,E,R}-\rho_{2,P_\mathbb{Z}})\varphi\right| \leq C_\varphi(C+W(\bar{E}_R))^{\frac{1}{2}}\left(W(\bar{E}_R)-W(\mathbb{Z})\right)^{1/2}$$

and integrating this inequality against $dP_\varepsilon(E)$ gives (using Jensen's inequality in the last line)

$$\begin{aligned}\left|\int_{\mathbb{R}^2}(\rho^R_{2,P_{\Lambda,\varepsilon}}-\rho_{2,P_\mathbb{Z}})\varphi\right| &= \left|\int dP_\varepsilon(E)\int_{\mathbb{R}^2}(\rho_{2,E,R}-\rho_{2,P_\mathbb{Z}})\varphi\right| \\ &\leq C_\varphi(C+\overline{W}(P^R_\varepsilon))^{\frac{1}{2}}\left(\overline{W}(P^R_\varepsilon)-W(\mathbb{Z})\right)^{1/2}.\end{aligned}$$

By construction we know that for R large enough (depending on G_ε and α) we have P_ε-a.s.

$$W(\bar{E}_R) \leq W(E)+\eta$$

hence (5.2.4) gives, for R large enough

$$\left|\int_{\mathbb{R}^2}(\rho^R_{2,P_{\Lambda,\varepsilon}}-\rho_{2,P_\mathbb{Z}})\varphi\right| \leq C_\varphi(C+\overline{W}(P_\varepsilon)+\eta)^{\frac{1}{2}}\left(\overline{W}(P_\varepsilon)+\eta-W(\mathbb{Z})\right)^{1/2}.$$

Combinining (5.2.4) and (5.2.4), we get

$$\left|\int(\rho_{2,P_{\Lambda,\varepsilon}}-\rho_{2,P_\mathbb{Z}})\varphi\right| \leq \eta+\frac{C_{\varphi,\varepsilon,P}}{R}+C_\varphi(C+\overline{W}(P_\varepsilon)+\eta)^{\frac{1}{2}}\left(\overline{W}(P_\varepsilon)+\eta-W(\mathbb{Z})\right)^{1/2}.$$

Since $\int|W(E)|dP(E)$ is finite (because $\overline{W}(P)$ is finite and W is bounded below on $\mathcal{A}_1$), and since $P(G_\varepsilon^c)<\varepsilon$, by the uniform continuity of the integral we know that

$$\overline{W}(P_\varepsilon)=\overline{W}(P)+o_{\varepsilon\to0}(1).$$

Combining (5.2.4), (5.2.4) and (5.2.4), taking η arbitrarily small, ε arbitrarily small and then R arbitrarily large, we conclude the proof. □

5.2.5 Uniqueness results

We now turn to the proof of the uniqueness results for minimizers as stated in Corollary 5.1.10 and Corollary 5.1.12. First we observe that the invariance condition in the definition of admissible measures is equivalent to translation-invariance of the disintegration measures. Let us briefly recall (for a precise definition see [AGS08, Section 5.3.]) that in this context if $P \in \mathcal{P}(\Sigma \times \mathcal{A})$, the disintegration measures (with respect to the normalized Lebesgue measure on Σ) form a family $\{P^x\}_{x\in\Sigma}$ of probability measures (concentrated) on $\mathcal{A}$ such that for any measurable function $f \in L^\infty(\Sigma \times \mathcal{A})$ we have

$$\int_{\Sigma\times\mathcal{A}} f(x,E)dP(x,E) = \int_\Sigma \frac{dx}{|\Sigma|}\int f(x,E)dP^x(E).$$

Remark 5.2.4. *Let P be an admissible probability measure on $\Sigma \times \mathcal{A}$, and let $\{P^x\}_{x\in\Sigma}$ be the disintegration measures of P on $\mathcal{A}$ with respect to Σ. Since the first marginal of P is the normalized Lebesgue measure on Σ we have, by defintion of disintegration measures, for any continuous map $f \in L^1(dP)$:*

$$\mathbf{E}_P[f] = \fint_\Sigma dx \int f(x,E)dP^x(E).$$

For any smooth cut-off function χ on $\mathbb{R}$ and any $\lambda \in \mathbb{R}$ we have, by the invariance property of P:

$$\fint_\Sigma \chi(x)dx \int f(x,E)dP^x(E) = \fint_\Sigma \chi(x)dx \int f(x,E(\lambda+\cdot))dP^x(E).$$

A standard approximation argument (by taking a sequence $\{\chi_n\}$ converging to a Dirac mass at x_0) shows that for any $x_0 \in \Sigma$, any $\lambda \in \mathbb{R}$ we have

$$\int f(x_0,E)dP^{x_0}(E) = \int f(x_0,E(\lambda+\cdot))dP^{x_0}(E),$$

hence P^{x_0} is translation-invariant for all $x_0 \in \Sigma$.

Conversely it is easy to see that if $\{P^x\}_{x\in\Sigma}$ is a measurable family of translation-invariant probability measures such that each P^x is concentrated on $\mathcal{A}_{m_0(x)}$, then $\frac{dx_{|\Sigma}}{|\Sigma|}\otimes P^x$ is an admissible probability measure. In particular, P_0 as defined in (5.1.6) *is admissible.*

We now give the proof of Corollary 5.1.10 and Corollary 5.1.12.

Proof. It is clear, from the crystallization result of section 5.1.3, that $P_{\frac{1}{m}\mathbb{Z}}$ (resp. P_0) is indeed *a* minimizer of $\overline{W}$ (resp. of $\widetilde{W}$) on stationary measures $\mathcal{P}(\mathcal{A}_m)$ (resp. on admissible probability measures). It remains to show the uniqueness. By the scaling relation (5.1.4), it is enough to show that $P_\mathbb{Z}$ is the unique minimizer of $\overline{W}$ on $\mathcal{P}(\mathcal{A}_1)$ to prove the first claim. If $P \in \mathcal{P}(\mathcal{A}_1)$ is another minimizer we have $\overline{W}(P) = W(\mathbb{Z})$ hence by Theorem 15, if P_Λ denotes the push-forward of P by the map (5.1.2), we have

$$\rho_{2,P_\Lambda} = \rho_{2,P_\mathbb{Z}}$$

where ρ_{2,P_Λ} is the two-point correlation function of P_Λ and $\rho_{2,P_\mathbb{Z}}$ is given in distributional sense by:

$$\rho_{2,P_\mathbb{Z}} = \int_0^1 \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z},j\neq i} \delta_{(i-t,j-t)}dt.$$

Let us note that, in general, two point processes sharing the same two-point correlation function may be distinct (for conditions under which two point processes sharing *all* their k-point correlation functions are equal, see [Len73]), but here the rigidity of the lattice structure ensures that $P_\Lambda = P_{\mathbb{Z}}$.

Let $\{f_n\}_n$ be an increasing sequence of continuous functions converging pointwise to the characteristic function $\mathbf{1}_{]0,1[}$ and let $\varphi_{n,T}(x,y) := f_n(x-y)\mathbf{1}_{[-T,T]}(x)\mathbf{1}_{[-T,T]}(y)$ for some fixed $T > 0$. The sequence of continuous compactly supported functions $\{\varphi_{n,T}\}_n$ converges pointwise to $(x,y) \mapsto \mathbf{1}_{x-y\in]0,1[}\mathbf{1}_{[-T,T]}(x)\mathbf{1}_{[-T,T]}(y)$. We have for any integer n: $\int \varphi_{n,T}\rho_{2,P_\Lambda} = \int \varphi_{n,T}\rho_{2,\mathbb{Z}} = 0$ and the Lebesgue dominated convergence theorem ensures that:

$$\int \mathbf{1}_{x-y\in]0,1[}\mathbf{1}_{[-T,T]}(x)\mathbf{1}_{[-T,T]}(y)\rho_{2,P_\Lambda}(x,y) = 0.$$

Hence we get that P_Λ-a.s. there is no couple of points $x, y \in \mathcal{C} \cap [-T,T]$ such that $x - y \in]0,1[$. We may apply the same argument for any interval $]k, k+1[$ ($k \in \mathbb{Z}$) and for any $T \in \mathbb{N}$, and since a countable union of events with zero P_Λ-measure has zero P_Λ-measure it implies that there is P_Λ-a.s. no couple of points $x, y \in \mathcal{C}$ such that $x - y \notin \mathbb{Z}$. Hence $\mathcal{C}$ is P_Λ-a.s. a subset of (a translated copy of) $\mathbb{Z}$. Moreover we know from Lemma 5.2.1 that average number of points in $[-T,T]$ coincides with that of $P_{\mathbb{Z}}$ for all T. Since P_Λ is stationary this ensures that in fact $P_\Lambda = P_{\mathbb{Z}}$ and proves the first claim of uniqueness.

To prove the second claim, let $P \in \mathcal{P}(X)$ be a minimizer of $\widetilde{W}$ on the set of admissible probability measures, and let us write its disintegration $P = \frac{dx_{|\Sigma}}{|\Sigma|} \otimes P^x$ where x-a.e. in Σ, P^x is a probability measure on $\mathcal{A}_{m_0(x)}$, and since P is admissible we also know that P^x itself is translation-invariant, see remark 5.2.4. Since P minimizes $\widetilde{W}$, the stationary probability measure P^x minimizes $\overline{W}$ over $\mathcal{P}(\mathcal{A}_{m_0(x)})$ for almost every $x \in \Sigma$. By the first claim, this means that $P^x = P_{\frac{1}{m_0(x)}\mathbb{Z}}$ for almost every $x \in \Sigma$, which in turn ensures that

$$P = \frac{dx_{|\Sigma}}{|\Sigma|} \otimes P_{\frac{1}{m_0(x)}\mathbb{Z}} = P_0.$$

□

Chapitre 6

Étude du gaz de Coulomb à deux composantes

Ce chapitre est constitué de l'article "Large deviations for the two-dimensional two-component plasma" [LSZW15] écrit avec S. Serfaty, O. Zeitouni et avec une annexe par W.Wu.

Contents

6.1 Introduction

6.1.1 General setting

The two-dimensional two-component plasma is a standard ensemble of statistical mechanics, in which N particles of positive charge and N particles of negative charge interact logarithmically in the plane, cf [Frö76,GP77,DL74,For10]. The associated Gibbs measure at inverse temperature $\beta > 0$ is given by

$$d\mathbb{P}_N^\beta(\vec{X}_N, \vec{Y}_N) := \frac{1}{Z_{N,\beta}} e^{-\frac{\beta}{2} w_N(\vec{X}_N, \vec{Y}_N)} d\vec{X}_N \, d\vec{Y}_N, \tag{6.1.1}$$

where $Z_{N,\beta}$ is the normalizing constant, i.e. the *partition function*

$$Z_{N,\beta} := \int_{\Lambda^{2N}} e^{-\frac{\beta}{2} w_N(\vec{X}_N, \vec{Y}_N)} d\vec{X}_N d\vec{Y}_N, \tag{6.1.2}$$

and we have written

$$w_N(\vec{X}_N, \vec{Y}_N) := \sum_{1 \le i \ne j \le N} -\log|x_i - x_j| - \log|y_i - y_j| + \sum_{1 \le i,j \le N} \log|x_i - y_j|$$

for any $N \ge 1$ and any N-tuples $\vec{X}_N = (x_1, \dots, x_N)$ and $\vec{Y}_N = (y_1, \dots, y_N)$ of points in, say, the unit cube $\Lambda := [0,1]^2$ of $\mathbb{R}^2$. The notation $d\vec{X}_N \, d\vec{Y}_N$ refers to the Lebesgue measure on Λ^{2N}. The choice of $\beta/2$ instead of β in the exponent of (6.1.1) is made in order to match the existing literature. In physical terms, $w_N(\vec{X}_N, \vec{Y}_N)$ computes the two-dimensional electrostatic (or logarithmic) interaction of the point charges $(x_1, \dots, x_N)$ and $(y_1, \dots, y_N)$, the former carrying a $+1$ charge and the latter a -1 charge.

We are interested in proving a Large Deviation Principle (LDP) on the Gibbs measure $\mathbb{P}_N^\beta$, which is inspired by [LS15], where such a result was obtained for the one-component plasma in arbitrary dimension. The (say, two-dimensional) one-component plasma corresponds to a system of point charges which all have *same sign* and interact logarithmically, but that need to be confined by some external potential, acting in effect like a slowly varying neutralizing (opposite) charge distribution.

Motivations for studying two-component plasmas are numerous. Besides its intrinsic interest as a toy model for classical electrons and ions, it is also related to the so-called Sine-Gordon model: the grand canonical partition function of the two component plasma can be related to the Euclidean version of the sine-Gordon partition function, and the Coulomb gas on a lattice itself related to the XY model and the Kosterlitz-Thouless phase transition (see the review [Spe97] and references therein). Another motivation, which will be described in more details in the appendix, is the connection with the partition function of "complex multiplicative Gaussian chaos" (cf. [LRV15]), which may be formally written as $\int e^{i\beta h(x)} \, dx$, where $h(x)$ is the Gaussian Free Field. This question is itself related to height functions of dimer models and to the Lee-Yang theorem for the XY model. It turns out that computing moments of $e^{i\beta h}$ makes the Gibbs measure of the two-component plasma appear and, as described in the appendix, our results yield a rate of decay in terms of β of the tails of this partition function.

Due to the presence of point charges of opposite signs, the system is unstable at low temperature because the thermal excitation does not compensate the energetical trend for configurations with $-\infty$ energy, in which at least two particles of opposite signs collide. The domain of stability

of the system (i.e. the range of the parameter β for which the integral in (6.1.2) converges, so that (6.1.1) makes sense) was found to be $\beta < 2$ in [DL74], together with a first bound on $Z_{N,\beta}$. A more accurate estimate and the existence of the thermodynamic limit were proven in [Frö76] by Euclidean quantum field techniques, and [GP77] by classical methods (the question of obtaining it by classical methods was apparently first raised in [SM76]). The result can be summarized as follows.

Proposition 6.1.1 (Gunson-Panta, [GP77]). *For any $\beta < 2$,*

$$\log Z_{N,\beta} = \frac{\beta}{2} N \log N + C_\beta N + o(N), \tag{6.1.3}$$

with a constant C_β and an error $o(N)$ both depending on β.

Later, a number of other results, such as the asymptotics of two-point correlation functions and the thermodynamic properties of the two-component plasma, were obtained in the physics literature. We refer to the review [Šam03] and references therein.

For any $\vec{X}_N, \vec{Y}_N$, let μ_N^+ and μ_N^- be the empirical measures associated to the positive and negative charges

$$\mu_N^+ := \frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \quad \mu_N^- := \frac{1}{N} \sum_{i=1}^N \delta_{y_i}.$$

A natural question is to ask for the large N behavior of these *macroscopic* quantities in the space $\mathcal{P}(\Lambda)$ of probability measures on Λ. At the microscopic level, we may also ask whether the point process induced by $\mathbb{P}_N^\beta$ has a typical behavior. In this paper, we give an LDP for a spatially averaged microscopic behavior and as a consequence we show that both empirical measures μ_N^+, μ_N^- converge a.s. to the uniform measure on Λ. We remark that in case β scales as $1/N$, a LDP and Gaussian fluctuations limits for these quantities are derived in [BG99]; the techniques needed to handle the scaling in this paper are completely different.

6.1.2 Main result

Before stating our main result, we need to introduce some notation and concepts, which are all "signed" versions of those introduced in [SS15b, LS15] (in the latter papers, all the charges have the same sign). We denote by $\mathcal{X}$ the set of locally finite signed point configurations with the topology of local convergence (for more details see Section 6.2). If $(\vec{X}_N, \vec{Y}_N)$ is a pair of N-tuples of points in the square Λ, we may see it as an element of the space $\mathcal{X}$ by associating to $\vec{X}_N$ (resp. $\vec{Y}_N$) the point configuration $\nu_N^+ := \sum_{i=1}^N \delta_{x_i}$ (resp. $\nu_N^- := \sum_{i=1}^N \delta_{y_i}$). When starting from $(\vec{X}_N, \vec{Y}_N)$, we first rescale the associated finite signed configurations by a factor $\sqrt{N}$ to get

$$\hat{\nu}_N^+ := \sum_{i=1}^N \delta_{\sqrt{N} x_i} \quad \hat{\nu}_N^- := \sum_{i=1}^N \delta_{\sqrt{N} y_i},$$

and we then define the map

$$\begin{aligned} i_N : (\mathbb{R}^2)^N \times (\mathbb{R}^2)^N &\longrightarrow \mathcal{P}(\Lambda \times \mathcal{X}) \\ (\vec{X}_N, \vec{Y}_N) &\longmapsto \bar{P}_{(\vec{X}_N, \vec{Y}_N)} := \int_\Lambda \delta_{(x, \theta_{\sqrt{N}x} \cdot (\hat{\nu}_N^+, \hat{\nu}_N^-))} dx \end{aligned} \tag{6.1.4}$$

where for any Borel space X, $\mathcal{P}(X)$ denotes the set of Borel probability measures on X and θ_λ denotes the action of translation by a vector $\lambda \in \mathbb{R}^2$, that is, for $\nu \in \mathcal{P}(\mathbb{R}^2)$, we have $\theta_\lambda \nu(A) = \nu(A + \lambda)$ for any measurable set A. The variable x is a "tag" that is keeping track

of the point $x \in \Lambda$ around which the configuration was blown-up, and this way we build from any signed point configuration the law of a "tagged signed point process", $\bar{P}_{(\vec{X}_N, \vec{Y}_N)}$. The laws of these signed point processes will easily be shown to be tight, and any accumulation point as $N \to \infty$ is a stationary probability measure on $\Lambda \times \mathcal{X}^0$ (i.e. the law of a stationary tagged point process) whose first marginal is the Lebesgue measure on Λ. We will generally denote with bars the quantities corresponding to tagged point processes and without bars the quantitites corresponding to non-tagged point processes.

Throughout we will always consider the subset

$$\left\{\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X}), \bar{P}(A \times \mathcal{X}) = \mathbf{Leb}(A), \forall A \text{ Borel}\right\},$$

and continue, with some abuse of notation, to denote it by $\mathcal{P}(\Lambda \times \mathcal{X})$. This assumption allows us to consider the disintegration probability measures $\bar{P}^x \in \mathcal{P}(\mathcal{X})$ for any $x \in \Lambda$. We denote by $\mathcal{P}_{\mathrm{inv}}(\mathcal{X})$ the set of stationary laws of signed point processes, and we denote by $\mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$ the set of stationary laws of tagged signed point processes, that is those $\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X})$ so that the corresponding disintegration measure $\bar{P}^x$ is stationary for Lebesgue-a.e. $x \in \Lambda$. Finally we denote by $\mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$ the set of $\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X})$ such that $\bar{P}$ has total intensity 1 (i.e. there is, in average, one point of each sign per unit volume).

In Section 6.2.4 we will define an interaction energy functional $\widetilde{\mathbb{W}}$ on the space $\mathcal{P}_{\mathrm{inv}}(\mathcal{X})$. It can be understood as the expectation of the infinite-volume limit of the logarithmic interaction in the system of charges described by the signed configurations. We then define the interaction energy of $\bar{P} \in \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$ as

$$\overline{\mathbb{W}}(\bar{P}) := \int_\Lambda \widetilde{\mathbb{W}}(\bar{P}^x) dx. \tag{6.1.5}$$

Next, we define the *specific relative entropy* of the law of a signed point process as the infinite-volume limit of the usual relative entropy with respect to a reference measure.

Definition 6.1.2. *Let $P \in \mathcal{P}_{\mathrm{inv}}(\mathcal{X})$. The relative specific entropy $\mathsf{ent}[P]$ with respect to the signed Poisson point process of uniform intensity 1 is given by*

$$\mathsf{ent}[P] := \lim_{R \to \infty} \frac{1}{R^2} \mathrm{Ent}\left(P_R | \mathbf{\Pi}^s_R\right), \tag{6.1.6}$$

where P_R denotes the restriction of P to $C_R := [-R/2, R/2]^2$, and

$$\mathrm{Ent}(\mu|\nu) = \begin{cases} \int \log \frac{d\mu}{d\nu} d\mu & \text{if } \mu \text{ is absolutely continuous with respect to } \nu, \\ +\infty & \text{otherwise} \end{cases}$$

is the usual relative entropy. The reference measure is the law of a signed Poisson point process

$$\mathbf{\Pi}^s := \mathbf{\Pi}^1 \otimes \mathbf{\Pi}^1,$$

which is nothing but the law of two independent Poisson point processes of intensity 1.

The good definition of such an infinite-volume relative entropy ent is known in the "non-signed" case where one deals with standard point processes (see e.g. [RAS09], and [LS15] for an extension to the case of tagged point processes), and we recast its properties in the setting of signed point processes in Section 6.2.6. We may then define the specific relative entropy of $\bar{P} \in \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$ as

$$\overline{\mathsf{ent}}(\bar{P}) := \int_\Lambda \mathsf{ent}[\bar{P}^x] dx. \tag{6.1.7}$$

Using (6.1.5) and (6.1.7), we introduce the function $\overline{\mathcal{F}}_\beta$ defined on the space $\mathcal{P}_{\mathrm{inv},1}(\Lambda\times\mathcal{X})$,

$$\overline{\mathcal{F}}_\beta(\bar{P}) := \begin{cases} \frac{\beta}{2}\overline{\mathbb{W}}(\bar{P}) + \overline{\mathsf{ent}}[\bar{P}] & \text{if } \overline{\mathsf{ent}}[\bar{P}] < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{6.1.8}$$

We let $\overline{\mathcal{F}}^{\mathrm{sc}}_\beta$ be the lower semi-continuous regularization of $\overline{\mathcal{F}}_\beta$ on $\mathcal{P}_{\mathrm{inv},1}(\Lambda\times\mathcal{X})$ i.e.

$$\overline{\mathcal{F}}^{\mathrm{sc}}_\beta(\bar{P}) := \lim_{\varepsilon\to 0} \inf_{B(\bar{P},\varepsilon)} \overline{\mathcal{F}}_\beta.$$

In particular it is standard that $\overline{\mathcal{F}}_\beta$ and $\overline{\mathcal{F}}^{\mathrm{sc}}_\beta$ have the same infimum on $\mathcal{P}_{\mathrm{inv},1}(\Lambda\times\mathcal{X})$. We remark that we do not know whether $\overline{\mathcal{F}}_\beta$ is lower semi-continuous, and in particular we do not rule out the possibility that $\overline{\mathcal{F}}_\beta = \overline{\mathcal{F}}^{\mathrm{sc}}_\beta$.

When $\beta < 2$ is fixed, we let $\bar{P}_N$ be the random variable $i_N(\vec{X}_N, \vec{Y}_N)$ (as in (6.1.4)) when $(\vec{X}_N, \vec{Y}_N)$ are sampled according to $\mathbb{P}^\beta_N$, and we let $\overline{\mathfrak{P}}^\beta_N$ be its law. In other terms $\overline{\mathfrak{P}}^\beta_N$ is the push-forward of $\mathbb{P}^\beta_N$ by i_N.

We may now state our main result.

Theorem 16. *The sequence $\{\overline{\mathfrak{P}}^\beta_N\}_N$ satisfies a Large Deviations Principle at speed N with good rate function given by*

$$\overline{\mathcal{F}}^{\mathrm{sc}}_\beta - \inf_{\mathcal{P}_{\mathrm{inv},1}(\Lambda\times\mathcal{X})} \overline{\mathcal{F}}_\beta.$$

As a consequence we obtain the following expansion for $\log Z_{N,\beta}$ with $\beta < 2$:

Corollary 6.1.3. *For any $\beta < 2$ it holds*

$$\log Z_{N,\beta} = \frac{\beta}{2} N\log N - \left(\inf_{\mathcal{P}_{\mathrm{inv},1}(\Lambda,\mathcal{X})} \overline{\mathcal{F}}_\beta\right) N + o(N),$$

where the term $o(N)$ depends on β.

In comparison with Proposition 6.1.1, we now have a characterization of the constant C_β in front of the term of order N. We also have information on the asymptotic behavior of the empirical measures.

Theorem 17. *The sequence $\{(\mu^+_N, \mu^-_N)\}_N$ converges $\mathbb{P}^\beta_N$-a.s. to $\mathbf{Leb}_\Lambda \otimes \mathbf{Leb}_\Lambda$, where $\mathbf{Leb}_\Lambda$ is the uniform probability measure on Λ.*

We emphasize that in the present case, in contrast with the one-component case, the optimal macroscopic distribution of the points cannot be deduced from the leading order behavior of the system. Indeed, a leading order LDP (see Section 6.7) only shows that μ^+_N and μ^-_N have the same limit. The next order analysis of Theorem 16 allows to identify *at the same time* the macroscopic distribution of the particles, and their microscopic behavior.

6.1.3 Interpretation and method

a. Comparison with the one-component case

To explain these results and their proof, it is useful to return to the case of the one-component plasma, as was studied in [LS15] using tools introduced in [SS15b, RS15, PS15] (see also [Ser15]). We recall that in the one-component plasma, the particles have the same (positive) sign and are confined by an external potential, which can be shown to act like a neutralizing negative diffuse background charge. It is well-known that the macroscopic distribution of the particles

can be identified by a leading order LDP (at speed N^2) to be the so-called *equilibrium measure*, uniquely determined by the confining potential (cf. [HP00, BAZ98]). Then the next order LDP analysis performed in [LS15] allows to identify the behavior of the particles at the microscopic scale as minimizing a certain rate function, which is of similar nature to (6.1.8). This analysis relied on expressing the logarithmic interaction energy via the *electric field*, or *electric potential* that the set of charges generated. A crucial tool was then the so-called *screening result*, which allows, via the electric field formulation, to localize the interaction energy in microscopic boxes which can then be seen as (essentially) independent and non-interacting.

The main difference between the one-component and two-component cases is that in the one-component case, the interaction energy has a sign and is bounded below, hence no configuration can give a large contribution to the partition function. In the two-component case, the interaction energy is not bounded below, and it is only thanks to the *entropy term*, corresponding to the volume in phase-space, that configurations with very negative energy do not weigh too much in the partition function. Another heuristic way of saying this is that the Lebesgue measure (in phase-space) behaves like a "Lebesgue repulsion" which prevents particle of opposite signs from getting too close to each other too often, and this is only true because $\beta < 2$, i.e. when this Lebesgue repulsion is strong enough. In other words, energy and volume considerations always have to be worked with jointly, and we always need to exploit the fact that $\beta < 2$.

b. Positive part of the energy and dipole contributions

Because the number of positively charged particles and the number of negatively charged particles are the same, it is natural to see a configuration as a set of *dipoles* of particles of opposite sign, matched by nearest neighbor pairing (or minimal matching, also called minimal connection). It is tempting to try to prove directly an LDP on the pairing, but we have not been able to do so. Instead, we exploit the idea of matching (borrowed from [GP77]) in conjunction with simple computations originating in [SS15b, RS15], which relies on the expression of the interaction energy via the electric potential generated by the system of charges. We rewrite the interaction energy as the sum of a positive part and a part corresponding only to nearest neighbor interactions, which can be thought of as a "dipole contribution". More precisely for each x_i or y_i, we define

$$r(x_i) = \min\left(1, \frac{1}{2}\min_{j\neq i}|x_i - x_j|, \frac{1}{2}\min_j |x_i - y_j|\right)$$

to be the (half) nearest neighbor distance truncated at 1. We then prove the identity, valid for every pair of N-tuples $\vec{X}_N, \vec{Y}_N$ with distinct coordinates:

$$w_N(\vec{X}_N, \vec{Y}_N) = \frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 + \sum_{i=1}^N \log r(x_i) + \sum_{i=1}^N \log r(y_i). \tag{6.1.9}$$

where

$$V_{N,r} = \log * \left(\sum_{i=1}^N \delta_{x_i}^{(r(x_i))} - \sum_{i=1}^N \delta_{y_i}^{(r(y_i))}\right)$$

and where $\delta_x^{(\eta)}$ denotes the uniform measure of mass 1 on the sphere of center x and radius η (for $x \in \mathbb{R}^2$ and $\eta > 0$).

The identity (6.1.9) is similar to the "electrostatic inequality" found in [GP77] (however we do not discard the positive term as they do), and it allows us to use the method of [GP77], which controls the negative (and possibly unbounded) dipole contributions. The analysis of [GP77]

exploits in a quantitative way the fact that $\beta < 2$ and that the Lebesgue repulsion dominates the dipole attraction.

Similarly, the interaction energy $\overline{\mathbb{W}}$ is the sum of two terms, one positive part corresponding to the large N limit (after blow-up at the scale $\sqrt{N}$) of the positive quantity $\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2$, and a negative part corresponding to the large N limit of the nearest-neighbor contributions $\sum_{i=1}^N \log r(x_i) + \sum_{i=1}^N \log r(y_i)$.

c. Interpretation of Theorem 16.

The way to read our result at the microscopic scale is to say that the Gibbs measure must concentrate on minimizers of $\overline{\mathcal{F}}_\beta$. Unfortunately, we do not know whether a minimizer can be shown to be unique, but in any case it reduces to a minimization problem (with the structure of a *free energy* as in statistical physics) which should identify some optimal random signed point processes. For comparison, in the two-dimensional one-component case, the analogous result allows to say that the law of the well-known Ginibre point process minimizes the rate function for a certain value of β. We may interpret the minimization of $\overline{\mathcal{F}}_\beta$ heuristically as follows. The term $\overline{\mathbb{W}}(\bar{P})$ in (6.1.8) favors signed configurations which minimize the logarithmic interaction, hence we expect that it favors short dipoles (and as such, is clearly not bounded below). On the contrary, the *specific relative entropy* in (6.1.8) favors disorder, and thus tend to "separate" the dipole points. When $\beta < 2$, the sum of the two terms can be shown to be bounded below. The competition between the two terms depends of course on the value of β. When β is small (i.e. the temperature is large) then the entropy term (or "thermal agitation") dominates, whereas as β gets larger and approaches 2 the dipole attraction gets stronger, until the system can no longer sustain (or spontaneously generate) dipoles.

d. Plan of the paper

The rest of the paper is organized as follows: in Section 6.2, we gather all the definitions and notation that we use, and we present the rewriting of the energy which isolates the dipole contribution. In Section 6.3 we use the results of [GP77] to prove that $\overline{\mathcal{F}}^{\mathrm{sc}}_\beta$ is a good rate function and we establish the main result, postponing some of the main proofs. In Section 6.4 we recall, for the reader's convenience, the main computations of [GP77] (on which we rely heavily). In Section 6.5 we prove the LDP lower bound, and in Section 6.6 we prove the LDP upper bound. Section 6.7 is devoted to the proof of the large deviations principle for the empirical measures (μ_N^+, μ_N^-), at the leading order speed (which is N^2).

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6.2 Definitions, notation, and preliminary results

6.2.1 General notation

If X is a topological space we denote by $\mathcal{P}(X)$ the set of Borel probability measures on X, and if P is in $\mathcal{P}(X)$ we denote by $\mathbf{E}_P[\cdot]$ the expectation under P. We endow the space $\mathcal{P}(X)$ of Borel probability measures on X with the Dudley distance:

$$d_{\mathcal{P}(X)}(P_1, P_2) = \sup\left\{\int F(dP_1 - dP_2)|\ F \in \mathrm{Lip}_1(X)\right\}$$

where $\mathrm{Lip}_1(X)$ denotes the set of functions $F : X \to \mathbb{R}$ that are 1-Lipschitz with respect to d_X and such that $\|F\|_\infty \leq 1$. It is well-known that the distance $d_{\mathcal{P}(X)}$ metrizes the topology of weak convergence on $\mathcal{P}(X)$. We denote by $C^0(X)$ (resp. $C_c^0(X)$) the space of continuous functions (resp. with compact support on X), and by $C_b^0(X)$ the set of continuous, bounded functions on X.

If $R > 0$ we let $C_R := [-R/2, R/2]^2$ be a square of center 0 and sidelength R. If U is a Borel subset of $\mathbb{R}^2$ we denote by $|U|$ its Lebesgue measure (or area). If (X, d_X) is a metric space, $x \in X$ and $r > 0$ we denote by $B(x, r)$ the closed ball of center x and radius r for d_X. In $\mathbb{R}^2$ we will use the notation $D(x, r)$ for the closed disk of center x and radius r. If $A \subset X$ we denote by $\mathring{A}$ its interior and by $\bar{A}$ its closure.

a. (Signed) point configurations

If A is a Borel set of $\mathbb{R}^2$ we denote by $\mathcal{X}^0(A)$ the set of locally finite point configurations in A or equivalently the set of non-negative, purely atomic Radon measures on A giving an integer mass to singletons (see [DVJ88]). We let $\mathcal{X}^0 := \mathcal{X}^0(\mathbb{R}^2)$. We endow the sets $\mathcal{X}^0(A)$ (for A Borel) with the topology induced by the topology of weak convergence of Radon measure (also known as vague convergence or convergence against compactly supported continuous functions).

If B is a compact subset of $\mathbb{R}^2$ we endow $\mathcal{X}^0(B)$ with the following distance:

$$d_{\mathcal{X}^0(B)}(\mathcal{C}_1, \mathcal{C}_2) := \sup\left\{\int F(d\mathcal{C}_1 - d\mathcal{C}_2)|\ F \in \mathrm{Lip}_1(B)\right\}.$$

Similarly we endow $\mathcal{X}^0 := \mathcal{X}^0(\mathbb{R}^2)$ with the following distance:

$$d_{\mathcal{X}^0}(\mathcal{C}_1, \mathcal{C}_2) := \sum_{k\geq 1} \frac{1}{2^k}\left(\frac{d_{\mathcal{X}^0(C_k)}(\mathcal{C}_1, \mathcal{C}_2)}{(\mathcal{C}_1(C_k) + \mathcal{C}_2(C_k)) \vee 1}\right).$$

We now define the analogue of $\mathcal{X}^0(A)$ and $\mathcal{X}^0$ in the setting of signed point configuration.

Definition 6.2.1. *A signed point configuration in $\mathbb{R}^2$ (resp. in A) is defined as an element $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ of $\mathcal{X} := \mathcal{X}^0 \times \mathcal{X}^0$ (resp. $\mathcal{X}(A) := \mathcal{X}^0(A) \times \mathcal{X}^0(A)$). We say that $\mathcal{C}$ is simple if the mass of each singleton is exactly one and the supports of $\mathcal{C}^+$ and $\mathcal{C}^-$ are disjoint.*

We endow these product spaces with the product topology and the usual 1-product metric (the sum of distances componentwise). We will sometimes abuse notation and write $\int f d\mathcal{C}$ as the integral of a test function f against the signed measure $d\mathcal{C}^+ - d\mathcal{C}^-$. For $A \subset \mathbb{R}^2$ a measurable subset we let $|\mathcal{C}|(A)$ be the total number of points in A i.e. $|\mathcal{C}|(A) := \mathcal{C}^+(A) + \mathcal{C}^-(A)$.

Definition 6.2.2. *We define the "pruning" map $\mathrm{Pr} : \mathcal{X} \to \mathcal{X}$ by associating to any signed point configuration $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ the Jordan decomposition of the signed measure $\mathcal{C}^+ - \mathcal{C}^-$.*

The effect of Pr is to remove any dipole which would not be felt at the level of the signed measure. For example, if $\mathcal{C} = (\delta_{x_0} + \delta_{x_1}, \delta_{x_0} + \delta_{x_2})$ with $x_i \in \mathbb{R}^2$ distinct, we have $\mathrm{Pr}(\mathcal{C}) = (\delta_{x_1}, \delta_{x_2})$.

The additive group $\mathbb{R}^2$ acts on $\mathcal{X}^0$ by translations $\{\theta_t\}_{t\in\mathbb{R}^2}$: if $\mathcal{C} = \{x_i, i \in I\} \in \mathcal{X}^0$ we let

$$\theta_t \cdot \mathcal{C} := \{x_i - t, i \in I\}.$$

We extend this action (while keeping the same notation) to the setting of signed configurations by acting on both components $\mathcal{C}^+$ and $\mathcal{C}^-$ of a given $\mathcal{C} \in \mathcal{X}(\mathbb{R}^2)$.

For any integer N we identify a configuration $\mathcal{C}$ with N points with an unordered N-tuples of points in $\mathbb{R}^2$, which we still denote by $\mathcal{C}$. Denoting by π_N the projection from $(\mathbb{R}^2)^N$ to unordered N-tuples in $\mathbb{R}^2$, for a set A of configurations with N points we write $\mathbf{Leb}^{\otimes N}(A) = \mathbf{Leb}^{\otimes N}(\pi_N^{-1}(A))$.

b. Random tagged signed point configurations

The space $\mathcal{P}(\Lambda\times\mathcal{X})$ can be viewed as the space of laws of tagged signed point configurations, where we keep as a tag the point $x \in \Lambda$ around which the signed configuration was blown up. It is equipped with the topology of weak convergence of measures on $\Lambda \times \mathcal{X}$. Throughout we will always consider the subset

$$\left\{\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X}), \bar{P}(A \times \mathcal{X}) = \mathbf{Leb}(A), \forall A \text{ Borel}\right\},$$

and continue, with some abuse of notation, to denote it by $\mathcal{P}(\Lambda \times \mathcal{X})$. This assumption allows us to consider the disintegration probability measures $\bar{P}^x \in \mathcal{P}(\mathcal{X})$ for any $x \in \Lambda$, which satisfy by definition

$$\int F(x,\mathcal{C})d\bar{P}(x,\mathcal{C}) = \int_\Lambda \left(\int F(x,\mathcal{C})d\bar{P}^x(\mathcal{C})\right)dx,$$

for any function $F \in C_b^0(\Lambda \times \mathcal{X})$. We refer to [AGS08, Section 5.3] for a proof of the existence of disintegration measures.

c. Intensity

To any $P \in \mathcal{P}(\mathcal{X})$ we may associate two probability measures P^+, P^- on $\mathcal{X}^0$ as the push-forwards of P by the two canonical projections of $\mathcal{X}$ on $\mathcal{X}^0$, namely $(\mathcal{C}^+, \mathcal{C}^-) \mapsto \mathcal{C}^+$ and $(\mathcal{C}^+, \mathcal{C}^-) \mapsto \mathcal{C}^-$.

Let $P \in \mathcal{P}(\mathcal{X}^0)$. If there exists a measurable function $\rho_{1,P}$ such that for any function $\varphi \in C_c^0(\mathbb{R}^2)$ we have

$$\mathbf{E}_P\left[\sum_{x\in\mathcal{C}} \varphi(x)\right] = \int_{\mathbb{R}^2} \rho_{1,P}(x)\varphi(x)dx, \tag{6.2.1}$$

then we say that $\rho_{1,P}$ is the one-point correlation function (or *intensity*) of P. For $m \geq 0$ we say that P is of intensity m when the function $\rho_{1,P}$ of (6.2.1) exists and satisfies $\rho_{1,P} \equiv m$.

When $\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X}^0)$ we let $\rho_{\bar{P}}$ be the intensity measure of $\bar{P}$ defined by $\rho_{\bar{P}}(x) := \rho_{1,\bar{P}^x}$. If $\bar{P} \in \mathcal{P}(\Lambda \times \mathcal{X})$ we let $\rho^+_{\bar{P}}$, $\rho^-_{\bar{P}}$ be the respective intensity measure of $\bar{P}^+$, $\bar{P}^-$. We denote by $\mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$ the set of all $\bar{P} \in \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$ such that

$$\int_\Lambda \rho^+_{\bar{P}} = \int_\Lambda \rho^-_{\bar{P}} = 1.$$

6.2.2 Rewriting the interaction energy

Here we adapt computations from [RS15, PS15] to rewrite the interaction energy in terms of the electric potential generated by the points, seen as charges (this is also the analogue of the "electrostatic inequality" of [GP77]). Comparing with [RS15, PS15], instead of using a fixed (small) truncation distance we use the nearest neighbor distance.

a. Truncation of the logarithmic interaction

Following [PS15] we define $\delta_p^{(\eta)}$ to be the normalized surface measure on $\partial D(p,\eta)$ (it coincides with the Dirac mass at p if $\eta = 0$). We will also need the notion of truncated logarithmic kernel defined for $1 > \eta > 0$ and $x \in \mathbb{R}^2$ by

$$f_\eta(x) = (-\log|x| - \log(\eta))_+ , \tag{6.2.2}$$

and by $f_\eta \equiv 0$ if $\eta = 0$. We note that the function f_η vanishes outside the disk $D(0,\eta)$ and satisfies that

$$\frac{1}{2\pi}\mathrm{div}\,(\nabla f_\eta) + \delta_0 = \delta_0^{(\eta)}.$$

b. Nearest-neighbor distance

If $\vec{X}_N, \vec{Y}_N$ are two N-tuples of points, we define the nearest-neighbor (half-)distance as

$$r(x_i) = \min\left(1, \frac{1}{2}\min_{j\neq i}|x_i - x_j|, \frac{1}{2}\min_j |x_i - y_j|\right),$$

for any $i = 1, \dots, N$, and similarly for $r(y_i)$.

c. Rewriting of the energy functional

Let $\vec{X}_N, \vec{Y}_N$ be two N-tuples of points in Λ. We let $V_{N,r}$ be the electric potential generated by $\mathcal{C}$, after "smearing out" the charges on a distance r. More precisely,

$$V_{N,r} := 2\pi(-\Delta)^{-1}\left(\sum_{i=1}^N \delta_{x_i}^{(r(x_i))} - \sum_{i=1}^N \delta_{y_i}^{(r(y_i))}\right),$$

where $2\pi(-\Delta)^{-1}$ is the convolution by log. By the properties of $\delta_p^{(\eta)}$, this is the same as setting

$$V_{N,r}(x) = \sum_{i=1}^N \left(-\log|x - x_i| - f_{r(x_i)}(x - x_i)\right) + \sum_{i=1}^N \left(\log|x - y_i| + f_{r(y_i)}(x - y_i)\right). \tag{6.2.3}$$

We also write

$$V_{N,0} := \sum_{i=1}^N \left(-\log|x - x_i| + \log|x - y_i|\right).$$

The next crucial identity expresses the fact that the interaction energy $w_N(\vec{X}_N, \vec{Y}_N)$ can be computed using the electric potential $V_{N,r}$ (more precisely its gradient, the truncated electric field) even if the interaction has been truncated at distance r.

Lemma 6.2.3. *Let $\vec{X}_N, \vec{Y}_N$ be such that the associated global point configuration $\mathcal{C}$ is simple. Then,*

$$w_N(\vec{X}_N, \vec{Y}_N) = \frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 + \sum_{i=1}^N \log r(x_i) + \sum_{i=1}^N \log r(y_i). \tag{6.2.4}$$

Proof. This can be seen as a simple application of Newton's theorem. Since the system is globally neutral (there are N positive charges and N negative charges), the electric potential $V_{N,r}$ decays like $1/|x|$ as $|x| \to \infty$, and $\nabla V_{N,r}$ decays like $1/|x|^2$. We may thus integrate by parts and find that the boundary terms tend to zero, and therefore, using (6.2.3), we obtain

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 = \int_{\mathbb{R}^2} -\frac{1}{2\pi}\Delta V_{N,r} V_{N,r} = \sum_{i=1}^N \int_{\mathbb{R}^2} V_{N,r}\left(\delta_{x_i}^{(r(x_i))} - \delta_{y_i}^{(r(y_i))}\right)$$
$$= \sum_{i,j=1}^N \left(-\log|x-x_j| - f_{r(x_j)}(x-x_j) + \log|x-y_j| + f_{r(y_j)}(x-y_j)\right)\left(\delta_{x_i}^{(r(x_i))} - \delta_{y_i}^{(r(y_i))}\right).$$

Next, we use the fact that the disks $D(x_i, r(x_i))$ and $D(y_i, r(y_i))$ are disjoint by the definition of r, and that for any p, η, the measure $\delta_p^{(\eta)}$ is supported on $\partial D(p,\eta)$ where $f_\eta(x-p)$ vanishes, to obtain

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 = \sum_{i\neq j}\left(-\log|x-x_j|\delta_{x_i}^{(r(x_i))} - \log|x-y_j|\delta_{y_i}^{(r(y_i))}\right)$$
$$+\sum_{i,j}\left(\log|x-x_j|\delta_{y_i}^{(r(y_i))} + \log|x-y_j|\delta_{x_i}^{(r(x_i))}\right) - \sum_{i=1}^N \left(\log r(x_i) + \log r(y_i)\right).$$

In addition, by Newton's theorem (or by the mean-value property), the average of $\log|x-p|$ over any disk $D(q,\eta)$ not containing p is $\log|x-q|$. Since $\delta_p^{(\eta)}$ is precisely the uniform measure of mass 1 on $\partial D(p,\eta)$, and using again the fact that the disks $D(x_i, r(x_i))$ and $D(y_i, r(y_i))$ are disjoint, we conclude that

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V_{N,r}|^2 = \sum_{i\neq j}\left(-\log|x_i-x_j| - \log|y_i-y_j|\right)$$
$$+\sum_{i,j}\left(\log|y_i-x_j| + \log|x_i-y_j|\right) - \sum_{i=1}^N \left(\log r(x_i) + \log r(y_i)\right),$$

which is the desired result. □

6.2.3 Blow-up coordinates

In view of Lemma 6.2.3, and since the finite point configurations $\vec{X}_N, \vec{Y}_N$ are simple $\mathbb{P}_N^\beta$-a.s., we may rewrite the Gibbs measure as the probability measure whose density with respect to the Lebesgue measure on Λ^{2N} is given by

$$\frac{1}{Z_{N,\beta}} \exp\left(-\frac{\beta}{2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla V_{N,r}|^2 + \sum_{i=1}^N \log r(x_i) + \log r(y_i)\right)\right). \tag{6.2.5}$$

We rescale the finite configurations by a factor $\sqrt{N}$ and use a prime symbol for the new quantities. In particular we let $\vec{X}'_N = (x'_1, \dots, x'_N)$ with $x'_i = \sqrt{N}x_i$ (and the same for $\vec{Y}'_N$), and we have of course $r(x'_i) = \sqrt{N}r(x_i)$. We let $V'_{N,r}$ be the electric potential generated by the rescaled point configuration after truncation

$$V'_{N,r} := V_{N,r}\left(\frac{\cdot}{\sqrt{N}}\right),$$

and

$$V'_N := V_{N,0}\left(\frac{\cdot}{\sqrt{N}}\right). \tag{6.2.6}$$

We have, by a change of variables

$$\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 = \int_{\mathbb{R}^2} |\nabla V_{N,r}|^2,$$

whereas the nearest-neigbor distance term scales as

$$\sum_{i=1}^{N} \log r(x_i) + \log r(y_i) = \sum_{i=1}^{N} \log r'(x_i) + \log r'(y_i) + N \log N.$$

Combining these identities with (6.1.3) and (6.2.5) we may write the Gibbs measure as the probability measure whose density with respect to the Lebesgue measure on Λ^{2N} is given by

$$\frac{1}{K_{N,\beta}} \exp\left(-\frac{\beta}{2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 + \sum_{i=1}^{N} \log r(x'_i) + \log r(y'_i)\right)\right), \tag{6.2.7}$$

where the new normalizing constant $K_{N,\beta}$ satisfies $\log K_{N,\beta} = C_\beta N + o(N)$, with C_β as in (6.1.3).

The computations in the last two subsections serve as a motivation for the upcoming definition of the interaction energy for the infinite configurations which arise after taking the limit $N \to \infty$. We note in particular that $V_{N,r}$ solves

$$-\Delta V_{N,r} = 2\pi\left(\sum_{i=1}^{N} \delta_{x_i}^{(r(x_i))} - \sum_{i=1}^{N} \delta_{y_i}^{(r(y_i))}\right),$$

a relation which will pass to the limit (in the sense of distributions) as $N \to \infty$. The electric field associated to the potential $V_{N,r}$ is $\nabla V_{N,r}$ and its divergence is $\Delta V_{N,r}$.

6.2.4 Interaction energy for signed configurations

a. Electric fields and electric processes

We may thus define the class Elec of "electric" vector fields on $\mathbb{R}^2$ to be the set of vector fields E which belong to $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for some $p < 2$ and satisfy

$$-\mathrm{div}\ E = 2\pi(\mathcal{C}^+ - \mathcal{C}^-) \quad \text{in } \mathbb{R}^2 \tag{6.2.8}$$

for some signed point configuration $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$. When E satisfies (6.2.8) we write $E \in \mathsf{Elec}(\mathcal{C})$ and say that E is *compatible* with $\mathcal{C}$. We note that two elements of $\mathsf{Elec}(\mathcal{C})$ differ by a divergence-free vector field. Unless stated otherwise, we endow the space $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with the weak topology. If $E \in \mathsf{Elec}$ we let $\mathrm{Conf}(E)$ be the underlying signed point configuration i.e. the signed point configuration $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ where $(\mathcal{C}^+, \mathcal{C}^-)$ is the Jordan decomposition of $\frac{-1}{2\pi}\mathrm{div}\ E$. In particular, if $\mathcal{C}$ is a signed point configuration with $\mathcal{C} = \mathrm{Pr}(\mathcal{C})$ (the latter is implied if $\mathcal{C}$ is simple), and such that $E \in \mathsf{Elec}(\mathcal{C})$, then $\mathcal{C} = \mathrm{Conf}(E)$.

We define an electric field law as an element of $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$ where $p < 2$, concentrated on Elec. It will usually be denoted by P^{elec}. We say that P^{elec} is stationary when it is invariant under the (push-forward by) translations $\theta_x \cdot E = E(\cdot - x)$ for any $x \in \mathbb{R}^2$. A tagged electric field law is an element of $\mathcal{P}(\Lambda \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$ whose first marginal is the normalized Lebesgue measure on Λ and whose disintegration slices are electric field laws. It will be denoted by $\bar{P}^{\mathrm{elec}}$. We say that $\bar{P}^{\mathrm{elec}}$ is stationary if for a.e. $z \in \Lambda$, the disintegration measure $\bar{P}^{\mathrm{elec},z}$ is stationary.

b. Nearest-neighbor distance and truncation

If $\mathcal{C} = (\mathcal{C}^+, \mathcal{C}^-)$ is a signed point configuration we define the nearest-neighbor (half-)distance as

$$r(x) = \min\left(1, \frac{1}{2}\min_{x' \in \mathcal{C}^+, x' \neq x} |x - x'|, \frac{1}{2}\min_{y \in \mathcal{C}^-} |x - y|\right),$$

for any $x \in \mathcal{C}^+$ and we define similarly $r(y)$ for $y \in \mathcal{C}^-$.

For any electric field E we define its truncation

$$E_r := E - \sum_{p \in \mathcal{C}^+} \nabla f_{r(p)}(x - p) + \sum_{p \in \mathcal{C}^-} \nabla f_{r(p)}(x - p), \tag{6.2.9}$$

where $\mathcal{C} = \mathrm{Conf}(E)$ and r is defined above, computed with respect to $\mathrm{Conf}(E)$. This is the "infinite configuration" equivalent of the truncated electric field $\nabla V_{N,r}$ as in (6.2.4). We may now define the interaction energy of an admissible electric field in a similar fashion as what arises in Lemma 6.2.3.

A control on the L^2-norm of E_r can be translated into a bound of the L^p-norm on E as follows.

Lemma 6.2.4. *Let $\mathcal{C}$ be a point configuration, let $E \in \mathsf{Elec}(\mathcal{C})$ and let $R > 0$. We have*

$$\|E\|_{L^p(C_R)} \leq L_1 \|E_r\|_{L^2(C_R)} + L_2 |\mathcal{C}|(C_{R+1}) \tag{6.2.10}$$

for some $L_1 > 0$ depending only on R and p and some universal constant L_2.

Proof. From the definition (6.2.9) we get

$$|E| \leq |E_r| + \sum_{q \in \mathcal{C}^+} |\nabla f_{r(q)}(x - q)| + \sum_{q \in \mathcal{C}^-} |\nabla f_{r(q)}(x - q)|,$$

and (6.2.10) follows by using the triangle inequality for the L^p-norm, Hölder's inequality which yields $\|E_r\|_{L^p(C_R)} \leq L_1 \|E_r\|_{L^2(C_R)}$ for some L_1 depending on R and p, and by observing that the terms $\|\nabla f_{r(q)}\|_{L^p}$ are uniformly bounded by some $L_2 > 0$ and that the number of such terms is bounded by the total number of points of $\mathcal{C}$ in C_{R+1}. □

c. Positive part of the energy

For any $E \in \mathsf{Elec}$ we define $\mathcal{W}^o(E)$ as

$$\mathcal{W}^o(E) := \limsup_{R \to \infty} \frac{1}{R^2} \int_{C_R} |E_r|^2,$$

and we call it the "positive part" of the energy of E. Recall that the truncation E_r of E at nearest-neighbor distance is defined with respect to the "minimal" underlying signed point configuration $\mathrm{Conf}(E)$.

Next, if $\mathcal{C}$ is a signed point configuration we let $\mathbb{W}^o(\mathcal{C})$ be the infimum of $\mathcal{W}^o(E)$ over the electric fields E compatible with $\mathcal{C}$

$$\mathbb{W}^o(\mathcal{C}) := \inf\left\{\mathcal{W}^o(E), E \in \mathsf{Elec}(\mathcal{C})\right\}.$$

We emphasize that the definition of $\mathbb{W}^o(\mathcal{C})$ proceeds by considering the energy of associated electric fields, and that the truncation of an electric field is defined with respect to the "minimal" underlying signed point configuration $\mathrm{Conf}(E)$. In particular if $\mathrm{Pr}(\mathcal{C}_1) = \mathrm{Pr}(\mathcal{C}_2)$ then $\mathbb{W}^o(\mathcal{C}_1) = \mathbb{W}^o(\mathcal{C}_2) = \mathbb{W}^o(\mathrm{Pr}(\mathcal{C}_1))$, where Pr is the pruning map of Definition 6.2.2.

d. Negative part of the energy

If $\chi : \mathbb{R}^2 \to \mathbb{R}$ is a nonnegative measurable function with compact support and $\mathcal{C}$ a signed point configuration we let

$$\mathbb{W}^*(\chi, \mathcal{C}) := -\int \chi(x) \log(r(x))(d\mathcal{C}^+ + d\mathcal{C}^-)(x)$$
$$\mathbb{W}^*(\mathcal{C}) := \limsup_{R\to\infty} \frac{1}{R^2} \mathbb{W}^*(\mathbf{1}_{C_R}, \mathcal{C}).$$

Similarly, for any $0 < \tau < 1$, we let

$$\mathbb{W}^*_\tau(\chi, \mathcal{C}) := -\int \chi \log(r(x) \vee \tau)(d\mathcal{C}^+ + d\mathcal{C}^-)(x)$$
$$\mathbb{W}^*_\tau(\mathcal{C}) := \limsup_{R\to\infty} \frac{1}{R^2} \mathbb{W}^*_\tau(\mathbf{1}_{C_R}, \mathcal{C}).$$

The function $\mathbb{W}^*$ is a non-negative quantity, corresponding to the expected "dipole contribution" to the energy. We will call $-\mathbb{W}^*$ the negative part of the energy. It can be obtained as the monotone limit of $\mathbb{W}^*_\tau(\mathcal{C})$.

We could now try to define the interaction energy of a signed point configuration as the difference $\mathbb{W}^o(\mathcal{C}) - \mathbb{W}^*(\mathcal{C})$. However it turns out that for good definition it is preferable to consider such an object at the level of signed point processes, as we do below.

e. A compatibility lemma

Lemma 6.2.5. *Let $\{E_N\}_N$ be a sequence of electric fields, let $E \in \mathsf{Elec}$ and let $\mathcal{C} \in \mathcal{X}$. Assume that*

1. *$\{E^{(N)}\}_N$ converges to E weakly in $L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$.*
2. *$\{\mathrm{Conf}(E^{(N)})\}_N$ converges to $\mathcal{C}$ in $\mathcal{X}$.*

Then $\mathrm{Conf}(E) = \mathrm{Pr}(\mathcal{C})$, and in particular E is an electric field. Moreover,

$$\liminf_{N\to\infty} \int \chi |E^{(N)}_r|^2 \geq \int \chi |E_r|^2 \tag{6.2.11}$$

for any smooth, compactly supported, nonnegative function χ.

Proof. Let B_R be the ball of radius R, and let $f \in C^0_b(W^{-1,p}(B_R))$. One may check that $(C^0_0(B_R))^*$ embeds continuously into $W^{-1,p}(B_R)$ (indeed, $W^{-1,p}(B_R)$ is the dual of the Sobolev space $W^{1,q}_0(B_R)$ where q is the conjugate exponent to p, and the latter embeds continuously into $C^0_0(B_R)$ since $q > 2$). It thus follows that f is also bounded and continuous on $(C^0_0(B_R))^*$. The convergence of $\mathrm{Conf}(E^{(N)})$ to $\mathcal{C}$ and the fact that $\mathcal{C}$ is locally finite thus imply that

$$\lim_{N\to\infty} f\left(-\frac{1}{2\pi}\mathrm{div}\ E^{(N)}\right) = f(\mathcal{C}^+ - \mathcal{C}^-), \quad \forall f \in C^0_b(W^{-1,p}(B_R)).$$

Since the function $E \mapsto -\frac{1}{2\pi}\mathrm{div}\ E$ is continuous from $L^p_{\mathrm{loc}}(B_R)$ to $W^{-1,p}(B_R)$, we may use the first assumption to get

$$\lim_{N\to\infty} f\left(-\frac{1}{2\pi}\mathrm{div}\ E^{(N)}\right) = f\left(-\frac{1}{2\pi}\mathrm{div}\ E\right).$$

Since this is true for all $f \in C^0_b(W^{-1,p}(B_R))$ and for all $R > 1$, we conclude that $-\frac{1}{2\pi}\mathrm{div}\ E = \mathcal{C}^+ - \mathcal{C}^-$ and thus $\mathrm{Conf}(E) = \mathrm{Pr}(\mathcal{C})$.

We next prove (6.2.11). We may assume that the left-hand side is finite, otherwise there is nothing to prove. This implies that, up to an extraction of a subsequence, $\sqrt{\chi}E_r^{(N)}$ converges weakly in $L^2(\mathbb{R}^2,\mathbb{R}^2)$ to some vector field, which we claim can only be $\sqrt{\chi}E_{r'}$ where r' denotes the nearest-neighbour distance computed in $\mathcal{C}$ (and not in $\mathrm{Conf}(E)$). Indeed, it suffices to check that $\sqrt{\chi}E_r^{(N)}$ converges to $\sqrt{\chi}E_{r'}$ weakly in L^p. In view of the first assumption and of (6.2.9), it suffices to check that

$$\sum_{p\in\mathcal{C}^{(N),+}} \nabla f_{r(p)}(x-p) - \sum_{q\in\mathcal{C}^{(N),-}} \nabla f_{r(q)}(x-q) \rightharpoonup \sum_{p\in\mathcal{C}^{+}} \nabla f_{r(p)}(x-p) - \sum_{q\in\mathcal{C}^{-}} \nabla f_{r(q)}(x-q)$$

weakly in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$, where $\mathcal{C}^{(N)} = (\mathcal{C}^{(N),+},\mathcal{C}^{(N),-}) := \mathrm{Conf}(E^{(N)})$. But from (6.2.2), we have $\nabla f_\eta(x-p) = -\frac{x-p}{|x-p|^2}\mathbf{1}_{|x-p|<\eta}$ (and 0 if $\eta=0$), which is continuous in L^p with respect to both p and η. So, the stated convergence follows from the second assumption and the definition of r', using also the fact that the point configurations are locally finite. We conclude that $\sqrt{\chi}E_r^{(N)}$ converges weakly in $L^2(\mathbb{R}^2)$ as claimed, so by the lower semi-continuity of the L^2 norm,

$$\liminf_{N\to\infty} \int \chi |E_r^{(N)}|^2 \geq \int \chi |E_{r'}|^2.$$

We may finally observe that since $r' < r$ (the nearest neighbor distances are smaller in $\mathcal{C}$ than in $\mathrm{Pr}(\mathcal{C})$) we have $|E_{r'}|^2 \geq |E_r|^2$ pointwise, which concludes the proof. □

At the level of laws of electric fields, the result of Lemma 6.2.5 translates into

Lemma 6.2.6. *Let $\{P_N\}_N$ be a sequence of random signed point processes and $\{P_N^{\mathrm{elec}}\}_N$ be a sequence of laws of electric fields. Let P be a random signed point process and P^{elec} be a law of electric fields. Let us assume that:*

1. *For any $N\geq 1$, the push-forward of P_N^{elec} by* Conf *coincides with P_N.*
2. *The sequence $\{P_N\}_N$ converges to P as $N\to\infty$ in $\mathcal{P}(\mathcal{X})$.*
3. *The sequence $\{P_N^{\mathrm{elec}}\}_N$ converges to P^{elec} as $N\to\infty$ in $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2))$.*

Then we have

1. *The push-forward of P^{elec} by $E\mapsto -\frac{1}{2\pi}\mathrm{div}\ E$ is concentrated on signed measures of the type $\mathcal{C}^+ - \mathcal{C}^-$ for some $(\mathcal{C}^+,\mathcal{C}^-)$ in $\mathcal{X}$. In particular P^{elec} is concentrated on* Elec.
2. *The push-forward of P^{elec} by* Conf *coincides with the push-forward of P by* Pr.

Moreover for any smooth, compactly supported, nonnegative function χ we have

$$\liminf_{N\to\infty} \mathbf{E}_{P_N^{\mathrm{elec}}}\left[\int \chi|E_r|^2\right] \geq \mathbf{E}_{P^{\mathrm{elec}}}\left[\int \chi|E_r|^2\right].$$

6.2.5 Process level energy

a. Energy of signed point processes

Let P (resp. $\bar{P}$) be the law of a signed point process (resp. of a tagged signed point process). We define

$$\widetilde{\mathbb{W}}^o(P) := \mathbf{E}_P\left[\mathbb{W}^o(\mathcal{C})\right], \quad \overline{\mathbb{W}}^o(\bar{P}) := \mathbf{E}_{\bar{P}}\left[\mathbb{W}^o(\mathcal{C})\right] \text{ (positive part of the energy)}$$
$$\widetilde{\mathbb{W}}^*(P) := \mathbf{E}_P\left[\mathbb{W}^*(\mathcal{C})\right], \quad \overline{\mathbb{W}}^*(\bar{P}) := \mathbf{E}_{\bar{P}}\left[\mathbb{W}^*(\mathcal{C})\right] \text{ (contribution of dipoles)}$$
$$\widetilde{\mathbb{W}}^*_\tau(P) := \mathbf{E}_P\left[\mathbb{W}^*_\tau(\mathcal{C})\right], \quad \overline{\mathbb{W}}^*_\tau(\bar{P}) := \mathbf{E}_{\bar{P}}\left[\mathbb{W}^*_\tau(\mathcal{C})\right] \text{ (dipoles at truncated distance } \geq\tau).$$

Finally we define the interaction energy of P (resp. $\bar{P}$)

$$\widetilde{\mathbb{W}}(P) := \widetilde{\mathbb{W}}^o(P) - \widetilde{\mathbb{W}}^*(P) \quad \text{and} \quad \overline{\mathbb{W}}(\bar{P}) := \overline{\mathbb{W}}^o(\bar{P}) - \overline{\mathbb{W}}^*(\bar{P}). \tag{6.2.12}$$

It is yet unclear whether the right-hand sides in (6.2.12) have an actual meaning, because it could be the difference of two infinite quantities. However we will see in Section 6.3 that for a certain class of point processes, which are the only candidates for describing the microscopic behavior, the quantity $\widetilde{\mathbb{W}}^*(\bar{P})$ is in fact finite.

b. Stationary lifting with minimal energy

The following useful lemma shows that we may associate to any stationary tagged point process the law of a *stationary* tagged electric field which is compatible with it and whose energy is minimal.

Lemma 6.2.7. *Assume $\bar{P} \in \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X})$ is such that $\overline{\mathbb{W}}^o(\bar{P})$ is finite, and such that signed point configuration are $\bar{P}$-a.s. simple. Then there exists a law $\bar{P}^{\mathrm{elec}}$ of tagged electric fields such that*

1. *The push-forward of $\bar{P}^{\mathrm{elec}}$ by $(z, E) \mapsto (z, \mathrm{Conf}(E))$ is $\bar{P}$.*
2. *We have*

$$\overline{\mathbb{W}}^o(\bar{P}) = \mathbf{E}_{\bar{P}^{\mathrm{elec}}}\left[\mathcal{W}^o\right]. \tag{6.2.13}$$

Proof. The proof of Lemma 6.2.7 is very similar to that of [LS15, Lemma 2.12] and we only sketch it here. For simplicity we present the argument in the non-tagged case, the extension to random tagged signed point processes being straightforward. Let $\varepsilon > 0$ be fixed. For any simple signed point configuration $\mathcal{C}$ such that $\mathbb{W}^o(\mathcal{C})$ is finite, by definition of $\mathbb{W}^o(\mathcal{C})$ we may find an electric field $E^{(\mathcal{C},\varepsilon)}$ such that $\mathcal{W}^o(E^{(\mathcal{C},\varepsilon)}) \leq \mathbb{W}^o(\mathcal{C}) + \varepsilon$. For any $k \geq 1$ we let

$$P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)} := \frac{1}{|C_k|}\int_{C_k} \delta_{\theta_x \cdot E^{(\mathcal{C},\varepsilon)}}\, dx,$$

which is an electric field law (as defined in Section 6.2.4). For any $m \geq 1$ we have (using Fubini's theorem)

$$\int_{C_m} |E_r|^2 dP^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)}(E) \leq \frac{1}{|C_k|}\int_{C_{m+k}} |E_r^{(\mathcal{C},\varepsilon)}|^2,$$

and the right-hand side is bounded as $k \to \infty$ by the finiteness of $\mathcal{W}^o(E^{(\mathcal{C},\varepsilon)})$. Using Lemma 6.2.4 and a standard compactness result in L^p-spaces, it follows that the sequence $\{P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)}\}_k$ is tight in $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$ (for the weak L^p_{loc} topology). We let $P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon)}$ be a limit point as $k \to \infty$. Set

$$P_{(\mathcal{C},k)} := \frac{1}{|C_k|}\int_{C_k} \delta_{\theta_x \cdot \mathcal{C}}\, dx$$

Since the signed point configurations are simple P-a.s. we see that $P_{(\mathcal{C},k)}$ is the push-forward of $P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)}$ by Conf for P-a.e. $\mathcal{C}$. Moreover the ergodic theorem implies that for P-a.e. $\mathcal{C}$, the sequence $\{P_{(\mathcal{C},k)}\}_k$ converges to a stationary signed point process $P_{\mathcal{C}}$. By Lemma 6.2.5 we conclude that the push-forward of $P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon)}$ by Conf coincides with the push-forward of $P_{\mathcal{C}}$ by Pr and that the energy is lower semi-continuous in the following sense

$$\begin{aligned}\mathbf{E}_{P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon)}}\left[\frac{1}{|C_m|}\int_{C_m}|E_r|^2\right] &\leq \liminf_{k\to\infty} \mathbf{E}_{P^{\mathrm{elec}}_{(\mathcal{C},\varepsilon,k)}}\left[\frac{1}{|C_m|}\int_{C_m}|E_r|^2\right] \\ &\leq \liminf_{k\to\infty}\frac{1}{|C_k|}\int_{C_{m+k}}|E_r^{(\mathcal{C},\varepsilon)}|^2 \leq \mathcal{W}^o(E^{(\mathcal{C},\varepsilon)}).\end{aligned} \tag{6.2.14}$$

Define next $P_\varepsilon^{\rm elec} := \int P^{\rm elec}_{(\mathcal{C},\varepsilon)} dP(\mathcal{C})$, or by duality

$$\int f dP_\varepsilon^{\rm elec} := \int \left(\int f dP^{\rm elec}_{(\mathcal{C},\varepsilon)} \right) dP(\mathcal{C}),$$

for any test function $f \in C^0(L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2))$. It is not difficult to check that the electric field law $P_\varepsilon^{\rm elec}$ is stationary and that its push-forward by Conf is P (because $\mathcal{C} = {\rm Pr}(\mathcal{C})$, P-a.s.). Moreover we get from (6.2.14), for any $m \geq 1$,

$$\mathbf{E}_{P_\varepsilon^{\rm elec}} \left[\frac{1}{|C_m|} \int_{C_m} |E_r|^2 \right] \leq \mathbf{E}_P \left[\mathcal{W}^o(E^{(\mathcal{C},\varepsilon)}) \right] \leq \mathbb{W}^o(P) + \varepsilon.$$

Letting $\varepsilon \to 0$, we may thus find a limit point $P^{\rm elec}$ of $\{P_\varepsilon^{\rm elec}\}$ that is still compatible with P and such that $\mathbf{E}_{P^{\rm elec}}[\mathcal{W}^o] \leq \mathbb{W}^o(P)$. The converse inequality is always true by definition of $\mathbb{W}^o$. □

We also obtain the following useful lower semi-continuity property.

Lemma 6.2.8. *The map $\bar{P} \mapsto \overline{\mathbb{W}}^o(\bar{P})$ is lower semi-continuous on $\mathcal{P}_{\rm inv}(\Lambda \times \mathcal{X})$.*

Proof. Let $\{\bar{P}_k\}_k$ be a sequence in $\mathcal{P}_{\rm inv}(\Lambda \times \mathcal{X})$ converging to some stationary $\bar{P}$, and such that $\liminf_{k\to\infty} \overline{\mathbb{W}}^o(\bar{P}_k)$ is finite. Up to the extraction of a subsequence, we may assume that the $\liminf$ is a limit. For any $k \geq 1$ we may apply Lemma 6.2.7 and obtain a stationary electric field law $\bar{P}_k^{\rm elec}$ such that (6.2.13) holds. Using the stationarity of $\bar{P}_k^{\rm elec}$ and Fubini's theorem we may write, for any k

$$\mathbf{E}_{\bar{P}_k^{\rm elec}} [\mathcal{W}^o(E)] = \mathbf{E}_{\bar{P}_k^{\rm elec}} \left[\int_{C_1} |E_r|^2 \right],$$

and in fact the left-hand side is equal to $\mathbf{E}_{\bar{P}_k^{\rm elec}} \left[\frac{1}{R^2} \int_{C_R} |E_r|^2 \right]$ for any $R > 0$. The sequence of the push-forwards of $\bar{P}_k^{\rm elec}$ by $E \mapsto E_r$ is thus tight in $\mathcal{P}(L^2_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2))$, and using Lemma 6.2.4 we see that $\{\bar{P}_k^{\rm elec}\}_k$ itself is tight in $\mathcal{P}(L^p_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2))$. Using Lemma 6.2.6 we see that any limit point $\bar{P}^{\rm elec}$ is compatible with $\bar{P}$ and that

$$\liminf_{k\to\infty} \mathbf{E}_{\bar{P}_k^{\rm elec}} \left[\int_{C_1} |E_r|^2 \right] \geq \mathbf{E}_{\bar{P}^{\rm elec}} \left[\int_{C_1} |E_r|^2 \right],$$

and using again the stationarity we see that the right-hand side satisfies

$$\mathbf{E}_{\bar{P}^{\rm elec}} \left[\int_{C_1} |E_r|^2 \right] = \mathbf{E}_{\bar{P}^{\rm elec}} [\mathcal{W}^o(E)] \geq \overline{\mathbb{W}}^o(\bar{P}),$$

which concludes the proof. □

6.2.6 Specific relative entropy and large deviations

Let $\mathbf{\Pi}^1$ denote the law of the Poisson point process of intensity 1 on $\mathbb{R}^2$, and let $\mathbf{\Pi}^s := \mathbf{\Pi}^1 \otimes \mathbf{\Pi}^1$.

a. Existence and main properties

The following proposition is an adaptation of classical results concerning the existence and properties of the so-called specific relative entropy for stationary point processes.

Proposition 6.2.9. *Let $P \in \mathcal{P}_{\mathrm{inv}}(\mathcal{X})$. The following limit exists in $[0,+\infty]$*

$$\mathsf{ent}[P] := \lim_{R\to\infty} \frac{1}{R^2}\mathrm{Ent}\left(P_R|\mathbf{\Pi}^{\mathrm{s}}_R\right),$$

moreover the functional $P \mapsto \mathsf{ent}[P]$ is affine and lower semi-continuous on $\mathcal{P}_{\mathrm{inv}}(\mathcal{X})$.

Proof. The proofs of the corresponding results in the non-signed setting extend readily to our context, see e.g. [RAS09, Section 7.2]. □

b. Large deviations for signed empirical fields

Let $\bar{\mathfrak{Q}}^{\beta}_N$ be the "reference" empirical field defined as the push-forward by i_N of the Lebesgue measure $\mathbf{Leb}^N \otimes \mathbf{Leb}^N$ on $\Lambda^N \times \Lambda^N$, where we recall that i_N was defined in (6.1.4).

Proposition 6.2.10. *For any $A \subset \mathcal{P}_{\mathrm{inv}}(\Lambda\times\mathcal{X})$, we have (with $\overline{\mathsf{ent}}$ defined in (6.1.7))*

$$-\inf_{\bar{P}\in \mathring{A}\cap\mathcal{P}_{\mathrm{inv},1}} \overline{\mathsf{ent}}[\bar{P}] \le \liminf_{N\to\infty}\frac{1}{N}\log \bar{\mathfrak{Q}}^{\beta}_N(A) \le \limsup_{N\to\infty}\frac{1}{N}\log \bar{\mathfrak{Q}}^{\beta}_N(A) \le -\inf_{\bar{P}\in\bar{A}} \overline{\mathsf{ent}}[\bar{P}].$$

The proof of Proposition 6.2.10 is almost identical to that of [LS15, Proposition 1.6], see [LS15, Section 7].

6.2.7 Tightness and discrepancy estimates

a. Compactness and exponential tightness

Lemma 6.2.11. *The sequence $\{\overline{\mathfrak{P}}^{\beta}_N\}_N$ is exponentially tight.*

Proof. Let $\mathcal{N}_R : (\Lambda\times\mathcal{X}^0) \to \mathbb{R}_+$ be the map $\mathcal{N}_R(x,\mathcal{C}) := \mathcal{C}^+(D(0,R)) + \mathcal{C}^-(D(0,R))$ which gives the total number of points in the disk $D(0,R)$ of a signed point configuration $\mathcal{C} = (\mathcal{C}^+,\mathcal{C}^-)$. (Although it may seem not to depend on x, in applications we will always use $\mathcal{N}_R(x,\mathcal{C}_x)$ where $\mathcal{C}_x$ is the blow-up around x of a signed point configuration.) By construction it is clear that $\overline{\mathfrak{P}}^{\beta}_N$ is concentrated on

$$\left\{\bar{P}_N \in \mathcal{P}(\Lambda\times\mathcal{X}), \mathbf{E}_{\bar{P}_N}[\mathcal{N}_R] \le 2\pi R^2\right\}.$$

This set is easily seen to be compact in $\mathcal{P}(\Lambda\times\mathcal{X})$, see e.g. [LS15, Lemma 4.1]. □

b. Discrepancy estimate, equality of intensities

Here we prove that we may control the difference between the number of positive and negative charges in terms of the two-component interaction energy of the signed point configuration. In particular, we show that if P is stationary, the finiteness of $\overline{\mathcal{F}}_{\beta}(P)$ implies that the intensities of positive and negative charges coincide.

Lemma 6.2.12. *Let $P \in \mathcal{P}_{\mathrm{inv}}(\mathcal{X})$ be such that $\overline{\mathbb{W}}^{o}(P)$ and $\mathsf{ent}[P]$ are finite. Then we have $\rho^+_P = \rho^-_P$.*

Proof. In this proof C denotes a constant depending only on P. Let $\mathcal{C}$ be a signed point configuration and let $\mathcal{D}_R$ be the discrepancy in the square C_R, i.e. the difference between the number of positive and negative charges in C_R

$$\mathcal{D}_R := \int_{C_R} (d\mathcal{C}^+ - d\mathcal{C}^-).$$

Assume that E is an electric field compatible with $\mathcal{C}$. Using the relation (6.2.9) and (6.2.8) and integrating by parts over C_R, we have

$$\int_{\partial C_R} E_r \cdot \vec{n} = 2\pi \left(\mathcal{D}_R + d_R\right),$$

with $\vec{n}$ the outer unit normal, where the error term d_R is bounded by the number of points of $\mathcal{C}$ in a 1-neighborhood of ∂C_R. Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be a map such that $\psi(x) = x \log\log x$ for x large and such that ψ is convex, nonnegative, nondecreasing. We have

$$\psi(|\mathcal{D}_R|) = \psi\left(|\frac{1}{2\pi}\left(\int_{\partial C_R} E_r \cdot \vec{n}\right) - d_R|\right) \leq C\psi\left(\left|\int_{\partial C_R} E_r \cdot \vec{n}\right|\right) + C\psi(|d_R|) + C.$$

Using Jensen's inequality and the stationarity of P we get

$$\mathbf{E}_P\left[\psi\left(\int_{\partial C_R} |E_r|\right)\right] \leq \mathbf{E}_P\left[\psi\left(4R\ |E_r|\right)\right].$$

By stationarity of P, $\mathbf{E}_P\left[|E_r|^2\right]$ is equal to the positive part energy of P, which is finite by assumption. We may thus bound

$$\mathbf{E}_P\left[\psi\left(4R\ |E_r|\right)\right] \leq CR\log\log R + CR + C. \tag{6.2.15}$$

On the other hand, again by stationarity of P and using the convexity of ψ we have

$$\mathbf{E}_P\left(\psi\left(|d_R|\right)\right) \leq CR\mathbf{E}_P\left(d_1 \log\log(Rd_1)\right) \leq CR\log\log R\,\mathbf{E}_P[d_1] + CR\,\mathbf{E}_P[d_1\log\log d_1].$$

The exponential moments of d_1 and $d_1\log\log d_1$ under a Poisson point process are finite, and P has finite entropy, hence both expectations under P are finite. We obtain

$$\mathbf{E}_P\left(\psi(|d_R|)\right) \leq CR\log\log R + CR + C\,, \tag{6.2.16}$$

where C depends on the entropy of P. Combining (6.2.15) and (6.2.16) we get $\mathbf{E}_P\left[\psi(|\mathcal{D}_R|)\right] = o(\psi(R^2))$ as $R \to \infty$. It implies by Jensen's inequality (and the fact that $\psi(x) = x\log\log x$ for x large) that $\mathbf{E}_P[|\mathcal{D}_R|] = o(R^2)$, but since P is stationary we have $\mathbf{E}_P[|\mathcal{D}_R|] = R^2\mathbf{E}_P[|\mathcal{D}_1|]$, hence we deduce that for all $R > 0$ we have $\mathbf{E}_P[|\mathcal{D}_R|] = 0$. In particular the mean density of positive and negative charges are equal. □

6.2.8 Scaling relations and optimal intensity

Let P be a stationary signed point process such that the intensity of P^+ and P^- are both equal to ρ. Let $\sigma_\rho(P)$ be the stationary point processes obtained as the push-forward of P by $\mathcal{C} \mapsto \sqrt{\rho}\mathcal{C}$. It is easy to see that both component of $\sigma_\rho\left(P\right)$ now have intensity 1.

Lemma 6.2.13. *We have the following scaling relations*

$$\begin{aligned}
\widetilde{\mathbb{W}}^o(P) &= \rho(\widetilde{\mathbb{W}}^o(\sigma_\rho(P))) \\
\widetilde{\mathbb{W}}^*(P) &= \rho(\widetilde{\mathbb{W}}^*(\sigma_\rho(P)) - \log\rho) \\
\mathsf{ent}[P] &= \rho\,\mathsf{ent}[\sigma_\rho(P)] + 1 - \rho + \rho\log\rho
\end{aligned}$$

Proof. The first two relations are deduced by a change of variable in the definitions, see also the scaling relations in [SS15b, Equation (2.4)]. The scaling relation for the entropy is proven in a elementary way as in [LS15, Lemma 4.2]. □

Lemma 6.2.14. *Let $\bar{P} \in \mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$ be a stationary tagged signed point process such that the intensity measures $\rho_{\bar{P}^+}$ and $\rho_{\bar{P}^-}$ are equal to the same function ρ on Λ with $\int_\Lambda \rho = 1$. Then*

$$\overline{\mathcal{F}}_\beta(\bar{P}) \geq \inf_{\mathcal{P}_{\mathrm{inv},1}} \overline{\mathcal{F}}_\beta,$$

with equality only if $\rho(x) = 1$ for Lebesgue-a.e. $x \in \Lambda$.

Proof. From Lemma 6.2.13 we deduce a scaling relation for $\overline{\mathcal{F}}_\beta$:

$$\overline{\mathcal{F}}_\beta(\bar{P}) = \frac{\beta}{2}\int_\Lambda \rho(x)\widetilde{\mathbb{W}}(\sigma_{\rho(x)}(\bar{P}^x))dx + \int_\Lambda \rho(x)\mathsf{ent}[\sigma_{\rho(x)}(\bar{P}^x)]dx + \int_\Lambda \left[(1-\frac{\beta}{2})\rho\log\rho - \rho + 1\right]dx.$$

In particular we have

$$\overline{\mathcal{F}}_\beta(\bar{P}) \geq \inf_{\mathcal{P}_{\mathrm{inv},1}} \overline{\mathcal{F}}_\beta + (1-\frac{\beta}{2})\int_\Lambda \rho\log\rho.$$

The total intensity being fixed, since $\frac{\beta}{2} < 1$ the expression above is minimized only if $\rho = 1$ Lebesgue-a.e. and we get

$$\overline{\mathcal{F}}_\beta(\bar{P}) \geq \inf_{\mathcal{P}_{\mathrm{inv},1}} \overline{\mathcal{F}}_\beta.$$

□

6.3 Study of the Gibbs measure and main conclusions

6.3.1 Bounds on the partition function

We rely on the work of Gunson-Panta [GP77] which gives a "classical" (in contrast to the Quantum Field Theory techniques of [Frö76]) approach to the study of the Gibbs measure $\mathbb{P}_N^\beta$ and of the partition function $Z_{N,\beta}$. For the reader's reference, a rewriting of their results can be found in Section 6.4 below.

In this subsection we mostly re-phrase the key points of their analysis in our notation.

a. Dipole contribution

The analysis in [GP77], recalled in Section 6.4, see (6.4.1) and (6.4.8), yields the following lemma.

Lemma 6.3.1. *For any integer N and any $\beta < 2$ we have*

$$\log\int_{\Lambda^{2N}} \exp\left(-\frac{\beta}{2}\left(\sum_{i=1}^N \log r(x_i) + \sum_{i=1}^N \log r(y_i)\right)\right) d\vec{X}_N d\vec{Y}_N \leq \frac{\beta}{2}N\log N + C_\beta N, \qquad (6.3.1)$$

with a constant C_β depending only on β.

b. Exponential moments

We give another consequence of the analysis in [GP77]. For any pair of integers N_+, N_-, and real $R > 0$ we denote by $\mathbf{B}_{N_+,N_-,R}$ the law of the signed point process on C_R obtained from two independent Bernoulli processes with N_+ and N_- points. The following lemma gives a bound on (the exponential moments of) the dipole contribution in the interaction energy.

Lemma 6.3.2. *For any $\beta < 2$ and any $R > 0$ we have*

$$\log \mathbf{E}_{\mathbf{B}_{N_+,N_-,R}}\left[e^{\frac{\beta}{2}\mathbb{W}^*(\mathbf{1}_{C_R},\mathcal{C})}\right] \leq \frac{\beta}{4}(N_+ + N_-)\log(N_+ + N_-) + (N_+ + N_-)C_\beta - \frac{\beta}{2}(N_+ + N_-)\log R. \quad (6.3.2)$$

Proof. Scaling the configuration by a factor R^{-1} changes the left-hand side by $\frac{\beta}{2}(N_+ + N_-)\log R$ and then we are left to prove the inequality for $R = 1$. With our notation, it reduces to the upper bound on [GP77, (2.4)] as expressed in [GP77, (2.9)] (cf. (6.4.8) and (6.4.1)). Let us emphasize that although the analysis of [GP77] initially deals with a system such that $N_+ = N_- = N$, the bound on [GP77, (2.4)] is not affected by the actual sign of each charge, as it is merely a bound on some given integral on $(\mathbb{R}^2)^{2N}$. We may thus follow the lines of [GP77, Section 2.2.] with $N_+ + N_-$ instead of $2N$ and [GP77, (2.9)] yields (6.3.2). □

6.3.2 Study of the rate function

In this subsection we show that $\overline{\mathcal{F}}_\beta$ is bounded below and is well-defined as a functional $\overline{\mathcal{F}}_\beta : \mathcal{P}_{\mathrm{inv}}(\Lambda \times \mathcal{X}^0) \to \mathbb{R} \cup \{+\infty\}$.

Lemma 6.3.3. *For any $\beta < 2$, any $\tau, R > 0$ and any $P \in \mathcal{P}_{\mathrm{inv}}(\mathcal{X})$ such that $\mathsf{ent}[P]$ is finite, it holds that*

$$-\frac{\beta}{2}\mathbf{E}_P\left[\mathbb{W}^*_\tau(\mathbf{1}_{C_R},\mathcal{C})\right] + \mathrm{Ent}[P_R|\mathbf{\Pi}^s_R] \geq -L_\beta R^2 \quad (6.3.3)$$

where L_β is a constant depending only on β. Consequently we get as $\tau \to 0$

$$-\frac{\beta}{2}\mathbf{E}_P\left[\mathbb{W}^*(\mathbf{1}_{C_R},\mathcal{C})\right] + \mathrm{Ent}[P_R|\mathbf{\Pi}^s_R] \geq -L_\beta R^2 \quad (6.3.4)$$

and finally in the limit $R \to +\infty$,

$$-\mathbf{E}_P\left[\frac{\beta}{2}\mathbb{W}^*(\mathcal{C})\right] + \mathsf{ent}[P] \geq -L_\beta. \quad (6.3.5)$$

Proof. By the variational characterization of the relative entropy, we know that

$$-\log \mathbf{E}_{\mathbf{\Pi}^s_R}\left[e^{\frac{\beta}{2}\mathbb{W}^*_\tau(\mathbf{1}_{C_R},\mathcal{C})}\right] \leq \mathrm{Ent}[P_R|\mathbf{\Pi}^s_R] - \mathbf{E}_P\left[\frac{\beta}{2}\mathbb{W}^*_\tau(\mathbf{1}_{C_R},\mathcal{C})\right]. \quad (6.3.6)$$

We evaluate the left-hand side of (6.3.6). Combining the definition of a Poisson point process with Lemma 6.3.2 we get

$$\log \mathbf{E}_{\mathbf{\Pi}^s_R}\left[e^{\frac{\beta}{2}\mathbb{W}^*_\tau(\mathbf{1}_{C_R},\mathcal{C})}\right] = \log \sum_{N_+,N_-=0}^{+\infty} \mathbf{\Pi}^s_R(N_+,N_-)\mathbf{E}_{\mathbf{B}_{N_+,N_-,R}}\left[e^{\frac{\beta}{2}\mathbb{W}^*_\tau(\mathbf{1}_{C_R},\mathcal{C})}\right]$$
$$\leq \log \sum_{N_+,N_-=0}^{+\infty} e^{-2R^2}\frac{R^{2(N_++N_-)}}{N_+!N_-!}e^{\frac{\beta}{4}(N_++N_-)\log(N_++N_-)+(N_++N_-)C_\beta-\frac{\beta}{2}(N_++N_-)\log R}.$$

Using the elementary inequality

$$(N_+ + N_-)\log(N_+ + N_-) \leq (N_+\log N_+ + N_-\log N_- + N_+ + N_-),$$

we may separate the variables N_+ and N_- (which play a symmetric role) and write, using the fact that $\frac{1}{N!} \leq e^{-N\log N+(C+1)N}$ for a certain constant C,

$$\log \mathbf{E}_{\mathbf{\Pi}_R^s}\left[e^{\frac{\beta}{2}\mathbb{W}_\tau^*(\mathbf{1}_{C_R},\mathcal{C})}\right] \leq 2\log \sum_{N=0}^{+\infty} e^{-R^2}\frac{R^{2N}}{N!}e^{\frac{\beta}{4}(N\log N+N)+N(C_\beta+C)-\frac{\beta}{2}N\log R}$$
$$\leq 2\log \sum_{N=0}^{+\infty} e^{-R^2}R^{2(1-\frac{\beta}{4})N}e^{(\frac{\beta}{4}-1)N\log N+(C_\beta+C+1)N} \leq L_\beta R^2$$

for a certain constant L_β depending only on β. Inserting this estimate into (6.3.6) yields (6.3.3), (6.3.4) follows by sending $\tau \to 0$ and (6.3.5) is obtained by dividing (6.3.4) by R^2 and then sending $R \to +\infty$ (together with the definition (6.1.6) of ent). □

In particular if $\overline{\mathsf{ent}}[\bar{P}]$ is finite, then for Lebesgue-a.e. $x \in \Lambda$ the disintegration measure $\bar{P}^x$ has finite entropy and satisfies (6.3.5), hence the functional $\overline{\mathcal{F}}_\beta$ is well-defined.

Conclusion

Lemma 6.3.4. *The functional $\overline{\mathcal{F}}_\beta^{\mathrm{sc}}$ is a good rate function.*

Proof. Lemma 6.3.3 shows that $\overline{\mathcal{F}}_\beta$ is well-defined as a functional $\overline{\mathcal{F}}_\beta : \mathcal{P}_{\mathrm{inv}}(\Lambda\times\mathcal{X}) \to \mathbb{R}\cup\{+\infty\}$. It also implies that the sub-level sets of $\overline{\mathcal{F}}_\beta$ are included in sub-level sets of $\bar{P} \mapsto \overline{\mathsf{ent}}[\bar{P}]$, which are compact.

Thus $\overline{\mathcal{F}}_\beta^{\mathrm{sc}}$ is well-defined and it is lower semi-continuous by definition. Since the sub-level sets of $\overline{\mathcal{F}}_\beta$ are pre-compact, those of its lower semi-continuous regularization are compact. It proves that $\overline{\mathcal{F}}_\beta^{\mathrm{sc}}$ is a good rate function. □

6.3.3 Properties of the limit objects

One of the crucial points in order to get Theorem 17 from Theorem 16 is to show, by entropy arguments, that the intensities of the underlying point processes coincide with the limits of the empirical measures, while by the scaling argument of Lemma 6.2.14 the rate function is minimized only when these intensities equal 1.

In this subsection we use the preliminary bounds on $Z_{N,\beta}$ available for $\beta < 2$ thanks to the analysis of Gunson-Panta to derive some *a priori* properties of the possible limits of μ_N^+ and $\bar{P}_N$. In particular we wish to show that the intensity of the limits of $\bar{P}_N$ equals, most of the time, the density of the limits of μ_N^+. This is not obvious since, with the topology that we use for the convergence of $\bar{P}_N$, there can be a loss of mass when taking the limit.

To overcome this, we will show below that with overwhelming probability (i.e. up to neglecting events of $\mathbb{P}_N^\beta$-probability less than e^{-NT} with T arbitrarily large) the limiting objects must have finite entropy, which will yield a uniform integrability of the densities of points, which in turn ensures that no loss of mass occurs in the limit.

a. A priori bounds on the entropy

Lemma 6.3.5. *For any $\beta < 2$, the following holds with a constant C_β depending only on β.*

1. *For any $\mu^+ \in \mathcal{P}(\Lambda)$ we have*

$$\lim_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N}\log \mathbb{P}_N^\beta\left(\mu_N^+ \in B(\mu^+,\varepsilon)\right) \leq C_\beta - \frac{1}{C_\beta}\mathrm{Ent}[\mu^+|\mathbf{Leb}_\Lambda]. \tag{6.3.7}$$

2. *For any* $\bar{P} \in \mathcal{P}_{\rm inv}(\Lambda \times \mathcal{X})$ *we have*

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \overline{\mathfrak{P}}_N^\beta \left(B(\bar{P}, \varepsilon) \right) \leq C_\beta - \frac{1}{C_\beta} \overline{\mathsf{ent}}[\bar{P}]. \tag{6.3.8}$$

3. *For any* R, N *let* $\{C_i\}_{i \in I}$ *be a partition of* Λ *by squares of sidelength in* $(\frac{R}{2\sqrt{N}}, \frac{3R}{2\sqrt{N}})$*, and let* $n_i = \mu_N^+(C_i)$ *be the number of positive charges in the square* C_i*. When* $R > 0$ *is fixed we have*

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_N^\beta \left(\left(\frac{1}{\#I} \sum_{i \in I} \frac{n_i}{R^2} \max \left(1, \left(\log \frac{n_i}{R^2} \right)^{\frac{1}{2}} \right) \right) \geq M \right) \leq f_\beta(M, R), \tag{6.3.9}$$

with $\lim f_\beta(M, R) = -\infty$ *as* $M \to \infty$*.*

Proof. For any $\beta < 2$, let us fix some $p > 1$ such that $p\beta < 2$ and let q be the conjugate exponent of p.

Let $\mu \in \mathcal{P}(\Lambda)$. Let $A \subset \mathcal{P}(\Lambda)$ be measurable. For any N we obtain using Hölder's inequality that

$$\begin{aligned} \mathbb{P}_N^\beta \left(\Lambda^{2N} \cap \{\mu_N^+ \in A\} \right) &= \frac{1}{Z_{N,\beta}} \int_{\Lambda^{2N} \cap \{\mu_N^+ \in A\}} e^{-\frac{\beta}{2} w_N(\vec{X}_N, \vec{Y}_N)} d\vec{X}_N d\vec{Y}_N \\ &\leq \frac{1}{Z_{N,\beta}} \left(\int_{\Lambda^{2N}} e^{-p\frac{\beta}{2} w_N(\vec{X}_N, \vec{Y}_N)} d\vec{X}_N d\vec{Y}_N \right)^{\frac{1}{p}} \left(\int_{\Lambda^{2N} \cap \{\mu_N^+ \in A\}} d\vec{X}_N d\vec{Y}_N \right)^{\frac{1}{q}} \\ &= \frac{Z_{N,p\beta}^{\frac{1}{p}}}{Z_{N,\beta}} \left(\int_{\Lambda^{2N} \cap \{\mu_N^+ \in A\}} d\vec{X}_N d\vec{Y}_N \right)^{\frac{1}{q}} \end{aligned} \tag{6.3.10}$$

where p, q are as above. By (6.1.3) we have

$$\log Z_{N,p\beta} = p\frac{\beta}{2} N \log N + C_{p\beta} N + o(N) \text{ and } \log Z_{N,\beta} = \frac{\beta}{2} N \log N + C_\beta N + o(N) \tag{6.3.11}$$

where $C_\beta, C_{p\beta}$ depend only on β. On the other hand we have by Sanov's theorem

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \int_{\Lambda^{2N} \cap \{\mu_N^+ \in B(\mu^+, \varepsilon)\}} d\vec{X}_N d\vec{Y}_N = -\mathrm{Ent}[\mu^+ | \mathbf{Leb}_\Lambda]. \tag{6.3.12}$$

Combining (6.3.10) (with $A = B(\mu^+, \varepsilon)$), (6.3.11) and (6.3.12) yields (6.3.7). The proof of (6.3.8) is similar, using Proposition 6.2.10 instead of Sanov's theorem in the last step, where (6.3.12) is replaced by

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \int_{\Lambda^{2N} \cap \bar{P}_N \in B(\bar{P}, \varepsilon)} d\vec{X}_N d\vec{Y}_N = -\overline{\mathsf{ent}}[\bar{P}].$$

To see (6.3.9), we take $A = A(M, R)$ in (6.3.10) as the event inside the probability in (6.3.9). Using (6.3.11), the proof of (6.3.9) reduces to proving

$$\limsup_{N \to \infty} \frac{1}{N} \log \int_{\Lambda^{2N} \cap A} d\vec{X}_N d\vec{Y}_N \leq f_\beta(M, R). \tag{6.3.13}$$

The proof of (6.3.13) is simplified if one uses comparison to a Poisson process of intensity N on Λ, denoted $\mathbf{\Pi}_N$. Indeed, extend the event A in a natural way to apply to any collection of integers $\{n_i\}_{i \in I}$, and note that A is monotone increasing with respect to $K = \sum_{i \in I} n_i$. Since there exists a constant $\eta > 0$ independent of N so that $\mathbf{\Pi}_N(K \geq N) > \eta$, and since conditioned

on K the points of the Poisson process are independent and uniformly distributed in Λ, the proof of (6.3.13) reduces to proving that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbf{\Pi}_N(A) \leq f_\beta(M,R). \tag{6.3.14}$$

The advantage of working with $\mathbf{\Pi}_N$ is that the random variables n_i are now independent Poisson of parameter in $(R^2/4, 9R^2/4)$. In particular, the random variables $n_i \max(1, (\log(n_i/R^2))^{1/2})$ possess a finite exponential moment. Applying Markov's exponential inequality then yields (6.3.14) and completes the proof of (6.3.9). □

Of course (6.3.7) and (6.3.9) also hold when replacing μ_N^+ by μ_N^-.

b. Uniform integrability of the number of points

The bound (6.3.9) implies that under $\mathbb{P}_N^\beta$, the random number of points $\mu_N^+\left(B(x, \frac{R}{\sqrt{N}})\right)$ is uniformly (as $N \to \infty$) integrable on Λ with overwhelming probablity. More precisely we have

Lemma 6.3.6. *For any $T, R > 0$ and any $\varepsilon > 0$ there exists $M' > 0$ (depending on T, ε and on β) such that for N large enough we have*

$$\int_\Lambda \mu_N^+\left(B(x, \frac{R}{\sqrt{N}})\right) \mathbf{1}_{\{\mu_N^+\left(B(x,\frac{R}{\sqrt{N}})\right) \geq M'\}} dx \leq \varepsilon$$

with probability $\geq 1 - \exp(-NT)$ under $\mathbb{P}_N^\beta$.

Proof. Indeed from (6.3.9) we control the $L^1(\Lambda)$ norm of the superlinear map ψ_R defined as

$$\psi_R(x) := \frac{x}{R^2} \max\left(1, \left(\log \frac{x}{R^2}\right)^{\frac{1}{2}}\right)$$

by M with $\mathbb{P}_N^\beta$-probability $\geq 1 - \exp(N f_\beta(M))$ with $\lim_{M\to\infty} f_\beta(M) = -\infty$. □

c. Microscopic intensity *versus* macroscopic density

We emphasize the following abuse of notation: in Lemma 6.3.7 and its proof, the quantities μ_N^+ and $\bar{P}_N$ are elements of a *deterministic* sequence.

Lemma 6.3.7. *Let $\{\vec{X}_N, \vec{Y}_N\}_N$ be a sequence of points in Λ^{2N}, let $\mu_N^+ := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and let $\bar{P}_N := i_N(\vec{X}_N, \vec{Y}_N)$. Assume that up to extraction the sequence $\{(\mu_N^+, \bar{P}_N)\}_N$ converges to $(\mu^+, \bar{P})$ where $\mu^+ \in \mathcal{P}(\Lambda)$ and $\bar{P} \in \mathcal{P}_{\rm inv}(\Lambda \times \mathcal{X})$. Then we have $\rho_{\bar{P}}^+ \leq \mu^+$ in the sense of nonnegative measures.*

Moreover under the assumption that μ^+ does not charge $\partial\Lambda$ and that for any $R > 1$, $x \mapsto \mu_N^+\left(B(x, \frac{R}{\sqrt{N}})\right)$ is uniformly integrable on Λ as $N \to \infty$ then $\rho_{\bar{P}}^+ = \mu^+$.

Of course the same results hold for the quantities associated to the negative charges as well.

Proof. Let χ be a non-negative test function in $C^0(\Lambda)$ and for any $R > 1$ let f_R be a smooth function satisfying

$$\frac{1_{C_{R-1}}}{|C_R|} \leq f_R \leq \frac{1_{C_R}}{|C_R|},$$

we also define $f_{R,N}$ as $f_{R,N}(x) := Nf_R(\sqrt{N}x)$ for $x \in \Lambda$. Finally we define $\langle f_R, \mathcal{C}^+ \rangle$ as

$$\left\langle f_R, \mathcal{C}^+ \right\rangle := \int f_R d\mathcal{C}^+ \text{ for any } \mathcal{C}^+ \in \mathcal{X}^0(\mathbb{R}^2).$$

We compute (with $*$ being the convolution product)

$$\begin{aligned}\int \chi * f_{R,N} d\mu_N^+ &= \int \chi f_{R,N} * d\mu_N^+ = \int \chi(y) \int f_{R,N}(y-z) d\mu_N^+(z) dy \\ &= \int \chi(y) \sum_{i=1}^N f_R(\sqrt{N}(x_i - y)) dy \geq \int \chi(y) \left\langle f_R, \mathcal{C}^+ \right\rangle d\bar{P}_N(y, \mathcal{C}), \quad (6.3.15)\end{aligned}$$

where the inequality is due to the possible loss of mass at the boundary. For any $M > 0$ we may write

$$\begin{aligned}\int \chi(y) \left\langle f_R, \mathcal{C}^+ \right\rangle d\bar{P}_N(y, \mathcal{C}) &\geq \int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}_N(y, \mathcal{C}) \\ &\underset{N\to\infty}{\longrightarrow} \int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}(y, \mathcal{C})\end{aligned}$$

and we have, by definition of f_R

$$\int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}(y, \mathcal{C}) \geq \frac{1}{R^2} \int \chi(y) \left(\mathcal{N}(0, R-1)(\mathcal{C}^+) \wedge M \right) d\bar{P}(y, \mathcal{C}).$$

By definition of the intensity it holds that

$$\lim_{R\to\infty} \lim_{M\to\infty} \frac{1}{R^2} \int \chi(y) \left(\mathcal{N}(0, R-1)(\mathcal{C}^+) \wedge M \right) d\bar{P}(y, \mathcal{C}) = \int \chi(y) \rho_{\bar{P}}^+(y).$$

Moreover, since μ_N^+ converges to μ^+ we have

$$\lim_{N\to\infty} \int \chi * f_{R,N} d\mu_N^+ = \int \chi d\mu^+ + o_R(1).$$

Finally, sending $R \to \infty, M \to \infty, N \to \infty$ we get

$$\int \chi d\mu^+ \geq \int \chi(y) \rho_{\bar{P}}^+(y)$$

for any non-negative continuous test function χ, which proves $\rho_{\bar{P}}^+ \leq \mu^+$.

We next prove the equality under the additional assumption that μ^+ does not charge $\partial\Lambda$ and that that $x \mapsto \mu_N^+ \left(B(x, \frac{R}{\sqrt{N}}) \right)$ is uniformly integrable on Λ as $N \to \infty$. First, the difference between the last two terms in (6.3.15) is bounded as follows:

$$\begin{aligned}&\int \chi(y) \sum_{i=1}^N f_R(\sqrt{N}(x_i - y)) dy \\ &\qquad \leq \int \chi(y) \left\langle f_R, \mathcal{C}^+ \right\rangle d\bar{P}_N(y, \mathcal{C}) + ||\chi||_\infty \mu_N^+ \left(\{x \in \Lambda, \operatorname{dist}(x, \partial\Lambda) \leq 2\frac{R}{\sqrt{N}}\} \right).\end{aligned}$$

Since μ_N^+ converges to μ^+ which does not charge the boundary, the error term satisfies

$$||\chi||_\infty \mu^+ \left(\{x \in \Lambda, \operatorname{dist}(x, \partial\Lambda) \leq 2\frac{R}{\sqrt{N}}\} \right) = o(1)$$

as $N \to \infty$, for any χ and R fixed.

Moreover, the uniform integrability assumption implies that $\mathcal{C} \mapsto \langle f_R, \mathcal{C}^+ \rangle$ is uniformly integrable against $d\bar{P}_N$ as $N \to \infty$ and we may for any $\delta > 0$ choose M large enough such that

$$\int \chi(y) \left\langle f_R, \mathcal{C}^+ \right\rangle d\bar{P}_N(y, \mathcal{C}) \leq \int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}_N(y, \mathcal{C}) + \delta$$

uniformly in N. Arguing as above we see that

$$\lim_{R\to\infty} \lim_{N\to\infty} \int \chi(y) \left(\left\langle f_R, \mathcal{C}^+ \right\rangle \wedge M \right) d\bar{P}_N(y, \mathcal{C}) \leq \int \chi(y) \rho^+_{\bar{P}}(y) \leq \int \chi(y) \rho^+_{\bar{P}}(y).$$

Eventually we get $\mu^+ \leq \rho^+_{\bar{P}} + \delta$ and we conclude by letting $\delta \to 0$. □

d. Total intensity of the limit random point process

From the previous lemmas we deduce that in the LDP we may restrict ourselves to random point processes with total intensity 1.

Lemma 6.3.8. *Let $\bar{P}$ be the law of a stationary tagged signed point process such that the intensity of positive charges satisfies $\int_\Lambda \rho^+_{\bar{P}} < 1$. Then we have*

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \overline{\mathfrak{P}}^\beta_N(B(\bar{P}, \varepsilon)) = -\infty. \tag{6.3.16}$$

Proof. Assume that (6.3.16) does not hold and that we have for some T

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \overline{\mathfrak{P}}^\beta_N(B(\bar{P}, \varepsilon)) \geq -T.$$

Using the relative compactness of $i_N(\Lambda^{2N})$ we may find a sequence $\{\vec{X}_N\}_N$ of points in Λ^{2N} such that $i_N(\vec{X}_N)$ converges to some $\bar{Q} \in B(\bar{P}, \varepsilon)$. Up to extraction we may also assume that μ^+_N converges to $\mu^+ \in \mathcal{P}(\Lambda)$ and the point (6.3.7) of Lemma 6.3.5 ensures that we may assume that μ^+ has finite entropy, hence does not charge the boundary $\partial\Lambda$. Then, using Lemma 6.3.6 and Lemma 6.3.7 we obtain that $\rho^+_{\bar{Q}} = \mu^+$ and in particular $\rho^+_{\bar{Q}}$ has total mass 1. Thus in any ball $B(\bar{P}, \varepsilon)$ we may find a random tagged point process $\bar{Q}_\varepsilon$ such that $\rho^+_{\bar{Q}_\varepsilon}$ has total mass 1. Moreover, again by Lemma 6.3.6, we may assume that the number of points in any disk is uniformly integrable under $\bar{Q}_\varepsilon$ as $\varepsilon \to 0$. Passing to the limit $\varepsilon \to 0$, it implies that $\rho^+_{\bar{P}}$ has total mass 1, which yields a contradiction. □

6.3.4 Conclusion

We now show how Theorems 16 and 17 follow once we have proven the following lower and upper bounds:

Proposition 6.3.9. *Let $\bar{P} \in \mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$. We have*

$$\lim_{\varepsilon\to 0} \liminf_{N\to\infty} \frac{1}{N} \log \int_{i_N^{-1}(B(P,\varepsilon))} e^{-\frac{\beta}{2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 + \sum_{i=1}^N \log r(x'_i) + \log r(y'_i)\right)} d\vec{X}_N d\vec{Y}_N \geq -\overline{\mathcal{F}}_\beta(\bar{P}).$$

Proposition 6.3.10. *Let $\bar{P} \in \mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$. We have*

$$\lim_{\varepsilon\to 0} \limsup_{N\to\infty} \frac{1}{N} \log \int_{i_N^{-1}(B(P,\varepsilon))} e^{\left(-\frac{\beta}{2}\left(\frac{1}{2\pi}\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 + \sum_{i=1}^N \log r(x'_i) + \log r(y'_i)\right)\right)} d\vec{X}_N d\vec{Y}_N \leq -\overline{\mathcal{F}}_\beta(\bar{P}).$$

a. Proof of Theorem 16

Since we have exponential tightness, the proof of Theorem 16 reduces to proving a weak LDP. Thanks to Lemmas 6.2.14 and 6.3.8, the latter is easily deduced by combining Proposition 6.3.9 and Proposition 6.3.10. Indeed in view of (6.2.7) it only remains to show that $\log K_{N,\beta} = -\inf \overline{\mathcal{F}}_\beta + o(N)$, but combining the upper and lower bounds with the exponential tightness (and Lemma 6.3.8) we have

$$-\inf_{\mathcal{P}_{\mathrm{inv},1}} \overline{\mathcal{F}}_\beta \leq \liminf_{N\to\infty} \frac{1}{N} \log K_{N,\beta} \leq \limsup_{N\to\infty} \frac{1}{N} \log K_{N,\beta} \leq -\inf_{\mathcal{P}_{\mathrm{inv},1}} \overline{\mathcal{F}}_\beta. \tag{6.3.17}$$

Hence $\lim_{N\to\infty} \frac{1}{N} \log K_{N,\beta} = -\inf_{\mathcal{P}_{\mathrm{inv},1}} \overline{\mathcal{F}}_\beta$, which concludes the proof of Theorem 16.

We also get Corollary 6.1.3 from (6.3.17) and the fact that $\log K_{N,\beta} + \frac{\beta}{2} N \log N = \log Z_{N,\beta}$ as seen in Section 6.2.3.

b. Proof of Theorem 17

From Lemmas 6.2.12 and 6.2.14 we see that minimizers of $\overline{\mathcal{F}}_\beta$ are such that the intensity of both components of $\bar{P}^x$ are equal to 1 (for Lebesgue-a.e. $x \in \Lambda$). Thus the limit points of $\{\bar{P}_N\}_N$ have $\mathbb{P}_N^\beta$-a.s. both intensity measures equal to the uniform measure on Λ, and by Lemma 6.3.7 we see that any limit point of $\{\mu_N^+, \mu_N^-\}_N$ must be the uniform measure on Λ, almost surely under $\mathbb{P}_N^\beta$.

The rest of the paper is organized as follows: in Section 6.4 we give for the reader's convenience the proof of the main result of [GP77], in Section 6.5 we prove Proposition 6.3.9, and in Section 6.6 we prove Proposition 6.3.10.

6.4 The method of Gunson-Panta

In this section we recall the main steps of the analysis of Gunson-Panta as presented in [GP77] while keeping our notation when it is in conflict with that of [GP77]. In [GP77] the charges have absolute value $q > 0$, and for our concerns q should always be taken equal to 1.

Recall that the partition function is defined as

$$Z_{N,\beta} := \int_{\Lambda^{2N}} e^{-\frac{\beta}{2} w_N(\vec{X}_N, \vec{Y}_N)} d\vec{X}_N d\vec{Y}_N.$$

This is almost exactly what is denoted by Q_{2N}^* in [GP77, (2.2)], up to the fact that the domain of integration Λ is a square, in contrast to [GP77] where it is a disk. We have a factor $\beta/2$ but the definition of w_N counts each pairwise interaction twice, whereas in [GP77, (2.2)] the temperature factor is β but each pairwise interaction is counted only once.

In [GP77, (2.3)] an "electrostatic inequality" is used to bound below the interaction energy in terms of the quantity

$$\sum_{i=1}^{N} (\log r(x_i) + \log r(y_i)),$$

this is the same computation as in our Lemma 6.2.3. It yields the bound

$$Z_{N,\beta} \leq \int_{\Lambda^{2N}} e^{-\frac{\beta}{2} \sum_{i=1}^{N} (\log r(x_i) + \log r(y_i)} d\vec{X}_N d\vec{Y}_N, \tag{6.4.1}$$

as expressed in [GP77, (2.4)] (up to notation, and the fact that in the latter, a minus sign is missing in the exponential).

Henceforth the signs of the charge will not play any role. For any M-tuple of points $\vec{S}_M = (S_1, \dots, S_M)$, let us define the map $F : \{1, \dots, M\} \to \{1, \dots, M\}$ such that

$$|S_i - S_{F(i)}| = \min_{j \in \{1,\dots,M\}} |S_i - S_j|.$$

With this notation we may rewrite (6.4.1) as

$$Z_{N,\beta} \leq \int_{\Lambda^{2N}} e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log(\frac{1}{2}|S_i - S_{F_i}|)} d\vec{S}_{2N}. \tag{6.4.2}$$

To any M-tuple $\vec{S}_M$ we associate the (directed) graph $\hat{\gamma}(\vec{S}_M)$ of "nearest-neighbors", whose set of vertices is $\{1, \dots, M\}$ and such that there is a directed arrow from i to F_i for any $i \in \{1, \dots, M\}$. We observe the following

Lemma 6.4.1. *For any $\vec{S}_M$, the associated graph $\hat{\gamma}(\vec{S}_M)$ has between 1 and $M/2$ connected components. Each connected component is composed of a cycle of length 2, together with trees attached to the two vertices of the cycle.*

The graphs satisfying these properties are called "functional digraphs" (or "functional directed graphs") such that each connected component contains a cycle of order 2. For any even $M \geq 1$ and $1 \leq K \leq M/2$ let us denote by $\mathbf{D}_{M,K}$ the set of (isomorphism classes of) labeled functional digraphs with M vertices and K connected components, each possessing a cycle of order 2. A combinatorial computation (as the one leading to [GP77, (2.8)]) shows that

Lemma 6.4.2. *For any $M \geq 1$ and $1 \leq K \leq M/2$ the cardinality of $\mathbf{D}_{M,K}$ is bounded by*

$$|\mathbf{D}_{M,K}| \leq \frac{\Gamma(M+1) M^{M-2K}}{2^K \Gamma(K+1)\Gamma(M-2K+1)}.$$

If $\gamma \in \mathbf{D}_{M,K}$ is an isomorphism class, we denote by $\hat{\gamma}(\vec{S}_M) \equiv \gamma$ the event "$\hat{\gamma}(\vec{S}_M)$ is isomorphic to γ". We may then rewrite (6.4.2) by splitting the domain of integration according to the isomorphism class of $\hat{\gamma}(\vec{S}_{2N})$, this reads

$$Z_{N,\beta} \leq \sum_{K=1}^{N} \sum_{\gamma \in \mathbf{D}_{N,K}} \int_{\Lambda^{2N} \cap \{\hat{\gamma}(\vec{S}_{2N}) \equiv \gamma\}} e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log(\frac{1}{2}|S_i - S_{F_i}|)} d\vec{S}_{2N}.$$

Let $1 \leq K \leq N$ and $\gamma \in \mathbf{D}_{2N,K}$ be fixed, we turn to evaluating the quantity

$$\int_{\Lambda^{2N} \cap \{\hat{\gamma}(\vec{S}_{2N}) \equiv \gamma\}} e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log(\frac{1}{2}|S_i - S_{F_i}|)} d\vec{S}_{2N}. \tag{6.4.3}$$

Let $L_1, \dots, L_K$ be the K subsets of vertices associated to each connected component of the isomorphism class γ of graphs. For $k \in \{1, \dots, K\}$ we perform a change of variables on the variables S_i for $i \in L_k$. We denote by $c_k := \{i_k^a, i_k^b\}$ the two vertices on the cycle. We let for $i \in L_k$ such that $i \notin c_k$

$$u_i := \frac{1}{2}(S_i - S_{F_i}),$$

and we let

$$u_{i_k^a} := \frac{1}{2}(S_{i_k^a} - S_{i_k^b}), \quad u_{i_k^b} := \frac{1}{2} S_{i_k^b}.$$

With respect to the new variables, the integral in (6.4.3) is bounded by

$$4^{2N} \prod_{k=1}^{K} \int_{D_k} e^{-\frac{\beta}{2}\left(\sum_{i \in L_k \setminus c_k} \log u_i + 2\log u_{i_k^a}\right)} \prod_{i \notin L_k} du_i \; du_{i_k^a} \; du_{i_k^b}, \tag{6.4.4}$$

where D_k is a (suitably enlarged) domain of integration for the new variables. It may be observed that the new variables satisfy

$$\sum_{i\in L_k\setminus c_k} |u_i|^2 + |u_{i_k^a}|^2 + |u_{i_k^b}|^2 \leq C.$$

for a certain universal constant C, thus each integral term in (6.4.4) can be viewed as an integral over a simplex i.e. a multiple Dirichlet integral. Using classical results about such integrals following [GP77, Equation (2.9)] we have

Lemma 6.4.3. *For any integer $M \geq 1$ and $K \leq M/2$, $\gamma \in \mathbf{D}_{M,K}$, we have*

$$\prod_{k=1}^{K} \int_{D_k} e^{-\frac{\beta}{2}\left(\sum_{i\in L_k\setminus c_k} \log u_i + 2\log u_{i_k^a}\right)} \prod_{i\notin L_k} du_i \; du_{i_k^a} \; du_{i_k^b}$$
$$\leq Diri_{M,K} := \frac{\left(Y(1-\frac{\beta}{4})\right)^{M-2K}\left(X(1-\frac{\beta}{2})\right)^{K}}{\Gamma\left((M-K)-\frac{M}{2}\frac{\beta}{2}+1\right)}, \quad (6.4.5)$$

where X and Y are two functions independent of M and K and the bound (6.4.5) *depends only on M, K, β and not on the isomorphism class inside $\mathbf{D}_{M,K}$.*

With that lemma we deduce

$$Z_{N,\beta} \leq \sum_{K=1}^{N} |\mathbf{D}_{2N,K}| Diri_{2N,K}$$
$$\leq \sum_{K=1}^{N} \frac{\Gamma(2N+1)(2N)^{2N-2K}}{2^K\Gamma(K+1)\Gamma(2N-2K+1)} \frac{\left(Y(1-\frac{\beta}{4})\right)^{2N-2K}\left(X(1-\frac{\beta}{2})\right)^{K}}{\Gamma\left((2N-K)-N\frac{\beta}{2}+1\right)}. \quad (6.4.6)$$

The last step is the evaluation of the right-hand side in (6.4.6). From [GP77, Equation (2.9)],

$$Z_{N,\beta} \leq (2N)^{\beta N/2} \sum_{K=1}^{N} \frac{\Gamma(2N+1)\exp(2N)}{\Gamma(K+1)\Gamma(2N-K+1)} \left(Y(1-\frac{\beta}{4})\right)^{2N-K} \left(\frac{X(1-\frac{\beta}{2})}{Y(1-\frac{\beta}{4})}\right)^{K}, \quad (6.4.7)$$

which gives in turn, using Newton's formula

$$Z_{N,\beta} \leq N^{\beta N/2} C_\beta^N,$$

where C_β depends only on β. The value of C_β is not important and it is thus enough to prove (6.4.7) up to a multiplicative constant of order C^N. In particular it yields an upper bound on the partition function

$$\log Z_{N,\beta} \leq \frac{\beta}{2} N \log N + C_\beta N. \quad (6.4.8)$$

Passing from (6.4.6) to (6.4.7) (up to a multiplicative constant of order C^N) is simple after observing that the summands in (6.4.6) and (6.4.7) differ by a factor

$$\frac{(2N)^{2N-2K}\Gamma(2N-K+1)}{\Gamma((2N-K)-N\frac{\beta}{2}+1)\Gamma(2N-2K+1)}.$$

Using Stirling's estimate for the Gamma function we see that the logarithm of the previous expression is equal to

$$(2N-2K)\log N + (2N-K)\log N - (2N-K-N\frac{\beta}{2})\log N - (2N-2K)\log N + O(N).$$

After simplifying we see that the ratio of the two summands in (6.4.6) and (6.4.7) is bounded by $C_\beta^N N^{\frac{\beta N}{2}}$ for some constant C_β depending on β, whose precise value is not important here.

The thermodynamic limit for $\log Z_{N,\beta}$ (as expressed in our Proposition 6.1.3) is proved in [GP77, Sections 3 and 4] using an interesting "conjugation" trick. In this paper we only need to use an upper bound (as (6.4.8)) and more generally to follow the method of [GP77, Section 2] that we have just recalled. *A posteriori* our large deviation principle at scale N implies in particular that Proposition 6.1.1 holds.

6.5 Next order large deviations: lower bound

In this section, we use the blow-up coordinates as introduced in Section 6.2.3 and we prove the LDP lower bound announced in Proposition 6.3.9.

In the rest of this section $\bar{P}$ is a fixed stationary tagged signed point process in $\mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$ such that $\overline{\mathsf{ent}}[\bar{P}]$ is finite, otherwise there is nothing to prove.

6.5.1 Negative part of the energy

First we observe that the negative part of the energy is semi-continuous in the suitable direction.

Lemma 6.5.1. *For any sequence $\{(\vec{X}_N, \vec{Y}_N)\}_N$ such that $i_N(\vec{X}_N, \vec{Y}_N) \in B(\bar{P}, \varepsilon)$, we have*

$$\liminf_{N\to\infty} -\frac{1}{N}\sum_{i=1}^{N}\left(\log r(x_i') + \log r(y_i')\right) \geq \overline{\mathbb{W}}^*(\bar{P}) - o_\varepsilon(1) \quad \text{as } \varepsilon \to 0.$$

Proof. We fix a family $\{\chi_\tau\}_{\tau\in(0,1)}$ of non-negative bounded by 1 smooth functions such that $\chi_\tau \equiv 1$ on $C_{1-\tau}$, $\chi_\tau \equiv 0$ outside C_1, and such that for any $x \in \mathbb{R}^2$, $\chi_\tau(x)$ is nonincreasing with respect to τ. We also set $I_\tau := \int \chi_\tau$, and we have $I_\tau \to 1$ as $\tau \to 0$.

For any $M > 0$ we have

$$-\frac{1}{N}\sum_{i=1}^{N}\left(\log r(x_i') + \log r(y_i')\right) \geq -\frac{1}{N}\sum_{i=1}^{N}\left(\log(r(x_i')\vee\tau) + \log(r(y_i')\vee\tau)\right)$$
$$\geq -\int\left(\mathbb{W}_\tau^*(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) \wedge M\right) d\bar{P}_N.$$

The map $\mathcal{C} \mapsto \mathbb{W}_\tau^*(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) \wedge M$ is continuous for the topology we use, hence

$$-\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\left(\log r(x_i') + \log r(y_i')\right) \geq \int\left(\mathbb{W}_\tau^*(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) \wedge M\right) d\bar{P} - o_\varepsilon(1).$$

The monotone convergence theorem implies

$$\lim_{M\to\infty}\int\left(\mathbb{W}_\tau^*(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) \wedge M\right) d\bar{P} = \int \mathbb{W}_\tau^*(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) d\bar{P}.$$

Let us observe that $\mathbb{W}_\tau^*(\cdot,\cdot)$ is linear in the first variable, in particular $\mathbb{W}_\tau^*(\frac{1}{I_\tau}\chi_\tau, \mathcal{C}) = \frac{1}{I_\tau}\mathbb{W}^*(\chi_\tau, \mathcal{C})$. The family of functions $\{\mathbb{W}_\tau^*(\chi_\tau, \cdot)\}_{\tau\in(0,1)}$ is monotone in τ, and the monotone convergence theorem implies

$$\lim_{\tau\to 0}\int \frac{1}{I_\tau}\mathbb{W}_\tau^*(\chi_\tau, \mathcal{C}) d\bar{P} = \int \mathbb{W}^*(\mathbf{1}_{C_1}, \mathcal{C}) d\bar{P}.$$

By stationarity of $\bar{P}$ we have $\int \mathbb{W}^*(\mathbf{1}_{C_1}, \mathcal{C}) d\bar{P} = \overline{\mathbb{W}}^*(\bar{P})$. Chosing then τ small enough, M large enough and ε small enough depending on τ, M yields the result. □

To prove the LDP lower bound, it thus suffices to prove a lower bound for

$$\int_{\Lambda^{2N}\cap\{i_N(\vec{X}_N,\vec{Y}_N)\in B(\bar{P}^s,\varepsilon)\}} e^{-\frac{\beta}{2}\frac{1}{2\pi}\int_{\mathbb{R}^2}|\nabla V'_{N,r}|^2} d\vec{X}_N d\vec{Y}_N$$

(for notation see Section 6.2.3). This then becomes similar to the question treated in [LS15] and we follow the same strategy: we need to show that there is a large enough volume of configurations (in fact one which is logarithmically very close to the relative entropy of $\bar{P}^s$) on which the energy $\int_{\mathbb{R}^2}|\nabla V'_{N,r}|^2$ is not too large. For that we will split the box Λ into squares of microscopic size, and we will draw signed configurations independently at random in a Poissonian way in these squares, so that the total number of points in each square is the expected one. Sanov's theorem will guarantee that this generates a set of configurations with the right volume. Then we need to estimate the energy $\int_{\mathbb{R}^2}|\nabla V'_{N,r}|^2$ generated by each such configuration. We in fact need to make these energies restricted to each square depend only on the configuration in the square, which is a priori not the case. For that we use the idea of "screening" the electric field E_K generated in each square K, by modifying the configuration in a small neighborhood of the boundary of the square, to make the electric field energies independent and summable. This is accomplished by enforcing the boundary condition $E_K\cdot\vec{n}=0$ on each boundary, which ensures that when pasting together these electric fields, the relation

$$-\text{div } E = 2\pi\mathcal{C} \quad \text{in } \mathbb{R}^2 \tag{6.5.1}$$

holds globally. Indeed, a vector field which is discontinuous across an interface has a distributional divergence concentrated on the interface and equal to the jump of the normal derivative. With the choice (6.5.1), there is no extra divergence created across the interfaces between the squares. Even if the E_K's were gradients, the global E is in general no longer a gradient. This does not matter however, since the energy of the true electric field $\nabla V'_{N,r}$ generated by the configuration $\mathcal{C}$ will be shown to be smaller than that of E by Helmholtz projection. In the case of positive charges with a neutralizing background, this procedure was introduced in [SS12] and further refined in [SS15b, RS15, PS15, LS15], to which we refer for more detail. We implement this program, with the appropriate adaptations needed for controlling dipoles, in the rest of this section.

6.5.2 The screening lemma

Proposition 6.5.2. *Let $R>0$ and let $\mathcal{C}=(\mathcal{C}^+,\mathcal{C}^-)$ be a simple signed configuration in C_R, and E an electric field satisfying*

$$-\text{div } E = 2\pi(\mathcal{C}^+-\mathcal{C}^-) \quad \textit{in } C_R.$$

Let E_r be its truncation at nearest-neighbour half-distance as defined in (6.2.9).

Let $n^+:=\mathcal{C}^+(C_R)$, $n^-:=\mathcal{C}^-(C_R)$, and $n:=n^++n^-$, and for any $0<\varepsilon<1$ let

$$n^{+,\varepsilon}_{\text{int}}:=\mathcal{C}^+(C_{R(1-\varepsilon)}),\quad n^{-,\varepsilon}_{\text{int}}:=\mathcal{C}^-(C_{R(1-\varepsilon)}),\quad n^{\varepsilon}_{\text{int}}:=n^{+,\varepsilon}_{\text{int}}+n^{-,\varepsilon}_{\text{int}},$$

and define

$$M:=\frac{1}{R^2}\int_{C_R}|E_r|^2.$$

Then for any $0<\varepsilon<1$, if R is large enough, there exists a (measurable) family of signed configurations $\Phi^{\text{scr}}_{\varepsilon,R}(\mathcal{C},E)$ such that for any $\mathcal{C}^{\text{scr}}\in\Phi^{\text{scr}}_{\varepsilon,R}(\mathcal{C},E)$ we have

1. *$\mathcal{C}$ and $\mathcal{C}^{\text{scr}}$ coincide in $C_{R(1-\varepsilon)}$.*

2. There exists a vector field $E^{\rm scr}$ *satisfying*

(a) $E^{\rm scr}$ *is compatible with* $\mathcal{C}^{\rm scr}$ *in* C_R *and is screened in the sense that*

$$\begin{cases} -\mathrm{div}\ E^{\rm scr} = 2\pi\mathcal{C}^{\rm scr} & \text{in } C_R \\ E^{\rm scr}\cdot\vec{n} = 0 & \text{on } \partial C_R \end{cases}.$$

(b) The energy of $E^{\rm scr}$ *(after truncation) is controlled by that of* E

$$\int_{C_R}|E_r^{\rm scr}|^2 \le \int_{C_R}|E_r|^2 + C\left(\frac{MR}{\varepsilon}+\varepsilon R^2+\frac{n}{\varepsilon R}\left|\log\frac{n}{\varepsilon R}\right|\right).$$

3. We have

$$n_{\rm int}^{+,\varepsilon}(\mathcal{C}) \le n^+(\mathcal{C}^{\rm scr}) \le n^+(\mathcal{C}) + C\left(\frac{MR}{\varepsilon}+1+\frac{n}{\varepsilon R}\left|\log\frac{n}{\varepsilon R}\right|\right) \tag{6.5.2}$$

and the same holds for $n_{\rm int}^{-,\varepsilon}, n^-$.

Moreover the map $\mathcal{C}\mapsto\Phi^{\rm scr}_{\varepsilon,R}(\mathcal{C},E)$ *is such that if* A *is a (measurable) subset of signed point configurations in* $\mathcal{X}(C_R)$ *such that the quantities* n *and* $n^\varepsilon_{\rm int}$ *are constant on* A*, then we have*

$$\log\mathbf{Leb}^{\otimes 2R^2}\left(\bigcup_{\mathcal{C}\in A}\Phi^{\rm scr}_{\varepsilon,R}(\mathcal{C},E)\right) \ge \log\mathbf{Leb}^{\otimes n}(A) - C\left((n-n^\varepsilon_{\rm int})\log R + R + \frac{MR}{\varepsilon}+\varepsilon R^2+\frac{n}{\varepsilon R}\log\frac{n}{\varepsilon R}\right). \tag{6.5.3}$$

Proof. Step 1: choice of a good "annulus". Consider the disjoint "annuli" $A_k = C_{R-8(k+1)}\setminus C_{R-8k}$ for k ranging from 1 to the integer part of $\frac{1}{8}\varepsilon R$. There are $[\frac{1}{8}\varepsilon R]$ such disjoint sets. The proportion of such k's such that

$$\int_{A_k}|E_r|^2 \le \frac{20MR^2}{\varepsilon R}$$

is strictly larger than $\frac{1}{2}$. Similarly, the proportion of such k's such that

$$|\mathcal{C}|(A_k) \le \frac{20n}{\varepsilon R}$$

is strictly larger than $\frac{1}{2}$. We deduce that there exists a $k_0\in[1,[\varepsilon R]]$ such that

$$\int_{A_{k_0}}|E_r|^2 \le \frac{20MR}{\varepsilon} \qquad |\mathcal{C}|(A_{k_0}) \le \frac{20n}{\varepsilon R}. \tag{6.5.4}$$

For brevity, we set $A := A_{k_0}$ and $A_{\rm int} = \{x\in A, \mathrm{dist}(x,\partial A)\ge 2\}$.

Set next $\eta = \frac{\varepsilon R}{40n}$ and

$$E_{r,\eta} := E_r - \sum_{p\in\mathcal{C}^+\cap A_{\rm int}}\nabla(f_{\eta\wedge r(p)}-f_{r(p)})(x-p) + \sum_{p\in\mathcal{C}^-\cap A_{\rm int}}\nabla(f_{\eta\wedge r(p)}-f_{r(p)})(x-p).$$

In other words, we replace $\delta_p^{(r(p))}$ by $\delta_p^{(\eta\wedge r(p))}$ for all the points p in $A_{\rm int}$.

Computing as in [PS15, Lemma 2.3] we may write

$$\int_A|E_{r,\eta}|^2 = \int_A|E_r|^2+|E_{r,\eta}-E_r|^2 + 2\int_A\sum_{p\in\mathcal{C}^+\cap A_{\rm int}}\nabla(f_{\eta\wedge r(p)}-f_{r(p)})(x-p)\cdot E_r + \sum_{p\in\mathcal{C}^-\cap A_{\rm int}}\nabla(f_{\eta\wedge r(p)}-f_{r(p)})(x-p)\cdot E_r$$

Integrating by parts, and using that $-\mathrm{div}\, E_r = 2\pi(\sum_{x\in\mathcal{C}^+}\delta_x^{(r(x))} - \sum_{x\in\mathcal{C}^-}\delta_x^{(r(x))})$, we have

$$\begin{aligned}\int_A \sum_{p\in\mathcal{C}^+\cap A_{\rm int}} \nabla(f_{\eta\wedge r(p)} - f_{r(p)})(x-p)\cdot E_r + \sum_{p\in\mathcal{C}^-\cap A_{\rm int}} \nabla(f_{\eta\wedge r(p)} - f_{r(p)})(x-p)\cdot E_r \\ = 2\pi \sum_{p\in\mathcal{C}^+\cap A_{\rm int}} \int_A (f_{\eta\wedge r(p)} - f_{r(p)})(x-p)\left(\sum_{x\in\mathcal{C}^+}\delta_x^{(r(x))} - \sum_{x\in\mathcal{C}^-}\delta_x^{(r(x))}\right) \\ + 2\pi \sum_{p\in\mathcal{C}^-\cap A_{\rm int}} \int_A (f_{\eta\wedge r(p)} - f_{r(p)})(x-p)\left(\sum_{x\in\mathcal{C}^+}\delta_x^{(r(x))} - \sum_{x\in\mathcal{C}^-}\delta_x^{(r(x))}\right) \quad (6.5.5)\end{aligned}$$

and there are no boundary terms since f_r and $f_{\eta\wedge r}$ vanish outside of $B(p, r(p))$ with $r(p) \le 1$. By the same argument, and since all the balls of radius $r(p)$ are disjoint, all the terms in the right-hand side of (6.5.5) vanish. We are thus left with

$$\int_A |E_{r,\eta}|^2 \;=\; \int_A |E_r|^2 + |E_{r,\eta} - E_r|^2 \;=\; \int_A |E_r|^2 + \sum_{p\in\mathcal{C}\cap A_{\rm int}} \int_{\mathbb{R}^2} |\nabla(f_{\eta\wedge r(p)} - f_{r(p)})|^2.$$

and using again the same integration by parts argument, we have

$$\int_{\mathbb{R}^2} |\nabla(f_{\eta\wedge r(p)} - f_{r(p)})|^2 = 2\pi(\log r(p) - \log(\eta\wedge r(p)) \le 2\pi|\log\eta|.$$

It follows with (6.5.4) that

$$\int_A |E_{r,\eta}|^2 \le \int_A |E_r|^2 + \frac{40\pi n}{\varepsilon R}\left|\log\frac{\varepsilon R}{40n}\right|. \quad (6.5.6)$$

Step 2: choice of a good boundary. Consider all the ∂C_t where t is chosen so that

$$\partial C_t \subset \{x \in A_{\rm int}, \mathrm{dist}(x, \partial A_{\rm int}) \ge 1\}.$$

By the bound $r(x) \le 1$, ∂K_t cannot intersect any $B(p, r(p))$ for $p \notin A_{\rm int}$. Moreover, by the choice of η and (6.5.4), we have $\eta|\mathcal{C}|(A) \le \frac{1}{2}$. Thus, the total perimeter of the balls $B(p, \eta\wedge r(p))$ with $p \in \mathcal{C}\cap A_{\rm int}$ is bounded by $\frac{1}{2}$. We deduce that there exists t such that $\partial C_t \subset A_{\rm int}$ and ∂C_t intersects none of the $B(p,\eta)$ for $p \in \mathcal{C}\cap A_{\rm int}$ and none of the $B(p, r(p))$ for $p \in \mathcal{C}\backslash A_{\rm int}$. Applying a mean value argument to the integrand in (6.5.6), and using (6.5.4), we may also assume that t is such that the restriction (or trace) of $E_{r,\eta}$ on ∂C_t is well defined as an L^2 function, and denoting

$$g := (E_{r,\eta}\cdot\vec{n})\lfloor_{\partial C_t}$$

where $\vec{n}$ denotes the inner unit normal, we have

$$\int_{\partial C_t} |g|^2 \le 10\left(\int_A |E_r|^2 + \frac{40\pi n}{\varepsilon R}\left|\log\frac{\varepsilon R}{40n}\right|\right) \le 10\left(\frac{20MR}{\varepsilon} + \frac{40\pi n}{\varepsilon R}\left|\log\frac{\varepsilon R}{40n}\right|\right). \quad (6.5.7)$$

Step 3: construction outside C_t

We take the C_t given by the previous step and keep the configuration in C_t unchanged, and discard the configuration in $C_R\backslash C_t$. This way the first item will be verified.

Consider next ∂C_t and partition each of its sides into segments I_l of length $\in [\frac{1}{2}, \frac{3}{2}]$. This yields a natural partition of $C_{t+1}\backslash C_t$ into disjoint rectangles $\mathcal{R}_l$, $l = 1, \cdots, L$. Each $\partial\mathcal{R}_l$ has four sides: one is I_l (or does not exist if $\mathcal{R}_l$ is a corner square), one (or two in case of a corner) belongs to ∂C_{t+1}, one is adjacent to $\partial\mathcal{R}_{l-1}$, one to $\mathcal{R}_{l+1}$.

For each l, we let g_l denote the restriction of g to I_l. We also define $n_0 = 0$ and for each $l \in [1, L]$,

$$n_l = \left[\sum_{k=1}^{l} \int g_k\right] - \sum_{k=0}^{l-1} n_k, \qquad c_l = \sum_{k=1}^{l} \int g_k - \left[\sum_{k=1}^{l} \int g_k\right]$$

where $[\cdot]$ is the integer part. We observe that

$$|c_l| \leq 1, \qquad \sum_{k=1}^{l} n_k = \left[\sum_{k=1}^{l} \int g_k\right]$$

and

$$n_l = c_l - c_{l-1} - \int g_l. \tag{6.5.8}$$

In each $\mathcal{R}_l$ we let Λ_l be a set of $|n_l|$ points of sign equal to that of n_l, and which are a perturbation of a fixed regular square lattice Λ_l^0 of sidelength $1/\sqrt{|n_l|}$. More formally, to each $z_i \in \Lambda_l^0$ which is at distance $\geq 1/\sqrt{|n_l|}$ from $\partial\mathcal{R}_l$, we associate a point x_i satisfying $|x_i - z_i| \leq 1/(4\sqrt{|n_l|})$, and set $\Lambda_l = \{x_i\}_{i=1}^{|n_l|}$. We let h_l be the mean zero solution to

$$\begin{cases} -\Delta h_l = 2\pi sgn(n_l) \sum_{p\in\Lambda_l} \delta_p & \text{in } \mathcal{R}_l \\ \nabla h_l \cdot \vec{n} = g_l & \text{on } \partial\mathcal{R}_l \cap I_l \\ \nabla h_l \cdot \vec{n} = -c_{l-1} & \text{on } \partial\mathcal{R}_l \cap \partial\mathcal{R}_{l-1} \\ \nabla h_l \cdot \vec{n} = c_l & \text{on } \partial\mathcal{R}_l \cap \partial\mathcal{R}_{l+1} \\ \nabla h_l \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l \cap \partial C_{t+1}. \end{cases}$$

One may check that this equation is solvable, and has a unique solution with mean zero, in view of (6.5.8). We may also write $h_l = u_l + v_l$ where

$$\begin{cases} -\Delta u_l = 2\pi \frac{n_l}{|\mathcal{R}_l|} & \text{in } \mathcal{R}_l \\ \nabla u_l \cdot \vec{n} = g_l & \text{on } \partial\mathcal{R}_l \cap I_l \\ \nabla u_l \cdot \vec{n} = -c_{l-1} & \text{on } \partial\mathcal{R}_l \cap \partial\mathcal{R}_{l-1} \\ \nabla u_l \cdot \vec{n} = c_l & \text{on } \partial\mathcal{R}_l \cap \partial\mathcal{R}_{l+1} \\ \nabla u_l \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l \cap \partial C_{t+1}. \end{cases}$$

and

$$\begin{cases} -\Delta v_l = 2\pi sgn(n_l) \sum_{p\in\Lambda_l} \delta_p - 2\pi \frac{n_l}{|\mathcal{R}_l|} & \text{in } \mathcal{R}_l \\ \nabla v_l \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l. \end{cases}$$

Both equations have a unique solution with zero average. Since the points in Λ_l are well-separated from the boundary, using the same notation as in (6.2.9), we have

$$\begin{cases} -\text{div } (\nabla v_l)_r = 2\pi sgn(n_l) \sum_{p\in\Lambda_l} \delta_p^{(r(p))} - 2\pi \frac{n_l}{|\mathcal{R}_l|} & \text{in } \mathcal{R}_l \\ (\nabla v_l)_r \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l, \end{cases}$$

and we may write $(\nabla v_l)_r = \sum_{p\in\Lambda_l} \nabla G_p$ with

$$\begin{cases} -\Delta G_p = 2\pi sgn(n_l) \left(\delta_p^{(r(p))} - \frac{1}{|\mathcal{R}_l|}\right) & \text{in } \mathcal{R}_l \\ (\nabla G_p) \cdot \vec{n} = 0 & \text{on } \partial\mathcal{R}_l. \end{cases}$$

We have that for each $p \in \Lambda_l$,

$$\int_{\mathcal{R}_l} |\nabla G_p|^2 \leq C(|\log r(p)| + 1).$$

This can be proved by comparing G_p to $f_{r(p)}(x-p)$ (defined in (6.2.2)), for example as in [PS15, proof of (6.23)]. Using that the separation of the points is at least of order $1/\sqrt{|n_l|}$, we may then write

$$\int_{\mathcal{R}_l} |(\nabla v_l)_r|^2 \leq C|n_l| \log |n_l| + \sum_{p \neq p' \in \Lambda_l} \nabla G_p \cdot \nabla G_{p'},$$

and by arrangement of the points near a lattice, the second sum is seen to be comparable to $-\sum_{p \neq p' \in \Lambda_l} \log |p - p'|$. This term is itself known to be asymptotic as $|n_l| \to \infty$ to $-(n_l)^2 \int_{\mathcal{R}_l \times \mathcal{R}_l} \log |x-y| \, d\mu(x) \, d\mu(y)$ where μ is the limit of $\frac{1}{|n_l|} \sum_{p \in \Lambda_l} \delta_p$ (here the uniform measure on $\mathcal{R}_l$). We thus conclude that whether $|n_l| \to \infty$ or not, we have

$$\int_{\mathcal{R}_l} |(\nabla v_l)_r|^2 \leq C(n_l)^2. \tag{6.5.9}$$

On the other hand, by elliptic estimates (for example [RS15, Lemma 5.8]), we have

$$\int_{\mathcal{R}_l} |\nabla u_l|^2 \leq C \left(\int g_l^2 + c_{l-1}^2 + c_l^2 \right) \tag{6.5.10}$$

with C universal. Since $|c_l| \leq 1$ for every l, in view of (6.5.8) we have $(n_l)^2 \leq 3 + 2 \int g_l^2$ and thus, combining (6.5.9) and (6.5.10), we find

$$\int_{\mathcal{R}_l} |(\nabla h_l)_r|^2 \leq C \left(1 + \int g_l^2 \right). \tag{6.5.11}$$

We now define E^{scr} to be E in C_t and to be $\sum_l \mathbf{1}_{\mathcal{R}_l} \nabla h_l$ in $C_{t+1} \backslash C_t$, and $\mathcal{C}^{\mathrm{scr}}$ to be

$$\mathcal{C}^{\mathrm{scr}} := (\mathcal{C} \cap C_t) \cup (\cup_l \Lambda_l)$$

(with sign). We see that the normal components of E^{scr} agree on each interface of $\partial \mathcal{R}_l$, and thus

$$\begin{cases} -\mathrm{div}\ E^{\mathrm{scr}} = 2\pi \mathcal{C}^{\mathrm{scr}} & \text{in } \mathcal{R}_l \\ E^{\mathrm{scr}} \cdot \vec{n} = 0 & \text{on } \partial C_{t+1}. \end{cases}$$

Also, in view of (6.5.11) we have

$$\int_{C_t \backslash C_{t-1}} |E_r^{\mathrm{scr}}|^2 \leq C \left(1 + \int g^2 \right).$$

We conclude with (6.5.7) that

$$\int_{C_t} |E_r^{\mathrm{scr}}|^2 \leq \int_{C_R} |E_r|^2 + C \left(\frac{MR}{\varepsilon} + 1 + \frac{n}{\varepsilon R} \left| \log \frac{\varepsilon R}{n} \right| \right).$$

Next, we extend (if needed) the configuration to $C_R \backslash C_{t+1}$ by just adding squares with dipoles. More precisely, we partition $C_R \backslash C_{t+1}$ into rectangles $\mathcal{R}$ of sidelengths in $[1, 2]$. In each rectangle we place a positive charge p_+ and a negative charge p_- separated from each other and from the boundary of the rectangle by at least $1/4$. We then solve for

$$\begin{cases} -\Delta u = 2\pi(\delta_{p_+} - \delta_{p_-}) & \text{in } \mathcal{R} \\ \nabla u \cdot \vec{n} = 0 & \text{on } \partial \mathcal{R}. \end{cases}$$

We check as above that $\int_{\mathcal{R}} |(\nabla u)_r|^2 \leq C$ for each such rectangle, and pasting together the electric fields $(\nabla u)_r$ thus constructed and the one constructed in C_{t+1}, we find a vector field and a family of configurations satisfying all the desired conditions, and we see that this extension has added an energy at most proportional to the volume of $C_R \backslash C_{t+1}$, i.e. $C\varepsilon R^2$.

Step 4: control on the number of points

By construction, the point configuration has not been changed in $C_{R(1-\varepsilon)}$, hence the left-hand inequality in (6.5.2) holds. The number of points that have been added is given by $\sum_{l=1}^{L} n_l$. In view of (6.5.8) and (6.5.7) we obtain

$$\sum_{l=1}^{L} n_l \leq \sum_{l=1}^{L} n_l^2 \leq C\left(R + \frac{MR}{\varepsilon} + \frac{n}{\varepsilon R}\left|\log \frac{\varepsilon R}{n}\right|\right),$$

which yields the right-hand inequality in (6.5.2).

Step 5: volume estimate

We now turn to the proof of (6.5.3). Since we have discarded the point configuration in $C_R \backslash C_t$, in which there were at most $n - n_{\text{int}}^{\varepsilon}$ points, we have lost a logarithmic volume bounded (in absolute value) by

$$(n - n_{\text{int}}^{\varepsilon}) \log |C_R \backslash K_t|. \tag{6.5.12}$$

On the other hand, the points that are constructed in each rectangle $\mathcal{R}_l$ were allowed to move independently in a small perturbation of the lattice of sidelength $1/\sqrt{|n_l|}$, e.g. they may be chosen arbitrarily in a disk of radius $\frac{1}{4\sqrt{|n_l|}}$ up to a multiplicative constant in the estimates. This allows us to create a volume of configurations of order

$$\left(\frac{1}{4\sqrt{|n_l|}}\right)^{2|n_l|}$$

in each rectangle $\mathcal{R}_l$. Summing over l, we see that the (absolute value of the) logarithmic volume contribution of the points that are created is bounded by

$$\sum_{l=1}^{L} C|n_l| \log |n_l| \leq C \sum_{l=1}^{L} n_l^2 \leq C\left(L + \int g^2\right)$$

in view of (6.5.8). We have $L \leq R$ and $\int g^2$ is bounded as in (6.5.7), which allows us to bound the previous expression by

$$C\left(R + \frac{MR}{\varepsilon} + \frac{n}{\varepsilon R} \log \frac{n}{\varepsilon R}\right). \tag{6.5.13}$$

Combining (6.5.12) and (6.5.13) yields (6.5.3). □

6.5.3 Screening the best electric field

For any $R > 0$, for any $\mathcal{C} \in \mathcal{X}(C_R)$, we let $\mathcal{O}_R(\mathcal{C})$ be the set of electric fields which are compatible with $\mathcal{C}$ in C_R, i.e. such that $-\text{div } E = 2\pi\mathcal{C}$ in C_R.

We may then define $F_R(\mathcal{C})$ to be the "best energy" associated to $\mathcal{C}$ in C_R, i.e.

$$F_R(\mathcal{C}) := \min\left\{\frac{1}{R^2}\int_{C_R} |E_r|^2, E \in \mathcal{O}_R(\mathcal{C})\right\}. \tag{6.5.14}$$

Since we may always consider the local electric field associated to $\mathcal{C}$, the set $\mathcal{O}_R(\mathcal{C})$ is non empty. If $\{E^{(k)}\}_k$ is a sequence in $\mathcal{O}_R(\mathcal{C})$ such that $\frac{1}{R^2}\int_{C_R}|E_r|^2$ is bounded, then using Lemma 6.2.4 we see that $\{E^{(k)}\}_k$ converges weakly (up to a subsequencen extraction) to some E in $L^p_{\mathrm{loc}}(C_R,\mathbb{R}^2)$. Moreover we have $E\in\mathcal{O}_R(\mathcal{C})$ and the sequence $\{E_r^{(k)}\}_k$ converges weakly to E_r in L^2. By the weak lower semi-continuity of the L^2 norm we obtain

$$\int_{C_R}|E_r|^2\leq\liminf_{k\to\infty}\int_{C_R}|E_r^{(k)}|^2.$$

This ensures that the minimum in (6.5.14) exists.

We next let $\Gamma_R:\mathcal{X}(C_R)\to\mathcal{A}$ be a measurable choice of an optimal electric field, i.e. be such that $\Gamma_R(\mathcal{C})\in\mathcal{O}_R(\mathcal{C})$ and

$$\frac{1}{R^2}\int_{C_R}|\Gamma_{R,r}|^2=F_R(\mathcal{C}).$$

Let us also make the following observation: if $\bar{P}$ is a stationary tagged signed point process, we have, for any $R>0$

$$\mathbf{E}_{\bar{P}}\left[F_R(\mathcal{C})\right]\leq\overline{\mathbb{W}}^o(\bar{P}). \tag{6.5.15}$$

Indeed from Lemma 6.2.7 we know that we may find a stationary electric field process $\bar{P}^{\mathrm{elec}}$ which is compatible with $\bar{P}$ and such that

$$\mathbf{E}_{\bar{P}^{\mathrm{elec}}}\left[\left(\frac{1}{R^2}\int_{C_R}|E_r|^2\right)\right]=\overline{\mathbb{W}}^o(\bar{P}),$$

and the left-hand side is by definition $\geq\int F_R(\mathcal{C})d\bar{P}$.

Lemma 6.5.3. *The map F_R is upper semi-continuous on $\mathcal{X}(C_R)$.*

Proof. The proof goes as in [LS15, Lemma 5.8]. First, if $\mathcal{C}_1$ is a signed point configuration in C_R and $\mathcal{C}_2$ is close to $\mathcal{C}_1$, then they have the same number of points (for both components). We may then evaluate the energy of the "Neumann" electric field $\tilde{E}$ associated to $\mathcal{C}_1-\mathcal{C}_2$ (i.e. the solution to $-\mathrm{div}\,\tilde{E}=2\pi(\mathcal{C}_1-\mathcal{C}_2)$ with zero mean and vanishing Neumann boundary conditions and see that it can be made arbitrarily small if $\mathcal{C}_2$ is close enough to $\mathcal{C}_1$. By adding $\tilde{E}$ to a properly an electric field in $\mathcal{O}_R(\mathcal{C}_1)$ of almost minimal energy we may construct an element of $\mathcal{O}_R(\mathcal{C}_2)$ whose energy is bounded above by $F_R(\mathcal{C}_1)+o(1)$ as $d_{\mathcal{X}}(\mathcal{C}_2,\mathcal{C}_1)$ goes to zero. □

Henceforth for any $R>0$ and any $\varepsilon\in(0,1)$, if $\mathcal{C}$ is a signed point configuration in C_R we let $\Phi^{\mathrm{scr}}_{R,\varepsilon}(\mathcal{C})$ be the set of point configurations obtained by applying Proposition 6.5.2 to $\mathcal{C}$ and $\Gamma_R(\mathcal{C})$, in other words we let (with a slight abuse of notation)

$$\Phi^{\mathrm{scr}}_{R,\varepsilon}(\mathcal{C}):=\Phi^{\mathrm{scr}}_{R,\varepsilon}(\mathcal{C},\Gamma_R(\mathcal{C})).$$

To any configuration in $\Phi^{\mathrm{scr}}_{R,\varepsilon}(\mathcal{C})$ is associated a compatible electric field E^{scr} whose energy is bounded in terms of $F_R(\mathcal{C})$ according to the conclusions of Proposition 6.5.2

$$\int_{C_R}|E_r^{\mathrm{scr}}|^2\leq F_R(\mathcal{C})+C\left(\frac{R}{\varepsilon}F_R(\mathcal{C})+\varepsilon R^2+\frac{n}{\varepsilon R}\log\frac{n}{\varepsilon R}\right). \tag{6.5.16}$$

6.5.4 Construction of configurations

We now turn to the construction of configurations by cutting the domain into microscopic squares as announced.

For any $N\geq1$ we let $\Lambda':=\sqrt{N}\Lambda$.

a. Tiling the domain

For any integer $N \geq 1$ and any $R > 0$, we let

$$\bar{R} := R(1 + (\log R)^{-1/10}).$$

If R is such that $\sqrt{N}/\bar{R}$ is an integer, we let for convenience $m_{N,R} := N/\bar{R}^2$. We let $\mathcal{K}_{N,R} := \{\bar{K}_i\}_{i=1,\dots,m_{N,R}}$ be a collection of closed squares with disjoint interior which tile Λ' by translated copies of $C_{\bar{R}}$. For any i we denote by z_i the center of $\bar{K}_i$, and we let K_i be the square of center z_i and sidelength R (and whose sides are parallel to those of $\bar{K}_i$). Finally for any $\varepsilon \in (0,1)$ we let $K_{i,\varepsilon}$ be the square of center z_i and sidelength $R(1-\varepsilon)$. In particular we have $K_{i,\varepsilon} \subset K_i \subset \bar{K}_i$.

b. Generating approximating microstates

In the following lemma we show how to generate configurations with enough phase-space volume and ressembling any given $\bar{P}$.

Lemma 6.5.4. *Let $\left((\mathbf{C}_1^+, \mathbf{C}_1^-), \dots, (\mathbf{C}_{m_{N,R}}^+, \mathbf{C}_{m_{N,R}}^-)\right)$ be $m_{N,R}$ independent random variables such that $(\mathbf{C}_i^+, \mathbf{C}_i^-)$ is distributed as $\mathbf{\Pi}^s_{|K_i}$, in other words $\mathbf{C}_i^+$ and $\mathbf{C}_i^-$ are the restriction to K_i of a couple of independent Poisson point processes of intensity 1. We condition this $m_{N,R}$-tuple of random variables so that the total number of points of each sign is about $N\left(\frac{R}{\bar{R}}\right)^2$, more precisely*

$$\sum_{i=1}^{m_{N,R}} \mathbf{C}_i^+ = \sum_{i=1}^{m_{N,R}} \mathbf{C}_i^- = \lceil N\left(\frac{R}{\bar{R}}\right)^2 \rceil. \tag{6.5.17}$$

We define $\dot{\mathfrak{M}}_{N,R}$ as the law of the following random variable in $\Lambda \times \mathcal{X}$:

$$\frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/2} z_i, \theta_{z_i} \cdot (\mathbf{C}_i^+, \mathbf{C}_i^-))}.$$

Moreover let $\mathbf{C}$ be the signed point process obtained as the union of the signed point processes $(\mathbf{C}_i^+, \mathbf{C}_i^-)$ i.e.

$$\mathbf{C} := \left(\sum_{i=1}^{m_{N,R}} \mathbf{C}_i^+, \sum_{i=1}^{m_{N,R}} \mathbf{C}_i^- \right)$$

and define $\widehat{\mathfrak{M}}_{N,R}$ as the law of the random variable in $\Lambda \times \mathcal{X}^0$

$$\frac{1}{N} \int_{\Lambda'} \delta_{(N^{-1/2} z, \theta_z \cdot \mathbf{C})} dz.$$

Then for any $\bar{P} \in \mathcal{P}_{s,1}(\Lambda \times \mathcal{X})$ the following inequality holds

$$\liminf_{R\to\infty} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log \dot{\mathfrak{M}}_{N,R}\left(B(\bar{P}, \nu)\right) \geq -\overline{\mathsf{ent}}[\bar{P}], \tag{6.5.18}$$

moreover for any $\delta > 0$ we have

$$\liminf_{R\to\infty} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log \left(\dot{\mathfrak{M}}_{N,R}, \widehat{\mathfrak{M}}_{N,R}\right)\left(B(\bar{P}, \nu) \times B(\bar{P}, \delta)\right) \geq -\overline{\mathsf{ent}}[\bar{P}], \tag{6.5.19}$$

where $\left(\dot{\mathfrak{M}}_{N,R}, \widehat{\mathfrak{M}}_{N,R}\right)$ denotes the joint law of $\dot{\mathfrak{M}}_{N,R}$ and $\widehat{\mathfrak{M}}_{N,R}$ with the natural coupling.

Proof. First let us forget about the condition on the number of points (i.e. we consider independent Poisson point processes) and about the tags (i.e. let us replace $\bar{P}$ by a signed point process P in $\mathcal{P}_{\mathrm{inv}}(\mathcal{X})$). Then for any fixed R there holds a LDP for $\dot{\mathfrak{M}}_{N,R}$ at speed $m_{N,R}$ with rate function $\mathrm{Ent}[\cdot|\mathbf{\Pi}^s_R]$. This is a consequence of the classical Sanov theorem (see [DZ10, Section 6.2]) because the random variables $\theta_{z_i} \cdot (\mathbf{C}_i^+, \mathbf{C}_i^-)$ are i.i.d. Taking the limit $R \to \infty$ yields

$$\lim_{R\to\infty} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log \dot{\mathfrak{M}}_{N,R}\left(B(P,\nu)\right) \geq -\mathsf{ent}[P].$$

We may extend this LDP to the context of tagged (signed) point processes by arguing as in [LS15, Section 7], where it is also shown that the condition on the number of points does not alter the LDP. This leads to (6.5.18). The lower bound (6.5.19) follows from (6.5.18) by elementary manipulations as sketched in [LS15, Section 7]. □

c. Further conditioning on the points

Let $\varepsilon = (\log R)^{-3}$. For any $i \in \{1, \dots, m_{N,R}\}$ we let n_i be the number of points in K_i and $n_{i,\mathrm{int}}$ be the number of points in $K_{i,\varepsilon}$ (we add a $+$ or $-$ superscript in order to restrict ourselves to points with a positive or negative charge).

Lemma 6.5.5. *The conclusions of Lemma 6.5.4 hold after conditioning the random variables* $\left((\mathbf{C}_1^+, \mathbf{C}_1^-), \dots, (\mathbf{C}^+_{m_{N,R}}, \mathbf{C}^-_{m_{N,R}})\right)$ *to satisfy the following additional conditions:*

$$\sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R} \left|\log \frac{n_i}{\varepsilon R}\right| \leq (\varepsilon R)^{-1/4} N, \tag{6.5.20}$$

$$\sum_{i=1}^{m_{N,R}} (n_i^{\pm} - n_{i,\mathrm{int}}^{\pm}) \log R \leq (\log R)^{-1/2} N. \tag{6.5.21}$$

Proof. It is enough to show that both events occur with probability $1 - \exp(-NT)$ with T tending to ∞ as $R \to \infty$, when throwing points as a signed Poisson point process of intensity 1 in $\bigcup_{i=1}^{m_{N,R}} K_i$. The result then follows from standard large deviations estimates. Indeed, to prove that we may assume (6.5.20) without changing the volume of microstates, we may observe that the exponential moments of

$$\mathcal{C} \mapsto \sqrt{\varepsilon R} \frac{1}{\varepsilon R} n \left|\log \frac{n}{\varepsilon R}\right|$$

under the law of a Poisson point process in C_R are bounded by $O(R^2)$ as $R \to \infty$. In particular,

$$\begin{aligned}
\log \mathbf{\Pi}^s \left(\sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R} \left|\log \frac{n_i}{\varepsilon R}\right| > (\varepsilon R)^{-1/4} N \right) &= \log \mathbf{\Pi}^s \left(\sum_{i=1}^{m_{N,R}} \sqrt{\varepsilon R} \frac{n_i}{\varepsilon R} \left|\log \frac{n_i}{\varepsilon R}\right| > (\varepsilon R)^{1/4} N \right) \\
&\leq -(\varepsilon R)^{1/4} N + \log \mathbf{E}_{\mathbf{\Pi}^s} \exp \left(\sum_{i=1}^{m_{N,R}} \sqrt{\varepsilon R} \frac{n_i}{\varepsilon R} \left|\log \frac{n_i}{\varepsilon R}\right| \right) \\
&\leq -(\varepsilon R)^{1/4} N + \frac{N}{R^2} \log \mathbf{E}_{\mathbf{\Pi}^s_{|C_R}} \exp \sqrt{\varepsilon R} \frac{1}{\varepsilon R} n \left|\log \frac{n}{\varepsilon R}\right| \leq -(\varepsilon R)^{1/4} N + O(N).
\end{aligned}$$

In particular (6.5.20) indeed occurs with probability of order $1 - \exp(-(\varepsilon R)^{1/4} N)$.

To prove that we may assume (6.5.21) we may argue similarly, by first observing that the exponential moments of

$$\mathcal{C} \mapsto (n_i - n_{i,\mathrm{int}})(\log R)^2$$

under a standard Poisson point process (of intensity 1) in C_R are bounded by $O(R^2)$ as $R \to \infty$. Indeed the quantity $n_i - n_{i,\mathrm{int}}$ is nothing but the number of points in the thin layer $C_R \backslash C_{R(1-\varepsilon)}$ which has an area of order $R^2\varepsilon = R^2(\log R)^{-3}$. We then deduce that

$$\log \mathbf{\Pi}\left(\sum_{i=1}^{m_{N,R}} (n_i^{\pm} - n_{i,\mathrm{int}}^{\pm}) \log R > (\log R)^{-1/2} N\right) \leq -C(\log R)^{1/4} N,$$

which implies that (6.5.21) indeed occurs with large enough probability. □

d. Screening microstates

Lemma 6.5.6. *Let $\bar{P} \in \mathcal{P}_{s,1}(\Lambda \times \mathcal{X})$ and $\delta > 0$ be fixed. Let $N, R, \bar{R}$ be as above. For any $\nu > 0$ there exists a set A^{mod} of signed point configurations in Λ' of the form $\mathcal{C}^{\mathrm{mod}} = \sum_{i=1}^{m_{N,R}} (\mathcal{C}_i^{\mathrm{mod},+}, \mathcal{C}_i^{\mathrm{mod},-})$ where $(\mathcal{C}_i^{\mathrm{mod},+}, \mathcal{C}_i^{\mathrm{mod},-})$ is a signed point configuration in K_i, satisfying*

$$\sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{mod},+}(\Lambda') = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{mod},-}(\Lambda') = N \tag{6.5.22}$$

and such that the following holds

1. *If R is large enough, ν small enough and N large enough we have for any $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$*

$$\frac{1}{N}\int_{\Lambda'} \delta_{(N^{-1/2}z, \theta_z \cdot \mathcal{C}^{\mathrm{mod}})} dz \in B(\bar{P}, \frac{3\delta}{4}).$$

2. *For any $\mathcal{C}^{\mathrm{mod}}$ in A^{mod}, there exists an electric field E^{mod} satisfying*
   (a) *E^{mod} is compatible with $\mathcal{C}^{\mathrm{mod}}$*

$$\begin{cases} \mathrm{div}\ (E^{\mathrm{mod}}) = 2\pi \mathcal{C}^{\mathrm{mod}} & \text{in } \Lambda' \\ E^{\mathrm{mod}} \cdot \vec{n} = 0 & \text{on } \partial\Lambda' \end{cases}. \tag{6.5.23}$$

   (b) *The energy of E^{mod} is bounded by*

$$\limsup_{R\to\infty, \nu\to 0, N\to\infty} \frac{1}{2\pi}\int_{\mathbb{R}^2} |E_r^{\mathrm{mod}}|^2 \leq \overline{\mathbb{W}}^o(\bar{P}) + \delta, \tag{6.5.24}$$

   uniformly on $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$.
3. *There is a good volume of such microstates*

$$\liminf_{R\to\infty, \nu\to 0, N\to\infty} \frac{1}{N} \log \frac{\mathbf{Leb}^{2N}}{|\Lambda'|^{2N}}\left(A^{\mathrm{mod}}\right) \geq -\overline{\mathsf{ent}}[\bar{P}]. \tag{6.5.25}$$

Proof. For any $\nu > 0$, let us write the conditions for a signed point configuration $\mathcal{C} := \sum_{i=1}^{m_{N,R}} \mathcal{C}_i$

$$\frac{1}{\Lambda'}\int_{\Lambda'} \delta_{(N^{-1/2}z, \theta_z \cdot \mathcal{C})} dx \in B(\bar{P}, \delta) \tag{6.5.26}$$

$$\frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/2}z_i, \theta_{z_i} \cdot \mathcal{C}_i)} \in B(\bar{P}, \nu). \tag{6.5.27}$$

By Lemma 6.5.4 we know that given $\delta > 0$, for any N, R, ν (such that $N/R \in \mathcal{N}$) there exists a set A^{abs} ("abs" as "abstract" because we generate them abstractly - and not by hand - using Sanov's theorem as explained in the previous section) of configurations $\mathcal{C}^{\mathrm{abs,s}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{abs,s}}$ with

N points, where $\mathcal{C}_i^{\mathrm{abs,s}}$ is a point configuration in the square K_i, such that for any $\mathcal{C}^{\mathrm{abs,s}} \in A^{\mathrm{abs}}$ (6.5.26) and (6.5.27) hold and satisfying

$$\liminf_{R\to\infty,\nu\to 0,N\to\infty} \frac{1}{2N} \log \frac{\mathbf{Leb}^{2N}}{|\Lambda_{N,R}|^{2N}}(A^{\mathrm{abs}}) \geq -\int_\Lambda \mathsf{ent}[\bar{P}^x]dx. \tag{6.5.28}$$

To see how Lemma 6.5.4 yields (6.5.28) it suffices to note that the law of the $2N$-points signed point process $\mathbf{C}$ of Lemma 6.5.4 coincides with the law of the point process induced by the $2N$-th product of the normalized Lebesgue measure on Λ', and with this observation (6.5.19) gives (6.5.28).

We let A^{mod} be the set of configurations obtained after applying the screening procedure described in Section 6.5.2 with the parameter ε chosen as

$$\varepsilon = \frac{1}{\log^3 R}. \tag{6.5.29}$$

More precisely, for each $\mathcal{C}^{\mathrm{abs,s}}$ in A^{abs} we decompose $\mathcal{C}^{\mathrm{abs,s}}$ as $\sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{abs,s}}$ where $\mathcal{C}_i^{\mathrm{abs,s}}$ is a signed point configuration in K_i, and for any $i = 1, \dots, m_{N,R}$ we let $\Phi_i^{\mathrm{scr}}(\mathcal{C}^{\mathrm{abs,s}})$ be the set of signed point configurations obtained after screening the configuration $\mathcal{C}_i^{\mathrm{abs,s}}$ in K_i. Combining (6.5.2) with (6.5.17) and (6.5.20), (6.5.21) we see that any signed point configuration $\mathcal{C}^{\mathrm{mod}}$ has a total number of points with positive charge between $N - o(N)$ and N. We may then complete the point configurations by just adding squares with dipoles in the remaining layers $\bigcup_{i=1,\dots,m_{N,R}}(\bar{K}_i \backslash K_i)$, in such a way that (6.5.22) is satisfied.

We then let $\Phi^{\mathrm{mod}}(\mathcal{C}^{\mathrm{abs,s}})$ be the set of signed point configurations in Λ' obtained as the cartesian product of the $\Phi_i^{\mathrm{scr}}(\mathcal{C}^{\mathrm{abs,s}})$

$$\Phi^{\mathrm{mod}}(\mathcal{C}^{\mathrm{abs,s}}) := \prod_{i=1}^{m_{N,R}} \Phi_i^{\mathrm{scr}}(\mathcal{C}^{\mathrm{abs,s}})$$

and A^{mod} ("mod" as "modified") is defined as the image of A^{abs} by Φ^{mod}. Since $\bar{P}$ and δ are given, the set A^{mod} depends on the parameters N, R, ε, ν.

Let us now check that A^{mod} satisfies the conclusions of Lemma 6.5.6.

Distance to $\bar{P}$. To prove the first item we claim that the screening procedure preserves the closeness of the continuous average to $\bar{P}$ as expressed in (6.5.26) (however in general it does not preserve that of the discrete average (6.5.27)). The proof of such a claim was already given (in a slightly different setting) in [LS15, Section 6.3.2].

Since the topology on $\mathcal{X}^0$ is *local*, when comparing two random signed point processes we can localize the configurations to a square of fixed size R_0, up to a uniform error which goes to 0 as $R_0 \to \infty$. Now, the main point is that the screening procedure only modifies the configurations in a thin layer of size εR, (where ε has been chosen in (6.5.29)) in each square K_i. In particular, when R is large (hence ε is small), the vast majority of the translates of a given square R_0 by $z \in \Lambda'$ does not intersect any such thin layer, so that the configurations in them have not been modified when passing from $\mathcal{C}^{\mathrm{abs,s}}$ to $\mathcal{C}^{\mathrm{mod}}$. Fixing R_0 large enough and sending $R \to \infty$ we may thus bound the distance between the continuous average for $\mathcal{C}^{\mathrm{abs,s}}$ and the one for $\mathcal{C}^{\mathrm{mod}}$ by at most $\delta/4$.

Energy. First we associate to any $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$ a screened electric field E^{mod}. We know by definition of the map Φ^{scr} that for $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$, for any $i = 1 \dots m_{N,R}$ there exists an electric field E_i^{mod} such that

$$\begin{cases} \mathrm{div}\ (E_i^{\mathrm{mod}}) = 2\pi \mathcal{C}_i^{\mathrm{mod}} & \text{in } K_i \\ E_i^{\mathrm{mod}} \cdot \vec{n} = 0 & \text{on } \partial K_i \end{cases}$$

An electric field $\bar{E}_i^{\mathrm{mod}}$ satisfying the analogous relation on $\bar{K}_i \backslash K_i$ can also easily be constructed, and its energy can be bounded be the number of dipoles added in that layer. Setting $E^{\mathrm{mod}} := \sum_i E_i^{\mathrm{mod}} \mathbf{1}_{K_i} + \bar{E}_i^{\mathrm{mod}} \mathbf{1}_{\bar{K}_i \backslash K_i}$ provides an electric field satisfying (6.5.23). We now turn to bounding its energy. Let $\mathcal{C}^{\mathrm{abs,s}} \in A^{\mathrm{abs}}$ be such that $\mathcal{C}^{\mathrm{mod}}$ is obtained from $\mathcal{C}^{\mathrm{abs,s}}$ after screening. For any $i = 1, \dots, m_{N,R}$, the energy of E_i^{mod} is bounded as in (6.5.16) in terms of the "best energy" associated to $\mathcal{C}_i^{\mathrm{abs,s}}$. We thus have, by summing the contributions of each square K_i and $\bar{K}_i \backslash K_i$

$$\int_{\Lambda'} |E_r^{\mathrm{mod}}|^2 = \sum_{i=1}^{m_{N,R}} \int_{K_i} |E_{i,r}^{\mathrm{mod}}|^2 + \int_{\bar{K}_i \backslash K_i} |\bar{E}_{i,r}^{\mathrm{mod}}|^2 \leq \sum_{i=1}^{m_{N,R}} F_R(\mathcal{C}_i^{\mathrm{abs,s}}) + C \sum_{i=1}^{m_{N,R}} \frac{R}{\varepsilon} F_R(\mathcal{C}_i^{\mathrm{abs,s}}) + \sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R} \left| \log \frac{n_i}{\varepsilon R} \right| + N\varepsilon + o(N) \quad (6.5.30)$$

where n_i is the number of points $|\mathcal{C}_i^{\mathrm{abs,s}}|(K_i)$.

We now use the fact that the discrete average of the configurations in the square K_i is close to $\bar{P}$ and that F_R is upper semi-continuous (see Lemma 6.5.3). We thus have

$$\limsup_{R \to \infty, \nu \to 0} \frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} F_R(\mathcal{C}_i^{\mathrm{abs,s}}) \leq \int F_R(\mathcal{C}) d\bar{P} \leq \overline{\mathbb{W}}^o(\bar{P}) \quad (6.5.31)$$

where the last inequality follows from (6.5.15). Combining (6.5.30), (6.5.31) and (6.5.20) proves (6.5.24) .

Volume. We now wish to bound the volume loss between the set A^{abs} of microstates generated "abstractly" and the set A^{mod} of configurations obtained after modification by the screening procedure. From the conclusions Proposition 6.5.2 we may estimate the cost (in logarithmic volume) of screening the signed point configurations. According to (6.5.3) the loss can be controlled by

$$\int_{\mathcal{C}^{\mathrm{abs,s}} \in A^{\mathrm{abs}}} C \left(\sum_{i=1}^{m_{N,R}} ((n_i - n_{i,\mathrm{int}}) \log R + R) + \frac{R}{\varepsilon} \sum_{i=1}^{m_{N,R}} \left(F_R(\mathcal{C}_i^{\mathrm{abs,s}}) + \varepsilon R^2 \right) + \sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R} \left| \log \frac{n_i}{\varepsilon R} \right| \right).$$

We have $m_{N,R} R = o(N)$, and $\lim_{R,N \to \infty} \frac{1}{N} m_{N,R} \varepsilon R^2 = 0$ due to the choice (6.5.29).

We also have, from (6.5.20)

$$\lim_{R \to \infty} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{m_{N,R}} \frac{n_i}{\varepsilon R} \left| \log \frac{n_i}{\varepsilon R} \right| = 0.$$

Since (6.5.31) holds we obtain

$$\lim_{R \to \infty} \lim_{\nu \to 0} \lim_{N \to \infty} \frac{1}{N} \frac{R}{\varepsilon} \sum_{i=1}^{m_{N,R}} F_R(\mathcal{C}_i^{\mathrm{abs,s}}) = 0.$$

Finally from (6.5.21) we get that

$$\sum_{i=1}^{m_{N,R}} (n_i - n_{i,\mathrm{int}}) \log R = o(N)$$

where $n_i = |\mathcal{C}_i^{\mathrm{abs,s}}|(C_R)$ and $n_{i,\mathrm{int}} = |\mathcal{C}_i^{\mathrm{abs,s}}|(C_{R(1-\varepsilon)})$, which concludes the proof of (6.5.25).

□

Combining Lemmas 6.5.1 and 6.5.6 yields Proposition 6.3.9.

6.6 Next order large deviations: upper bound

In this section we prove the Large Deviations upper bound stated in Proposition 6.3.10.

6.6.1 Positive part of the energy

First we observe that the positive part of the energy is semi-continuous in the suitable direction.

Lemma 6.6.1. *Let $\bar{P} \in \mathcal{P}_{\mathrm{inv},1}(\Lambda \times \mathcal{X})$. For any sequence $\{(\vec{X}_N, \vec{Y}_N)\}_N$ such that $i_N(\vec{X}_N, \vec{Y}_N) \in B(\bar{P}, \varepsilon)$, we have*

$$\liminf_{N\to\infty} \frac{1}{2\pi N} \int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 \geq \overline{\mathbb{W}}^o(\bar{P}) - o_\varepsilon(1).$$

Proof. Define $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ as

$$\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)} := \int_\Lambda \delta_{(x, V'_N(\sqrt{N}x+\cdot))} dx,$$

i.e. $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ is the push-forward of the Lebesgue measure on Λ by $x \mapsto V'_N(\sqrt{N}x+\cdot)$, where V'_N was defined in (6.2.6). Assume that $\int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2 \leq CN$ (otherwise there is nothing to prove). For any $m > 0$ we have

$$\int_\Lambda \frac{1}{|C_m|} \int_{C_m} |E_r|^2 d\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}(x, E) \leq \frac{1}{N} \int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2. \tag{6.6.1}$$

The bound (6.6.1) implies that the push-forward of $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ by $(z, E) \mapsto (z, E_r)$ is tight in $\mathcal{P}(\Lambda \times L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$. Using Lemma 6.2.4 we also get the tightness of $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ in $\mathcal{P}(\Lambda \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$. The associated random tagged signed point process $\bar{P}_{(\vec{X}_N,\vec{Y}_N)}$ (i.e. the push-forward of $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ by $(z, E) \mapsto (z, \mathrm{Conf}(E))$ is also tight in $\mathcal{P}(\Lambda \times \mathcal{X})$, arguing as in Lemma 6.2.11. Up to a subsequence extraction, we may thus find $\bar{P}^{\mathrm{elec}}_0 \in \mathcal{P}(\Lambda \times L^p_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2))$ and $P_0 \in \mathcal{P}(\Lambda \times \mathcal{X})$ such that

1. $\bar{P}^{\mathrm{elec}}_{(\vec{X}_N,\vec{Y}_N)}$ converges to $\bar{P}^{\mathrm{elec}}_0$ as $N \to \infty$
2. $\bar{P}_{(\vec{X}_N,\vec{Y}_N)}$ converges to $\bar{P}_0$ as $N \to \infty$.

It is not hard to check that the first marginal of $\bar{P}_0$ is the Lebesgue measure on Λ, that its second marginal is stationary, and that $\bar{P}_0 \in B(\bar{P}, 2\varepsilon)$.

Using Lemma 6.2.6, we obtain from (6.6.1) that for any $m > 0$

$$\int_\Lambda \frac{1}{|C_m|} \int_{C_m} |E_r|^2 d\bar{P}^{\mathrm{elec}}_0(x, E) \leq \liminf_{N\to\infty} \frac{1}{N} \int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2.$$

In particular, letting $m \to \infty$ and using the definition of $\overline{\mathbb{W}}^o$ we obtain

$$\overline{\mathbb{W}}^o(\bar{P}_0) \leq \liminf_{N\to\infty} \frac{1}{N} \int_{\mathbb{R}^2} |\nabla V'_{N,r}|^2.$$

We conclude by letting $\varepsilon \to 0$ and using the lower semi-continuity of $\overline{\mathbb{W}}^o$ among random stationary tagged processes, as stated in Lemma 6.2.8. □

6.6.2 Bound on the nearest neighbor contributions

We are now left to bound from above

$$\int_{\Lambda^{2N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^N \log r(x_i')+\log r(y_i')} d\vec{X}_N d\vec{Y}_N.$$

For $0<\tau<1$ we will distinguish between the points whose nearest neighbor is at distance $\geq\tau$ and those with a very close neighbor, at distance $\leq\tau$. We thus write

$$\sum_{i=1}^N \log r(x_i')+\log r(y_i') = \sum_{i=1}^N \log(r(x_i')\vee\tau)+\log(r(y_i')\vee\tau) + \sum_{i=1}^N \log(\frac{r(x_i')}{\tau}\wedge 1)+\log(\frac{r(y_i')}{\tau}\wedge 1). \quad (6.6.2)$$

a. Points at distance $\geq\tau$

The contributions due to the interactions of points at distance $\geq\tau$ is continuous, as expressed by the following

Lemma 6.6.2. *Let $\bar{P}\in\mathcal{P}_{\mathrm{inv},1}(\Lambda\times\mathcal{X})$. For any sequence $\{\vec{X}_N,\vec{Y}_N\}_N$ such that $i_N(\vec{X}_N,\vec{Y}_N)\in B(\bar{P},\varepsilon)$ and such that $\mathcal{N}(\mathcal{C},C_1)$ is uniformly integrable against $d\bar{P}_N$ as $N\to\infty$, we have*

$$\liminf_{N\to\infty} -\sum_{i=1}^N \log(r(x_i')\vee\tau)+\log(r(y_i')\vee\tau) \geq -\mathbb{W}_\tau^*(\bar{P})+o_\varepsilon(1).$$

Proof. For any $t>0$ let χ_t be a smooth nonnegative function such that $\chi_t\equiv 1$ in C_{1-t} and $\chi_t\equiv 0$ outside C_{1+t} and such that $\int\chi_t=1$. Let Λ_t be the square $\{x\in\Lambda, d(x,\partial\Lambda)\geq t\}$. For N large enough we have

$$-\frac{1}{N}\sum_{i=1}^N \log(r(x_i')\vee\tau)+\log(r(y_i')\vee\tau) \geq -\int \mathbf{1}_{\Lambda_t}(x)\,(\mathbb{W}_\tau^*(\chi_t,\mathcal{C}))\,d\bar{P}_N.$$

The map $\mathcal{C}\mapsto -\mathbb{W}_\tau^*(\chi_t,\mathcal{C})$ is continuous and bounded by $(-\log(\tau))\mathcal{N}(\mathcal{C},C_{1+1})$, and since $\mathcal{N}(\mathcal{C},C_{1+t})$ is uniformly integrable against $d\bar{P}_N$ we have

$$\lim_{N\to\infty} -\int \mathbf{1}_{\Lambda_t}(x)\,(\mathbb{W}_\tau^*(\chi_t,\mathcal{C}))\,d\bar{P}_N = -\int \mathbf{1}_{\Lambda_t}(x)\,(\mathbb{W}_\tau^*(\chi_t,\mathcal{C}))\,d\bar{P}+o_\varepsilon(1).$$

Letting $t\to 0$ yields the result. □

b. Contribution of close dipoles

We now turn to bounding the contributions to the Boltzmann factor $e^{-\beta w_N}$ due to pairs of points which are at distance $\leq\tau$ from each other, and see that this quantity is negligible when $\tau\to 0$. Using (6.6.2) we see that we are left with bounding

$$\int_{i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^N \log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\,d\vec{X}_N\,d\vec{Y}_N.$$

We prove

Lemma 6.6.3. *We have*

$$\limsup_{\tau\to 0,\varepsilon\to 0,N\to\infty} \frac{1}{N}\log \int_{i_N^{-1}(B(\bar P,\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^N \log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\, d\vec X_N\, d\vec Y_N \le -\overline{\mathsf{ent}}(\bar P).$$

This will rely on the method of [GP77] described in Section 6.4.

Proof. For each configuration, we denote by n the number of points of any sign for which $r(z_i')\le\tau$, and separate over the value of n. Without loss of generality, we may assume that these points are the first n_x ones for the x's and n_y ones for the y's, with $n_x+n_y=n$. We may write

$$\begin{aligned}\int_{i_N^{-1}(B(\bar P,\varepsilon))} & e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\, d\vec X_N\, d\vec Y_N \\ &\le \sum_{n=0}^{2N}\sum_{n_x+n_y=n}\binom{N}{n_x}\binom{N}{n_y}\int_{\Lambda^n} e^{-\frac{\beta}{2}\sum_{i=1}^n \log\frac{r(z_i')}{\tau}\wedge 1}\, dz_1\dots dz_n \\ &\qquad\times\int_{i_N^{-1}(B(\bar P,\varepsilon))} dx_{n_x+1}\dots dx_N\, dy_{n_y+1}\dots dy_{2N}. \quad (6.6.3)\end{aligned}$$

For any $n\ge 0$, $1>\tau>0$ and $N\ge 1$, define $\Lambda_{N,\tau}^n$ as the set of n-tuples of points in Λ such that all the nearest-neighbor distances at blown-up (by $\sqrt{N}$) scale are smaller than τ. We define

$$Z(n,\tau,\beta,N)=\int_{\Lambda_{N,\tau}^n} e^{-\frac{\beta}{2}\sum_{i=1}^n \log\frac{r(z_i')}{\tau}}\, dz_1\dots dz_n,$$

with $z_i'=z_i\sqrt{N}$.

Fixing the parameter $\delta=1/\sqrt{|\log\tau|}$, we may rewrite (6.6.3) as

$$\begin{aligned}\int_{i_N^{-1}(B(\bar P,\varepsilon))} & e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\, d\vec X_N\, d\vec Y_N \le \sum_{n=\lfloor\delta N\rfloor+1}^{2N} Z(n,\tau,\beta,N)|\Lambda|^{2N-n} \\ &+\sum_{n=0}^{\lfloor\delta N\rfloor}\sum_{n_x+n_y=n}\binom{N}{n_x}\binom{N}{n_y} Z(n,\tau,\beta,N)\int_{\Lambda^{2N-n}\cap i_N^{-1}(B(\bar P,\varepsilon))} dx_{n_x+1}\dots dx_N\, dy_{n_y+1}\dots dy_{2N}.\end{aligned}$$

Next, we claim that if $n\le N$ (which is the case here), we have

$$Z(n,\tau,\beta,N)\le\tau^n C^n. \quad (6.6.4)$$

Assuming this claim is true, and observing that $i_N(\vec X_N,\vec Y_N)\in B(\bar P,\varepsilon)$ implies that

$$i_{N-n}(x_n,\dots,x_N,y_n,\dots,y_N)\in B(\bar P,2\varepsilon)$$

with obvious notation (for δ small enough depending on ε), so we may then write

$$\begin{aligned}\int_{i_N^{-1}(B(\bar P,\varepsilon))} & e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log\frac{r(x_i')}{\tau}\wedge 1+\log\frac{r(y_i')}{\tau}\wedge 1}\, d\vec X_N\, d\vec Y_N \\ &\le \sum_{n=\lfloor\delta N\rfloor+1}^{2N}\sum_{n_x+n_y=k}\binom{N}{n_x}\binom{N}{n_y} Z(n,\tau,\beta,N) \\ &\quad+\sum_{n=0}^{\lfloor\delta N\rfloor}\sum_{n_x+n_y=n}\binom{N}{n_x}\binom{N}{n_y} Z(n,\tau,\beta,N)\int_{i_N^{-1}(B(\bar P,\varepsilon))} dx_{n_x+1}\dots dx_N\, dy_{n_y+1}\dots dy_{2N}.\end{aligned} \quad (6.6.5)$$

The first term in the right-hand side of (6.6.5) is easily bounded above by $C^N\tau^{\delta N}$ in view of (6.6.4). For the second one, we note that by direct computations one may show that for $n \leq \delta N$,

$$\sum_{n_x+n_y=n} \left(\begin{array}{c} N \\ n_x \end{array} \right) \left(\begin{array}{c} N \\ n_y \end{array} \right) \leq C_\delta^N$$

with $\lim_{\delta\to 0} C_\delta = 1$. Combining this with (6.6.4) and inserting into (6.6.5) we thus are led to

$$\int_{i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log \frac{r(x_i')}{\tau}\wedge 1 + \log\frac{r(y_i')}{\tau}\wedge 1}\, d\vec{X}_N\, d\vec{Y}_N$$
$$\leq C^N\tau^{\delta N} + \delta N C_\delta^N C^{\delta N} \int_{\Lambda^{2N-2\delta N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} dx_{\delta N+1}\dots dx_N\, dy_{\delta N+1}\dots dy_{2N}.$$

By the choice of $\delta = 1/\sqrt{|\log\tau|}$ we find that the first term is logarithmically negligible and thus

$$\limsup_{N\to\infty} \frac{1}{N} \log \int_{\Lambda^{2N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} e^{-\frac{\beta}{2}\sum_{i=1}^{2N} \log \frac{r(x_i')}{\tau}\wedge 1 + \log\frac{r(y_i')}{\tau}\wedge 1}\, d\vec{X}_N\, d\vec{Y}_N$$
$$\leq \log C_\delta + C\delta + \limsup_{N\to\infty} \frac{1}{N} \log \int_{\Lambda^{2N-2\delta N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} dx_{\delta N+1}\dots dx_N\, dy_{\delta N+1}\dots dy_{2N}. \quad (6.6.6)$$

But we have, from Proposition 6.2.10

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty} \frac{1}{N} \log \int_{\Lambda^{2N-2\delta N}\cap i_N^{-1}(B(\bar{P},\varepsilon))} dx_{\delta N+1}\dots dx_N\, dy_{\delta N+1}\dots dy_{2N}$$
$$\leq -N(1-\delta)\overline{\mathsf{ent}}(\bar{P})$$

so letting $\varepsilon \to 0$ and $\tau \to 0$ (hence $\delta \to 0$) in (6.6.6), we obtain the conclusion.

We now check (6.6.4), following Section 6.4. We will establish the more general bound

$$Z(n,\tau,\beta,N) \leq \left(\frac{C\tau n}{N}\right)^n, \quad (6.6.7)$$

which implies (6.6.4) when $n \leq N$. We may assume that $n \geq 1$, otherwise (6.6.7) is obvious (both sides are equal to 1). First, we rewrite $Z(n,\tau,\beta,N)$ as

$$Z(n,\tau,\beta,N) = \int_{\Lambda_{N,\tau}^n} e^{-\frac{\beta}{2}\sum_{i=1}^n \log\frac{r(S_i)\sqrt{N}}{\tau}}\, d\vec{S}_n.$$

As in Section 6.4, for each configuration $(S_1,\dots,S_n)$, we form the nearest-neighbor function $i \mapsto F_i$, and associate to the configuration a directed graph with $K \in [1,n]$ connected components, each component comprising a cycle of length 2 together with trees attached to the two vertices of the cycle. We know moreover that the distances to the nearest neighbor $|S_i - S_{F_i}|$ are bounded by $\tau/\sqrt{N}$.

Splitting between isomorphisms classes of graphs and following the computations and using the notation of Section 6.4, we now have to bound

$$\int_{\Lambda_{N,\tau}^n\cap\{\hat{\gamma}(\vec{S_n})\equiv\gamma\}} e^{-\frac{\beta}{2}\sum_{i=1}^n \log\left(\frac{\frac{1}{2}|S_i - S_{F_i}|\sqrt{N}}{\tau}\right)} d\vec{S}_n. \quad (6.6.8)$$

Let $L_1, \dots, L_K$ be the K subsets of indices associated to γ. We make the change of variables: for $i \in L_k$ such that $i \notin c_k$

$$u_i := \sqrt{\frac{N}{n}} \frac{1}{2\tau}(S_i - S_{F_i}),$$

and

$$u_{i_k^a} := \sqrt{\frac{N}{n}} \frac{1}{2\tau}(S_{i_k^a} - S_{i_k^b}), \quad u_{i_k^b} := \sqrt{\frac{N}{n}} \frac{1}{2} S_{i_k^b}.$$

With respect to the new variables, the integral in (6.6.8) is bounded by

$$\left(\sqrt{\frac{n}{N}}\right)^{2n} e^{-\frac{\beta}{2} n \log \sqrt{\frac{n}{N}}} 4^{p_k} \tau^{2(p_k-1)} \left(\frac{1}{\sqrt{N}}\right)^{\frac{1}{2}\beta p_k} \int_{D_k'} e^{-\frac{\beta}{2}\left(\sum_{i \in L_k \setminus c_k} \log u_i + 2 \log u_{i_k^a}\right)} \prod_{i \notin L_k} du_i \, du_{i_k^a} \, du_{i_k^b},$$

where D_k' is a (suitably enlarged) domain of integration for the new variables where they satisfy

$$\sum_{i \in L_k \setminus c_k} |u_i|^2 \leq 1, \qquad |u_{i_k^a}| \leq \frac{1}{2\sqrt{n}}, \qquad |u_{i_k^b}| \leq 1$$

and p_k denotes the cardinality of L_k. Clearly D_k' is included in the set that was denoted D_k in Section 6.4, so we deduce using Lemma 6.4.3 that

$$\begin{aligned} \int_{\Lambda_{N,\tau}^n \cap \{\hat{\gamma}(\vec{S_n}) \equiv \gamma\}} e^{-\frac{\beta}{2}\sum_{i=1}^n \log\left(\frac{\frac{1}{2}|S_i - S_{F_i}|\sqrt{N}}{\tau}\right)} d\vec{S_n} &\leq \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \frac{\tau^{2\sum_{k=1}^K (p_k-1)}}{N^{\beta/4 \sum_{k=1}^K p_k}} Diri_{n,K} \\ &\leq \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \frac{\tau^{2(n-K)}}{N^{n\beta/4}} Diri_{n,K}. \end{aligned}$$

Summing over all isomorphism classes of γ, we deduce that

$$\begin{aligned} Z(n,\tau,\beta,N) &\leq \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \sum_{K=1}^{n/2} |\mathbf{D}_{n,K}| Diri_{n,K} \frac{\tau^{2(n-K)}}{N^{n\beta/4}} \\ &\leq \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \frac{\tau^n}{N^{n\beta/4}} \sum_{K=1}^{n/2} |\mathbf{D}_{n,K}| Diri_{n,K} \leq \left(\frac{n}{N}\right)^{n(1-\frac{\beta}{4})} \frac{\tau^n}{N^{n\beta/4}} \left(\frac{n}{2}\right)^{\beta n/4} C_\beta^{n/2}, \end{aligned}$$

which yields (6.6.7). □

6.7 Leading order large deviations

In this section, we prove a large deviations upper bound at the leading order (N^2) the joint law of the empirical measures μ_N^+ and μ_N^-, with rate function given by (6.7.1).

a. The limiting energy

For any two probability measures μ^+, μ^- in $\mathcal{P}(\Lambda)$ we let

$$H(\mu^+, \mu^-) := \min\left\{ \int_{\mathbb{R}^2} |E|^2, E \in L^2(\mathbb{R}^2, \mathbb{R}^2) \text{ s.t. } -\operatorname{div} E = 2\pi(\mu^+ - \mu^-) \right\}, \tag{6.7.1}$$

which represents the electrostatic interaction energy between μ^+ and μ^-. The infimum in (6.7.1) is achieved because the L^2-norm is coercive and lower semi-continuous for the weak-L^2 topology.

In fact it is not difficult to check that if $H(\mu^+,\mu^-)$ is finite, it is equal to $\int_{\mathbb{R}^2}|E^{\rm loc}|^2$ where $E^{\rm loc}$ is the "local electric field" defined as $E^{\rm loc}(x):=\int_{\mathbb{R}^2}-\nabla\log|x-t|(d\mu^+(t)-d\mu^-(t))$.

The functional H takes value in $[0,+\infty]$ and we have

$$H(\mu^+,\mu^-)=0 \iff \mu^+=\mu^-. \tag{6.7.2}$$

b. Large deviations upper bound

We give a large deviation upper bound at speed N^2 for the joint law of the empirical measures μ_N^+ and μ_N^-.

Proposition 6.7.1. *For any $\mu^+,\mu^-\in\mathcal{P}(\Lambda)$ we have*

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}_N^\beta\left(\{(\mu_N^+,\mu_N^-)\in B((\mu^+,\mu^-),\varepsilon)\}\right)\le-\frac{\beta}{2}H(\mu^+,\mu^-).$$

Together with (6.7.2) this essentially says that we must have $\mu_N^+\approx\mu_N^-$ for N large, except with very small (of order e^{-N^2}) probability.

Proof. Let μ^+,μ^- be in $\mathcal{P}(\Lambda)$ and $\varepsilon>0$. Using Lemma 6.2.3 we may write

$$\begin{aligned}&\mathbb{P}_N^\beta\left(\left\{(\mu_N^+,\mu_N^-)\in B((\mu^+,\mu^-),\varepsilon)\right\}\right)\\&\le\frac{1}{Z_{N,\beta}}\int_{\Lambda^{2N}\cap\{(\mu_N^+,\mu_N^-)\in B((\mu^+,\mu^-),\varepsilon)\}}d\vec{X}_N d\vec{Y}_N\exp\left(-\frac{\beta}{2}\left(\int|\nabla V_{N,r}|^2+\sum_{i=1}^N\log(r(x_i))+\sum_{i=1}^N\log(r(y_i))\right)\right).\end{aligned} \tag{6.7.3}$$

We claim that $\int|\nabla V_{N,r}|^2$ is lower-semi continuous in the following sense: if $\mu_N^+\to\mu^+$ and $\mu_N^-\to\mu^-$ then

$$\liminf_{N\to\infty}\frac{1}{N^2}\int_{\mathbb{R}^2}|\nabla V_{N,r}|^2\ge H(\mu^+,\mu^-). \tag{6.7.4}$$

Indeed, a uniform bound on the L^2-norm implies that $\{\frac{1}{N}\nabla V_{N,r}\}_N$ is tight in $L^2(\mathbb{R}^2,\mathbb{R}^2)$, let us denote by E a limit point. For any N we have by definition

$$-\text{div }(\nabla V_{N,r})=2\pi\left(\sum_{i=1}^N\delta_{x_i}^{r(x_i)}-\sum_{i=1}^N\delta_{y_i}^{r(y_i)}\right)$$

and it easily implies that

$$-\text{div }E=2\pi(\mu^+-\mu^-).$$

On the other hand, by lower semi-continuity of the L^2-norm with respect to weak convergence we have

$$\liminf_{N\to\infty}\frac{1}{N^2}\int_{\mathbb{R}^2}|\nabla V_{N,r}|^2\ge\int_{\mathbb{R}^2}|E|^2,$$

thus (6.7.4) holds, and together with (6.7.3) it yields

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}_N^\beta\left(\{(\mu_N^+,\mu_N^-)\in B((\mu^+,\mu^-),\varepsilon)\}\right)\\&\qquad\le-\frac{\beta}{2}H(\mu^+,\mu^-)-\liminf_{N\to\infty}\frac{1}{N^2}\log Z_{N,\beta}\\&\qquad+\limsup_{N\to\infty}\frac{1}{N^2}\log\int_{\Lambda^{2N}}\exp\left(\frac{\beta}{2}\left(\sum_{i=1}^N\log(r(x_i))+\sum_{i=1}^N\log(r(y_i))\right)\right)d\vec{X}_N d\vec{Y}_N.\end{aligned} \tag{6.7.5}$$

Combining (6.7.5) with the control (6.1.3) on the partition function and with (6.3.1) we get

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{1}{N^2}\log\mathbb{P}_N^\beta\left(\{(\mu_N^+,\mu_N^-)\in B((\mu^+,\mu^-),\varepsilon)\}\right)\leq-\frac{\beta}{2}H(\mu^+,\mu^-),$$

which concludes the proof of the proposition. □

c. Large deviations lower bound (sketch).

We remark (without providing details) that a complementary large deviations lower bound, with the same rate function, can be derived by adapting the approximation constructions in [BAZ98,Ser15]. Indeed, to construct an approximate configuration of points with enough volume (in the exponential in N^2 scale), one may proceed as follows. First, one cancels the common parts of μ_+ and μ_- by positioning pairs of positive and negative charges (henceforth referred to as dipoles) with intra-dipole separation of N^{-10} say and inter-dipole separation of at least N^{-1}. Then, one may use the construction of e.g. [Ser15, Theorem 2.3] and construct a sequence of "well-separated" configurations (separation at least $\eta N^{-1/2}$ between points with small enough η) so that $\limsup_{N\to\infty}N^{-2}w_N(\vec{X}_N,Y_N)\leq H(\mu_+,\mu_-)+C(\eta)$ where $C(\eta)\to_{\eta\to 0}0$; the argument in [BAZ98] shows that the volume of such configurations is large enough.

6.8 Tail estimates for the complex Gaussian multiplicative chaos (by Wei Wu)

In this appendix we apply Theorem 16 and Corollary 6.1.3 to obtain the tail asymptotics of subcritical complex Gaussian multiplicative chaos on $\mathbb{R}^2$. Let h be an instance of the Gaussian free field (GFF) on $\mathbb{R}^2$, which we define below. Let $D\subset\mathbb{R}^2$ be a bounded domain, we are interested in the distribution of $\left|\int_D e^{i\beta h(x)}dx\right|$, for $\beta\in(0,\sqrt{2})$. This object appears in different contexts, such as the partition function of complex random energy type models, the scaling limit of the compactified height function in two dimensional dimer model [Dub11], and the electric vertex operator in conformal field theory [Gaw97]. Another motivation comes from the conjecture that this object describes the scaling limit of the magnetization of XY model in the plasma phase [FS81]. The Lee-Yang Theorem was proved for the XY model (see [LS81]), therefore one may further conjecture that the characteristic function of $\int_D e^{i\beta h(x)}dx$ has pure imaginary zeros. Here we focus on another perspective, which is the tail behavior of $\int_D e^{i\beta h(x)}dx$, and our approach is based on an identity relating the moments of $\left|\int_D e^{i\beta h(x)}dx\right|$ to the partition function of a two component Coulomb gas (Lemma 6.8.1 below). For simplicity we set $D=\Lambda$, the unit cube.

Formally, let $H_0(\mathbb{R}^2)=\{\varphi:\varphi\in C_0^\infty(\mathbb{R}^2),\text{ s.t.}\int\varphi(x)\,dx=0\}$, and denote by $H(\mathbb{R}^2)$ the completion of $H_0(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$. The GFF is defined as a random distribution in $(H(\mathbb{R}^2))'$, such that for any $\rho_1,\rho_2\in H(\mathbb{R}^2)$, the covariances are given by

$$\mathrm{Cov}\left(\langle h,\rho_1\rangle,\langle h,\rho_2\rangle\right)=-\int\int\rho_1(x)\,\rho_2(y)\log|x-y|\,dxdy,$$

where $\langle h,\rho\rangle\doteq\int h(x)\,\rho(x)\,dx$. Therefore, to give a mathematical definition of $e^{i\beta h}$, one immediately runs into the issue that h is only defined up to an additive constant, so that one can only hope to define $e^{i\beta h}$ up to a multiplicative constant. However, as we explain below, $\left|\int_D e^{i\beta h(x)}dx\right|^2$ can be uniquely defined, and we then simply set $\left|\int_D e^{i\beta h(x)}dx\right|=\sqrt{\left|\int_D e^{i\beta h(x)}dx\right|^2}$.

Let h_r denote the GFF on a disk $D_r = \{z : |z| \leq r\}$ with zero boundary condition, and let $h^{(m)}$ be the massive GFF on $\mathbb{R}^2$ with mass m. h_r, $h^{(m)}$ are Gaussian processes with covariances given by

$$\begin{aligned} \mathrm{Cov}\left(h_r(x), h_r(y)\right) &\doteq g_r(x,y) \\ &= -\log|x/r - y/r| + \log\left|1 - x\bar{y}/r^2\right|, \end{aligned}$$

and $\mathrm{Cov}\left(h^{(m)}(x), h^{(m)}(y)\right) = \left(-\Delta + m^2\right)^{-1}(x,y)$, respectively. Then, as was explained in Section 6 of [LRV15], both $e^{i\beta h_r}$ and $e^{i\beta h^{(m)}}$ are well defined as random distributions in $(C_0^\infty(\mathbb{R}^2))'$ for $\beta \in (0, \sqrt{2})$. Therefore we can define the random variable $\left|\int_D e^{i\beta h(x)}dx\right|^2$ as the distributional weak limit of $\left|\int_D e^{i\beta h_r(x)}dx\right|^2$ as $r \to \infty$, or the distributional weak limit of $\left|\int_D e^{i\beta h^{(m)}(x)}dx\right|^2$ as $m \to 0$. Indeed, given $\rho_1, \rho_2 \in H(\mathbb{R}^2)$, since

$$\lim_{r\to\infty} \mathrm{Cov}\left(\langle h_r, \rho_1\rangle, \langle h_r, \rho_1\rangle\right) = \lim_{m\to 0} \mathrm{Cov}(\langle h^{(m)}, \rho_1\rangle, \langle h^{(m)}, \rho_1\rangle) = \mathrm{Cov}\left(\langle h, \rho_1\rangle, \langle h, \rho_1\rangle\right),$$

it is possible to prove that the two constructions lead to the same limiting object. We will not give a proof of the equivalence in this appendix, but instead construct the limiting object using the first approach.

As was explained in Section 6.2 of [LRV15], one needs a regularization procedure to define $e^{i\beta h_r}$. For $\varepsilon > 0$ and $x \in D_r$, let $h_r^\varepsilon(x)$ denote the average of h on the circle of radius ε centered at x (we assume h to be identically zero outside D_r). h_r^ε is a Gaussian process with covariances given by

$$\begin{aligned} \mathrm{Cov}\left(h_r^\varepsilon(x), h_r^\varepsilon(y)\right) &\doteq g_r^\varepsilon(x,y) \\ &= \frac{1}{(2\pi\varepsilon)^2}\int_{|z-x|=\varepsilon}\int_{|w-y|=\varepsilon} g_r(z,w)\,dzdw. \end{aligned}$$

It was shown in [LRV15] that as $\varepsilon \to 0$,

$$\varepsilon^{-\beta^2/2}\int_D e^{i\beta h_r^\varepsilon(x)}dx \to \int_D e^{i\beta h_r(x)}dx \text{ in } L_p, \text{ for } p > 1.$$

Some properties of the circle averaged field h_r^ε were summarized in Section 3.1 of [DS11]. In particular,

$$\mathrm{Var} h_r^\varepsilon(x) = \log C(x; D_r) - \log\varepsilon,$$

where $C(x; D_r) = r(1 - |x|^2/r^2)$ is the conformal radius of D_r from x. Also notice that when $|x - y| > 2\varepsilon$, $g_r^\varepsilon(x,y) = g_r(x,y)$, and $g_r^\varepsilon(x,y) \to 0$ if either x or y tends to ∂D_r. Moreover, for all $r > 2$ (which we will assume for the rest of the argument) and $\varepsilon < 1$,

$$\sup_{x,y\in D_r} |g_r^\varepsilon(x,y)| \leq -\log\varepsilon + C_1, \text{ for some } C_1 < \infty.$$

We may therefore compute, for $k \in \mathbb{N}$,

$$\begin{aligned}
& \mathbb{E}\left|\int_D e^{i\beta h_r(x)}dx\right|^{2k} \\
= & \lim_{\varepsilon\to 0}\mathbb{E}\left|\varepsilon^{-\beta^2/2}\int_D e^{i\beta h_r^\varepsilon(x)}dx\right|^{2k} \\
= & \lim_{\varepsilon\to 0}\varepsilon^{-\beta^2 k}\mathbb{E}\int_{D^{\otimes 2k}}\prod_{i=1}^k e^{i\beta h_r^\varepsilon(x_i)}\prod_{j=1}^k e^{-i\beta h_r^\varepsilon(y_j)}d\vec{x}d\vec{y} \\
= & \lim_{\varepsilon\to 0}\varepsilon^{-\beta^2 k}\int_{D^{\otimes 2k}} e^{-\frac{\beta^2}{2}\mathrm{Var}\left(\sum_{i=1}^k h_r^\varepsilon(x_i)-\sum_{i=1}^k h_r^\varepsilon(y_i)\right)}d\vec{x}d\vec{y} \\
= & \lim_{\varepsilon\to 0}\int_{D^{\otimes 2k}}\prod_{i=1}^k\left(\frac{1}{C\left(x_i;D_r\right)C\left(y_i;D_r\right)}\right)^{\beta^2/2}\left(\frac{\prod_{1<i<j<k}e^{-g_r^\varepsilon(x_i,x_j)}e^{-g_r^\varepsilon(y_i,y_j)}}{\prod_{i,j}e^{-g_r^\varepsilon(x_i,y_j)}}\right)^{\beta^2}d\vec{x}d\vec{y},
\end{aligned}$$

where we applied (6.8) to obtain the last equation. We now show that the limit above equals

$$\begin{aligned}
& \int_{D^{\otimes 2k}}\prod_{i=1}^k\left(\frac{1}{C\left(x_i;D_r\right)C\left(y_i;D_r\right)}\right)^{\beta^2/2}\left(\frac{\prod_{1<i<j<k}e^{-g_r(x_i,x_j)}e^{-g_r(y_i,y_j)}}{\prod_{i,j}e^{-g_r(x_i,y_j)}}\right)^{\beta^2}d\vec{x}d\vec{y} \\
= & \int_{D^{\otimes 2k}}\left(\frac{\prod_{1<i<j<k}\left|x_i-x_j\right|\left|y_i-y_j\right|}{\prod_{i,j}\left|x_i-y_j\right|}\right)^{\beta^2}F_r\left(\vec{x},\vec{y},\beta\right)d\vec{x}d\vec{y},
\end{aligned}$$

where

$$F_r\left(\vec{x},\vec{y},\beta\right)=\prod_{i=1}^k\left(1-\frac{|x_i|^2}{r^2}\right)^{-\frac{\beta^2}{2}}\left(1-\frac{|y_i|^2}{r^2}\right)^{-\frac{\beta^2}{2}}\left[\frac{\prod_{i,j}\left|1-x_i\bar{y}_j/r^2\right|}{\prod_{1<i<j<k}\left|1-x_i\bar{x}_j/r^2\right|\left|1-y_i\bar{y}_j/r^2\right|}\right]^{\beta^2}.$$

To show the equality of (6.8.1) and (6.8.2), given $\varepsilon>0$, set

$$D_\varepsilon=\left\{(\vec{x},\vec{y})\in D^{\otimes 2k}:\min_{i,j}\left|x_i-x_j\right|\geq 2\varepsilon,\min_{i,j}\left|y_i-y_j\right|\geq 2\varepsilon,\min_{i,j}\left|x_i-y_j\right|\geq 2\varepsilon\right\}.$$

For $(\vec{x},\vec{y})\in D_\varepsilon$, the integrand in ((6.8.1)) and ((6.8.2)) coincide. Since the integrand in ((6.8.2)) has integrable singularities when $\beta\in\left(0,\sqrt{2}\right)$, it suffices to prove

$$\lim_{\varepsilon\to 0}\int_{D^{\otimes 2k}\setminus D_\varepsilon}\prod_{i=1}^k\left(\frac{1}{C\left(x_i;D_r\right)C\left(y_i;D_r\right)}\right)^{\beta^2/2}\left(\frac{\prod_{1<i<j<k}e^{-g_r^\varepsilon(x_i,x_j)}e^{-g_r^\varepsilon(y_i,y_j)}}{\prod_{i,j}e^{-g_r^\varepsilon(x_i,y_j)}}\right)^{\beta^2}d\vec{x}d\vec{y}=0.$$

Given $(x_1,...,x_k),(y_1,...,y_k)\in\mathbb{R}^k$, one can define the Gale-Shapley matching $\sigma\left(\vec{x},\vec{y}\right)\in S_k$ of $\vec{x}$ with $\vec{y}$, by

1. Find x_i and y_j, such that

$$\forall i',j',\ \left|x_i-y_j\right|<\left|x_i-y_{j'}\right|\ \text{ and }\ \left|x_i-y_j\right|<\left|x_{i'}-y_j\right|,$$

   and set $\sigma\left(i\right)=j$.
2. Delete the points that have been matched in Step 1.
3. Iterate the procedure until all the points have been matched.

Set

$$B_\varepsilon = (D^{\otimes 2k} \backslash D_\varepsilon) \cap \{(\vec{x}, \vec{y}) : \sigma(\vec{x}, \vec{y}) = \mathrm{Id}\}.$$

And by symmetry, it suffices to prove the integration over B_ε vanishes as $\varepsilon \to 0$.

Using the properties of $g_r^\varepsilon(\cdot,\cdot)$, we have

$$\sup_{\varepsilon < e^{-C_1}} \sup_{\substack{x,y,z \in D \\ |x-y| \le 2|x-z|}} \frac{e^{-g_r^\varepsilon(x,y)}}{e^{-g_r^\varepsilon(x,z)}} < \infty.$$

Therefore, by the same argument as [LRV15, Lemma A.2], for $(\vec{x}, \vec{y}) \in B_\varepsilon$ we have the upper bound

$$\left(\frac{\prod_{1<i<j<k} e^{-g_r^\varepsilon(x_i,x_j)} e^{-g_r^\varepsilon(y_i,y_j)}}{\prod_{i,j} e^{-g_r^\varepsilon(x_i,y_j)}}\right)^{\beta^2} \le C(k,\beta) \prod_{i=1}^k e^{\beta^2 g_r^\varepsilon(x_i,y_i)},$$

thus

$$\begin{aligned} &\int_{B_\varepsilon} \prod_{i=1}^k \left(\frac{1}{C(x_i; D_r) C(y_i; D_r)}\right)^{\beta^2/2} \left(\frac{\prod_{1<i<j<k} e^{-g_r^\varepsilon(x_i,x_j)} e^{-g_r^\varepsilon(y_i,y_j)}}{\prod_{i,j} e^{-g_r^\varepsilon(x_i,y_j)}}\right)^{\beta^2} d\vec{x} d\vec{y} \\ \le\ & C(k,\beta) \prod_{i=1}^k \int_{B_\varepsilon} \left(\frac{1}{C(x_i; D_r) C(y_i; D_r)}\right)^{\beta^2/2} e^{\beta^2 g_r^\varepsilon(x_i,y_i)} dx_i dy_i. \end{aligned}$$

Now, for $(\vec{x}, \vec{y}) \in B_\varepsilon$, if for some i, $|x_i - y_i| < 2\varepsilon$, then because $g_r^\varepsilon(x_i, y_i) \le -\log \varepsilon + O(1)$, the integrand in (6.8.3) is bounded by $O(\varepsilon^{-\beta^2})$. The volume of the point configuration is at most $O(\varepsilon^2)$, thus (6.8.3) has integrable singularities. Since the volume of B_ε goes to zero as $\varepsilon \to 0$, we conclude that (6.8.1) equals (6.8.2).

From the explicit expression of F_r, we see that $F_r(\vec{x}, \vec{y}, \beta) \to 1$ uniformly for all $(\vec{x}, \vec{y}) \in D^{\otimes 2k}$, as $r \to \infty$. Finally, noting Corollary 6.1.3, we can send $r \to \infty$ and apply the dominated convergence theorem to obtain

Lemma 6.8.1. *For $\beta \in (0, \sqrt{2})$, $k \in \mathbb{N}$, and $\left|\int_D e^{i\beta h(x)} dx\right|$ defined in the sense above, we have*

$$\mathbb{E}\left|\int_D e^{i\beta h(x)} dx\right|^{2k} = Z_{k,2\beta^2},$$

where $Z_{k,2\beta^2}$ is defined as (6.1.2).

Corollary 6.8.2. *For $\beta \in (0, \sqrt{2})$,*

$$\mathbb{P}\left(\left|\int_D e^{i\beta h(x)} dx\right| > x\right) = \exp\left(-c^*(\beta) x^{\frac{2}{\beta^2}} + o(x^{\frac{2}{\beta^2}})\right), \quad \text{as } x \to \infty,$$

and

$$c^*(\beta) = \beta^2 \exp\left(-1 + \beta^{-2} \inf_{\mathcal{P}_{inv,1}(\Lambda, \mathcal{X}^s)} \bar{\mathcal{F}}_\beta\right).$$

(See (6.1.8) for the definition of $\bar{\mathcal{F}}_\beta$.)

Proof. By Chebyshev's inequality and Lemma 6.8.1,

$$\log \mathbb{P}\left(\left|\int_D e^{i\beta h(x)} dx\right| > x\right) \le \log \frac{\mathbb{E}\left|\int_D e^{i\beta h(x)} dx\right|^{2k}}{x^{2k}} = \log Z_{k,2\beta^2} - 2k \log x.$$

Apply Corollary 6.1.3, choose k^* to optimize the right hand side while neglecting the $o(k)$ term in $\log Z_{k,2\beta^2}$, we obtain $k^* = \lfloor \beta^{-2} c^*(\beta) x^{\frac{2}{\beta^2}} \rfloor$, and

$$\mathbb{P}\left(\left|\int_D e^{i\beta h(x)} dx\right| > x\right) \leq \exp\left(-c^*(\beta) x^{\frac{2}{\beta^2}} + o(x^{\frac{2}{\beta^2}})\right).$$

For the lower bound, fix constants C_1, d_1, such that $C_1 d_1 + \int_{C_1}^{\infty} \exp\left(-c^*(\beta) x^{\frac{2}{\beta^2}}\right) dx = 1$. Let Y be the non-negative random variable whose p.d.f is given by

$$f(x) = \begin{cases} d_1 & \text{if } 0 \leq x \leq C_1 \\ \exp\left(-c^*(\beta) x^{\frac{2}{\beta^2}}\right) & \text{if } x > C_1 \end{cases}.$$

An explicit computation gives $\log \mathbb{E} Y^{2k} = \log \mathbb{E}\left|\int_D e^{i\beta h(x)} dx\right|^{2k} + o(k)$. Given $\delta > 0$, denote $Y_\delta = (1-\delta) Y^2$. Then there exists $k_0(\delta)$, such that for all $k > k_0(\delta)$, $\mathbb{E} Y_\delta^k \leq \mathbb{E}\left|\int_D e^{i\beta h(x)} dx\right|^{2k}$. Take $C_2 = C_2(\delta) < \infty$, such that for all $k \in \mathbb{N}$, $\mathbb{E}(Y_\delta - C_2)^k \leq \mathbb{E}\left|\int_D e^{i\beta h(x)} dx\right|^{2k}$. Therefore the tail of $\left|\int_D e^{i\beta h(x)} dx\right|$ dominates the tail of $Y_\delta - C_2$. Then for $x > \sqrt{C_1}$,

$$\begin{aligned} \mathbb{P}\left(\left|\int_D e^{i\beta h(x)} dx\right|^2 > x^2\right) &\geq \mathbb{P}\left(Y_\delta - C_2 > x^2\right) \\ &= \mathbb{P}\left(Y > \sqrt{\frac{x^2 + C_2}{1-\delta}}\right) \\ &= \exp\left(-c^*(\beta)\left(\frac{x^2 + C_2}{1-\delta}\right)^{1/\beta^2}\right). \end{aligned}$$

Taking x sufficiently large yields the lower bound. □

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Une noix, qu'y a-t-il à l'intérieur d'une noix ?
Qu'est-ce qu'on y voit, quand elle est ouverte ?
- Quand elle est ouverte ?
On n'a pas le temps d'y voir, on la croque et puis bonsoir,
On n'a pas le temps d'y voir, on la croque,
et puis bonsoir...
Les découvertes !

Charles Trenet, *Une noix.*

Thomas LEBLÉ **05/02/2016**

Sujet : Comportement microscopique de particules en interaction : gaz de Coulomb, Riesz et log-gases.

Résumé : Cette thèse est consacrée à l'étude de systèmes modélisant des particules chargées en interaction, ou les valeurs propres de matrices aléatoires. On s'intéresse aux gaz avec interaction logarithmique en dimension 1 et 2, et aux interactions de Coulomb/Riesz en dimension générale. On en décrit le comportement microscopique par le biais d'un principe de grandes déviations que satisfait la loi des champs empiriques et qui est gouverné par une fonctionnelle d'énergie libre dans laquelle apparaît la dépendance en la température. Parmi les minimiseurs de cette énergie libre, on compte les processus ponctuels Sine-beta définis dans le contexte des matrices aléatoires. On démontre la convergence des minimiseurs vers un processus de Poisson à haute température et, en dimension 1, on prouve la cristallisation du système dans la limite de basse température. Dans le cas des interactions logarithmiques en dimension 2, on montre une loi locale qui contrôle les fluctuations à toute échelle mésoscopique. On traite aussi le cas du gaz de Coulomb bi-dimensionnel avec des charges de signes opposés.

Mots clés : physique statistique, matrices aléatoires, log-gases, interactions de Riesz, sytèmes de Coulomb, grandes déviations, champs empiriques, cristallisation

Subject : Microscopic behavior of interacting particles: Coulomb, Riesz and log-gases

Résumé : This thesis is devoted to the study of statistical physics systems which can represent charged interacting particles or eigenvalues of random matrices. We are interested in gases with logarithmic interaction in dimension 1 and 2, or with Coulomb/Riesz interactions in general dimension. We describe their microscopic behavior by showing that the law of the empirical fields obeys a Large Deviation Principle governed by a free energy functional in which the temperature dependence appears. Minimizers of this free energy include the Sine-beta point processes defined in random matrix theory. We show the convergence to a Poisson point process at high temperature and in dimension 1 we prove crystallization in the zero temperature limit. For two-dimensional log-gases we establish a local law which bounds the fluctuations at any mesoscopic scale. We also treat the case of a two-dimensional Coulomb gas with charges of opposite sign.

Keywords : statistical physics, random matrices, log-gases, Riesz interactions, Coulomb sytems, large deviations, empirical fields, crystallization