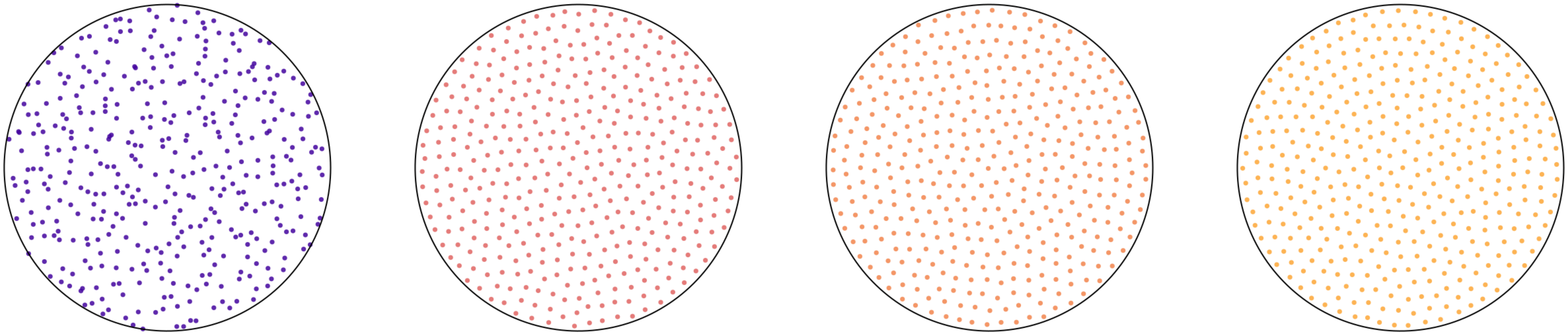

Random point configurations in statistical physics

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Point configurations: from statistical physics to potential theory (CIRM, May 2026)

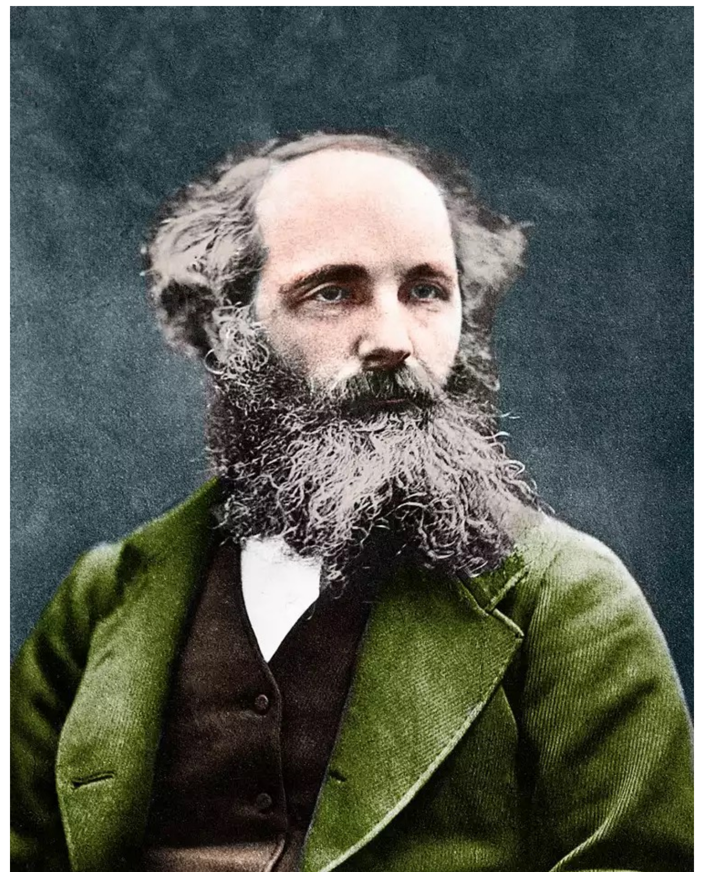
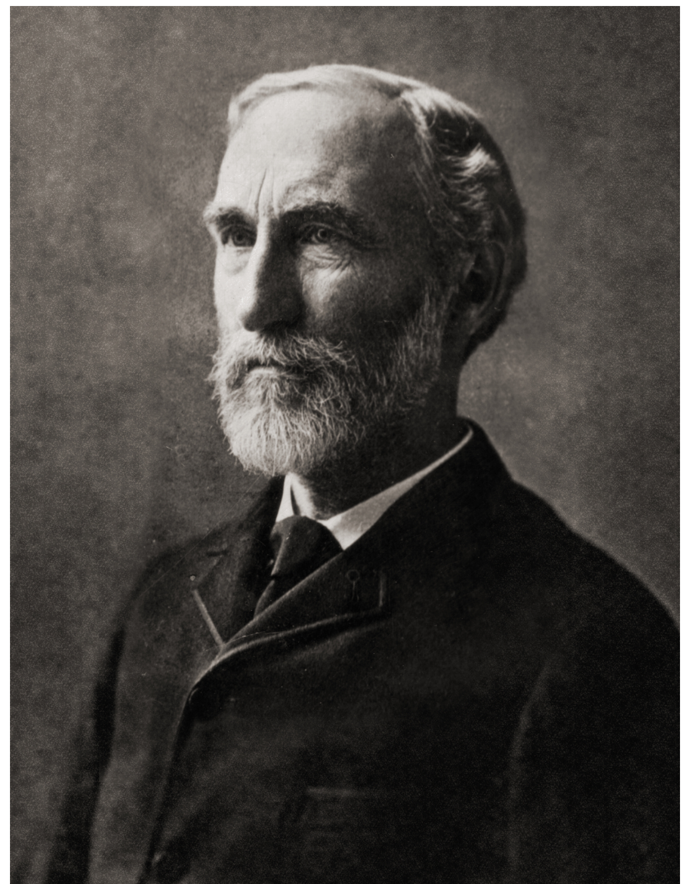
Statistical physics

- Statistical physics = physics of many, *many* elements in interaction
  e.g. H_2O in a pot, molecules in the air of the room…

- Very large number N of elements (1 mole = 10^{23})… almost « infinite » !

- Developed in late 19th / early 20th century (Maxwell, Boltzmann, Gibbs)

- *Thermodynamics* was phenomenological, *statistical mechanics* is reductionist

A reductionist approach?

Newton
Deterministic

Thermal agitation →

Langevin
Random

↓ Invariant measure

Gibbs measure
Microscopic
Random

Large scale behavior →

Phases
Macroscopic
Deterministic

Random distributions (*ensembles*)

Ingredients of a model

1. *State* space S (all possible states of the system)

   e.g. $S = (\mathbb{R}^d)^N$ for N particles in $\mathbb{R}^d$

2. Reference *measure* μ

   e.g. $\mu =$ Lebesgue, or Riemannian volume on manifold

3. *Energy* functional $H : S \to (-\infty, +\infty]$

   e.g. $H =$ sum of all interactions between particles

4. *Inverse temperature* parameter $\beta > 0$

   sets the *strength* of interactions

The basic recipe

The state $\mathbf{X} \in S$ is *random* and distributed as:

$$\mathrm{d}\mathbb{P}(\mathbf{X}) = \frac{1}{Z} e^{-\beta H(\mathbf{X})} \mathrm{d}\mu(\mathbf{X})$$

Gibbs measure

Boltzmann's factor

Partition function

$$Z = \int_S e^{-\beta H(\mathbf{X})} \mathrm{d}\mu(\mathbf{X})$$

Microstates, macrostates

- $\mathbf{X} \in S$ a *microstate* = a description of the system at microscopic scale
  e.g. **all** the positions and **all** the velocities of **all** the molecules in the room
  precise but not very *useful*

- *Observable* $\mathcal{O} : S \to Y$ = function of the state, preferably of *macroscopic* nature
  e.g. "the **average** kinetic energy of the molecules in **this corner** of the room"

- *Macrostate* = "all the microstates $\mathbf{X}$ such that $\mathcal{O}(\mathbf{X}) =$ a certain value".

- Goal: understand typical values of chosen $\mathcal{O}$, namely *typical macrostates*

Typical ?? Under the Gibbs measure!

Gibbs measure = distribution of *microstates*

$$\mathrm{d}\mathbb{P}(\mathbf{X}) = \frac{1}{Z} e^{-\beta H(\mathbf{X})} \mathrm{d}\mu(\mathbf{X})$$

Law of *observable*

$$\mathbb{P}\left(\mathcal{O}(\mathbf{X}) = \lambda\right) = \frac{1}{Z} \int_S e^{-\beta H(\mathbf{X})} \mathbf{1}_{\mathcal{O}(\mathbf{X})=\lambda} \mathrm{d}\mu(\mathbf{X})$$

Energy : from "$\mathcal{O}(\mathbf{X}) = \lambda$", can I say that $H(\mathbf{X}) \approx \tilde{H}_\lambda$?

$$\mathbb{P}\left(\mathcal{O}(\mathbf{X}) = \lambda\right) \approx \frac{1}{Z} e^{-\beta \tilde{H}_\lambda} \int_S \mathbf{1}_{\mathcal{O}(\mathbf{X})=\lambda} \mathrm{d}\mu(\mathbf{X})$$

Can I compute the *volume* $\int_S \mathbf{1}_{\mathcal{O}(\mathbf{X})=\lambda} \mathrm{d}\mu(\mathbf{X}) = \mu\left(\mathcal{O}(\mathbf{X}) = \lambda\right)$?

$$\mathbb{P}\left(\mathcal{O}(\mathbf{X}) = \lambda\right) \approx \frac{1}{Z} e^{-\beta \tilde{H}_\lambda} \times \mu\left(\mathcal{O}(\mathbf{X} = \lambda)\right)$$

$$\mathbb{P}\left(\mathcal{O}(\mathbf{X}) = \lambda\right) = \frac{1}{Z} e^{-\beta \tilde{H}_\lambda + \log \mu\left(\mathcal{O}(\mathbf{X}) = \lambda\right)}$$

Free energy

$$\mathbb{P}\left(\mathcal{O}(\mathbf{X}) = \lambda\right) \approx \frac{1}{Z} e^{-\beta \tilde{H}_\lambda + \log \mu\left(\mathcal{O}(\mathbf{X}) = \lambda\right)}$$

Most likely values λ are those for which $\beta \tilde{H}_\lambda - \log \mu\left(\mathcal{O}(\mathbf{X}) = \lambda\right)$ is as **small** as possible

Free energy: "$\beta \times$ Energy – logarithmic volume (Entropy)"

The **minimization** problem clearly depends on β (the inverse temperature)

Does it depend on β in a "dramatic" way? *Phase transition?*

Entropy = logarithmic volume

Entropy

Boltzmann's constant

Macrostate volume
Wahrscheinlichkeit

Models

Lattice models

- **Ising model** on a lattice Λ (+ many related models)
  State space = $\{-1, +1\}^{\Lambda}$ *"spins"*
  Energy = alignement vs. non-alignement of neighboring spins
  Observable = magnetization

Extremely well-studied
Non-trivial geometric questions (e.g. describe the +1/-1 regions)
…but not very interesting in terms of "point configurations"

Continuum models

- Particles $x_1, \ldots, x_N$ in physical space (a domain of $\mathbb{R}^d$, a manifold…)

- Interactions between each pair of particles via a *potential* φ
  $$Energy = \sum_{i \neq j} \varphi(x_i - x_j)$$

- Introduce the Gibbs measure = law of a **random point configuration in space !**
  + Choose an observable characterizing order / disorder.

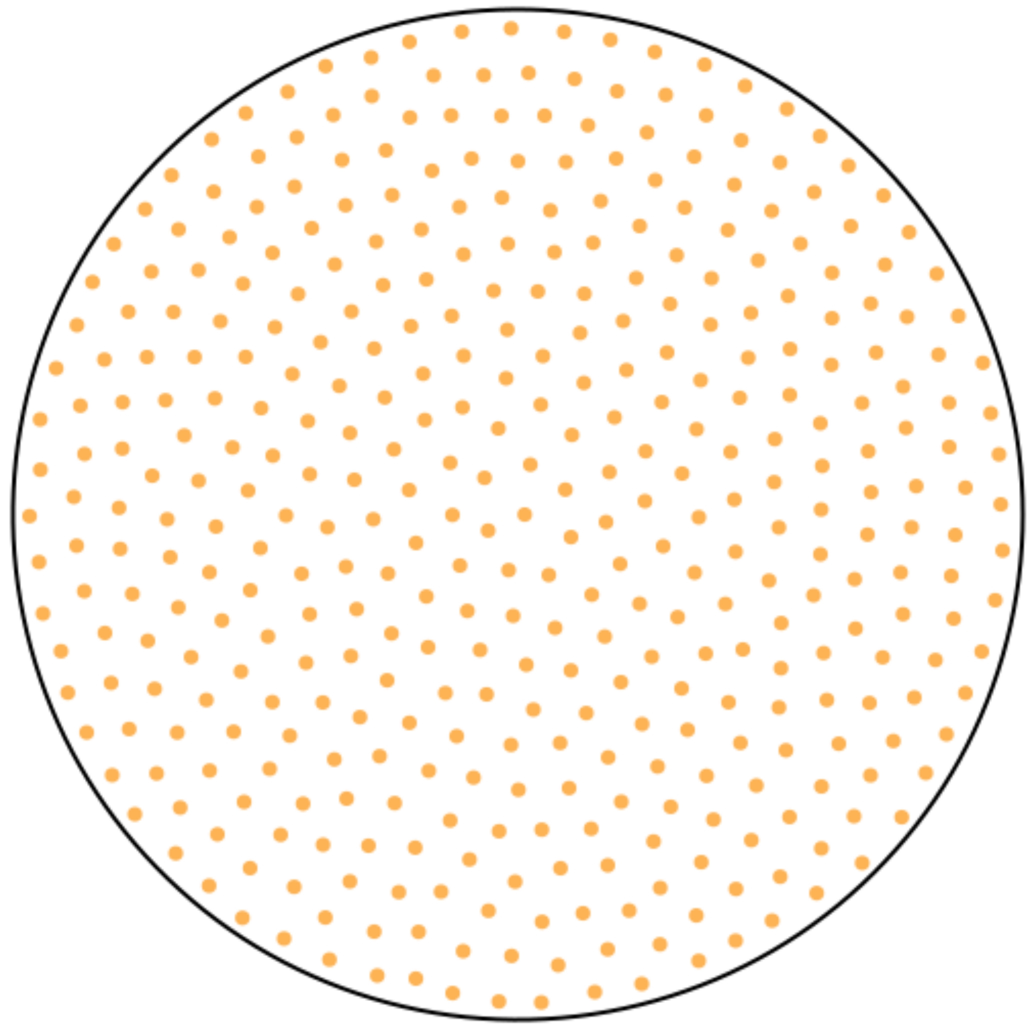
- Existence of *phase transitions* in the **continuum** is on B. Simon's 1984 list of « Fifteen problems in mathematical physics »

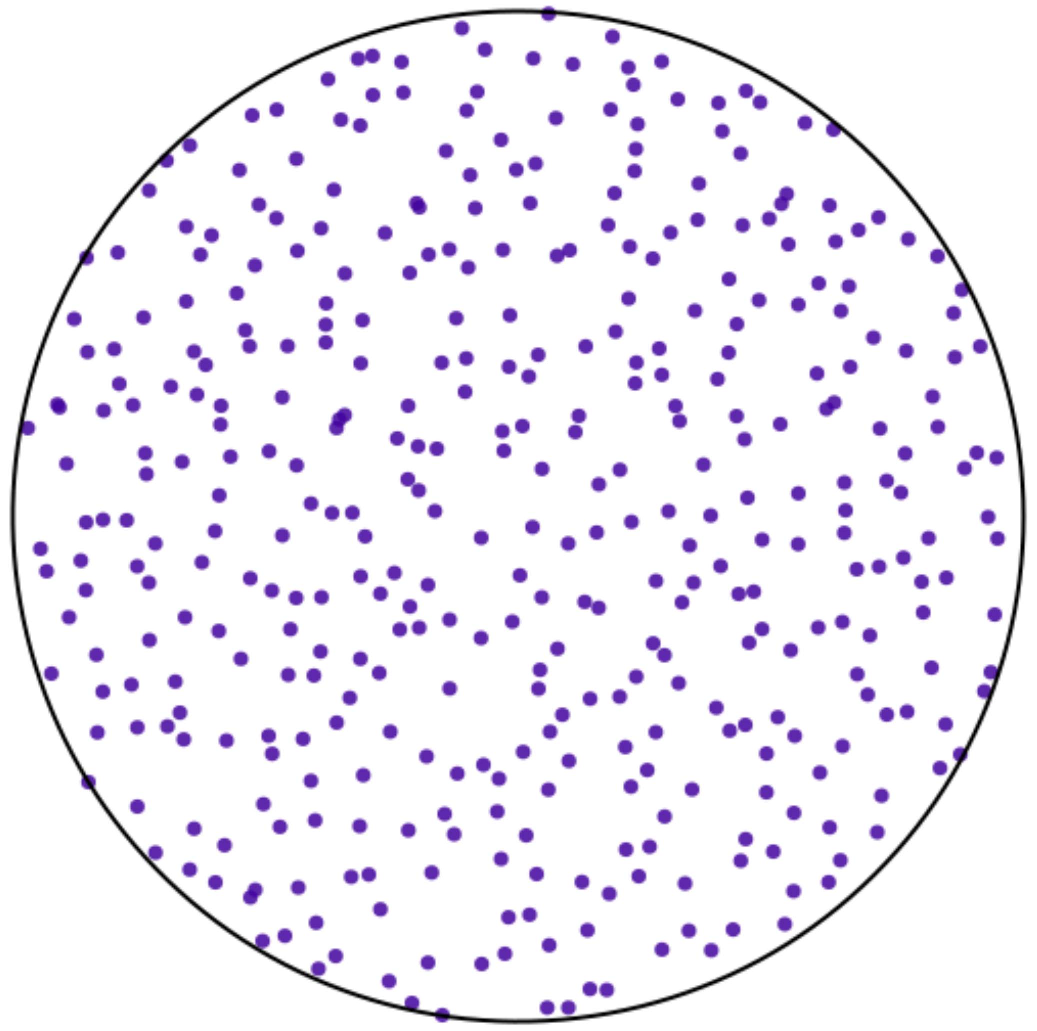
Goal: phase diagram

Will depend on dimension d and on interaction φ

$T = 0, \beta = +\infty$ — $T = +\infty, \beta = 0$

Lattice (?) | *Order* | *Transition?* | *Disorder* | **Independent particles**





$$Energy = \sum_{i \neq j} \varphi(x_i - x_j),$$ who is φ ?

Riesz interactions

Power-Law $\varphi(x) = \dfrac{1}{|x|^s}$, *Riesz potential*

In dimension d :

$s > d$ short-range

$s < d$ long-range

$s = d - 2$ Coulomb

$s = 0 \leftrightarrow \varphi(x) = -\log|x|$

Expect **major differences** between short- and long-range
e.g. when $s < d$, the energy ceases to be (almost-)additive w.r.t. space

Other interactions

- Lennard-Jones $\dfrac{1}{|x|^{12}} - \dfrac{1}{|x|^{6}}$ *non-monotonous*
- Gaussian interaction (*Gaussian core model*)
- Hardcore interactions ($\varphi(x) = +\infty$ if $|x| \leq R_0$)
- A general interaction + assumptions on *regularity and range* (60's…)

$$d\mathbb{P}_{N,\beta}(x_1, \ldots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \sum_{i<j} \varphi |x_i - x_j|} d\mu(x_1)\ldots d\mu(x_N)$$

Finite volume

Gaussian β-ensemble

RMT origin: take a $N \times N$ Hermitian matrix with Gaussian coefficients

Joint distribution of eigenvalues is computable **Dyson 60's**

$\rightarrow$ Gibbs measure with logarithmic interaction… on the line…at $\beta = 2$

$$d\mathbb{P}_{N,\beta}(x_1, \ldots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \sum_{i<j} -\log|x_i - x_j|} d\mu(x_1)\ldots d\mu(x_N)$$

μ is some Gaussian measure on the real line

Existence of a random matrix model for all $\beta > 0$ (Dumitriu-Edelman '00)

Called β-ensemble, GβE or ***one-dimensional log-gas***

Circular β-ensemble

Take a uniform $N \times N$ unitary matrix
Joint distribution of eigenvalues is computable

→ Gibbs measure with logarithmic interaction… on the circle…at $\beta = 2$
Again: RMT model for general β

$$d\mathbb{P}_{N,\beta}(x_1, \ldots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \sum_{i<j} -\log|x_i - x_j|} d\mu(x_1)\ldots d\mu(x_N)$$

μ **uniform measure on the unit circle**

Called CβE or *one-dimensional* ***periodic*** *log-gas*

Two-dimensional log-gas

Take a $N \times N$ matrix with Gaussian coefficients (no symmetry)

Joint distribution of eigenvalues is computable

→ Gibbs measure with logarithmic interaction… in the plane, at $\beta = 2$

$$d\mathbb{P}_{N,\beta}(x_1, \ldots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \sum_{i<j} -\log|x_i - x_j|} d\mu(x_1)\ldots d\mu(x_N)$$

μ **is some Gaussian measure on the plane**

General β ? No RMT interpretation, but **existence in statistical physics**.

Called (one-component) **Coulomb gas** or **one-component plasma**

Quick summary

- Gaussian β-ensemble/1d log-gas: finite random point configuration on the **line**
- Circular β-ensemble: finite random point configuration on the unit **circle**
- 2d log-gas: finite random point configuration in the **plane**

(Those three have logarithmic interaction, and some link with RMT)

- Riesz gases in arbitrary dimension: short/long range, Coulomb case $s = d - 2$
- General potential φ: finite random point configuration in **dimension** d

Some friends

2d log-gas
= Particles + logarithmic interaction in $\mathbb{R}^2$
= Random *complex* eigenvalues ($\beta = 2$)

1d log-gas
= Particles + logarithmic interaction in $\mathbb{R}$
= Random *real* eigenvalues ($\beta > 0$)

Random polynomials and Gaussian entire functions

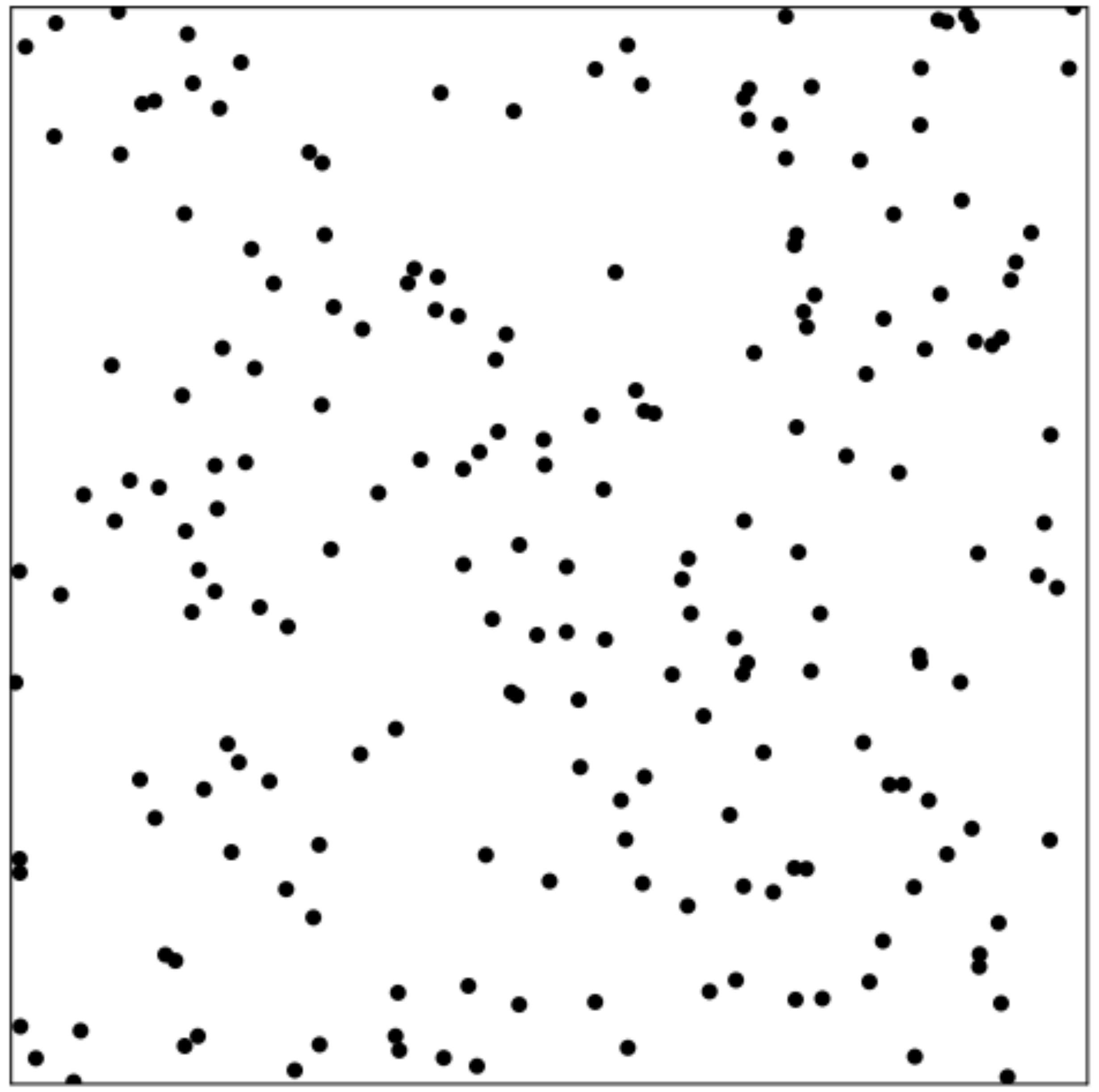
Take $(a_k)_{k\geq 0}$ independent complex Gaussian coefficients, and form:

$$f(z) := \sum_{k\geq 0} \frac{a_k}{\sqrt{k!}} z^k$$

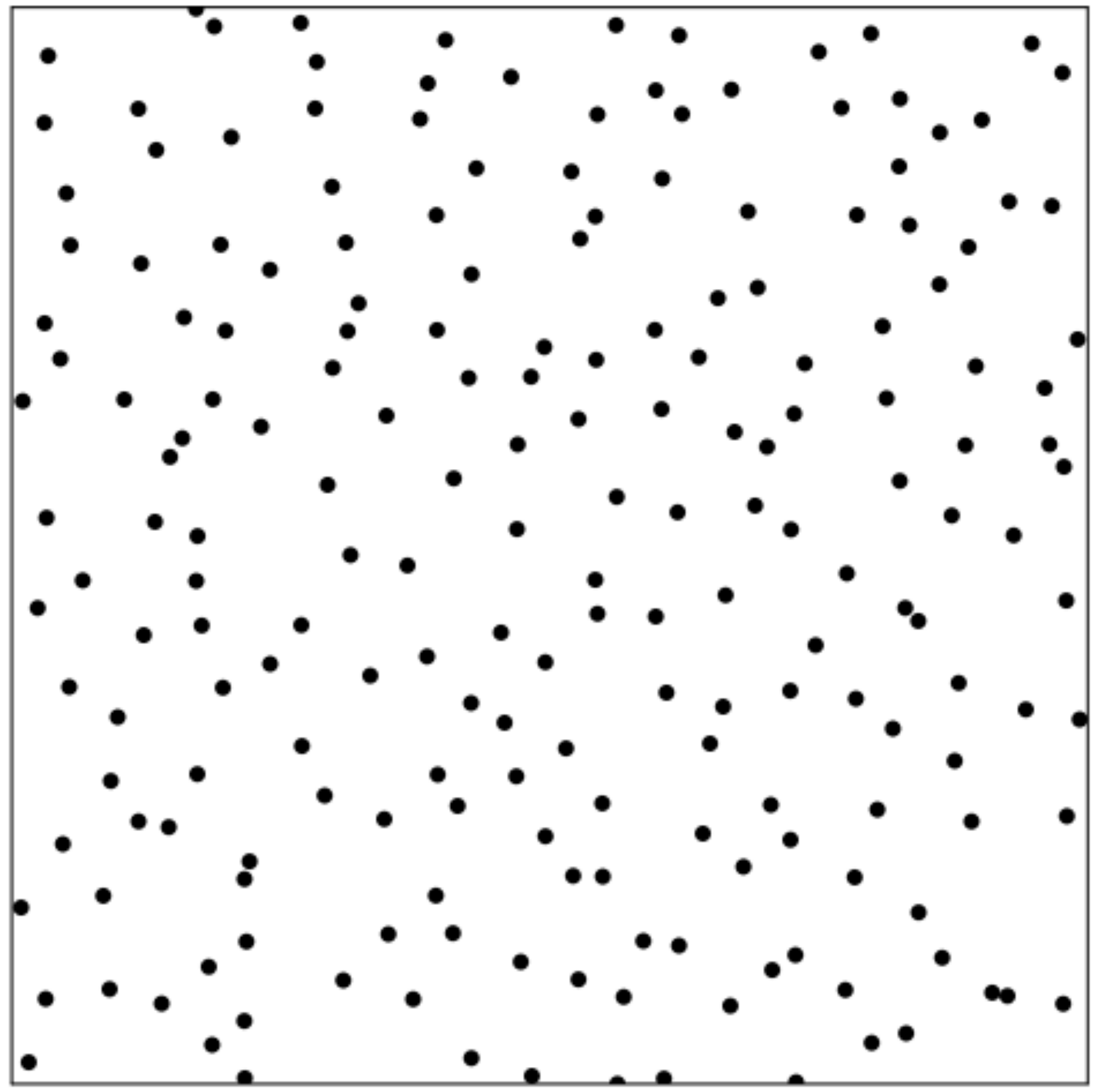
$$P_N(z) := \sum_{k=0}^{N} a_k z^k$$

Gaussian Entire Function (GEF)
Zeroes of f share many similarities with a 2d log-gas

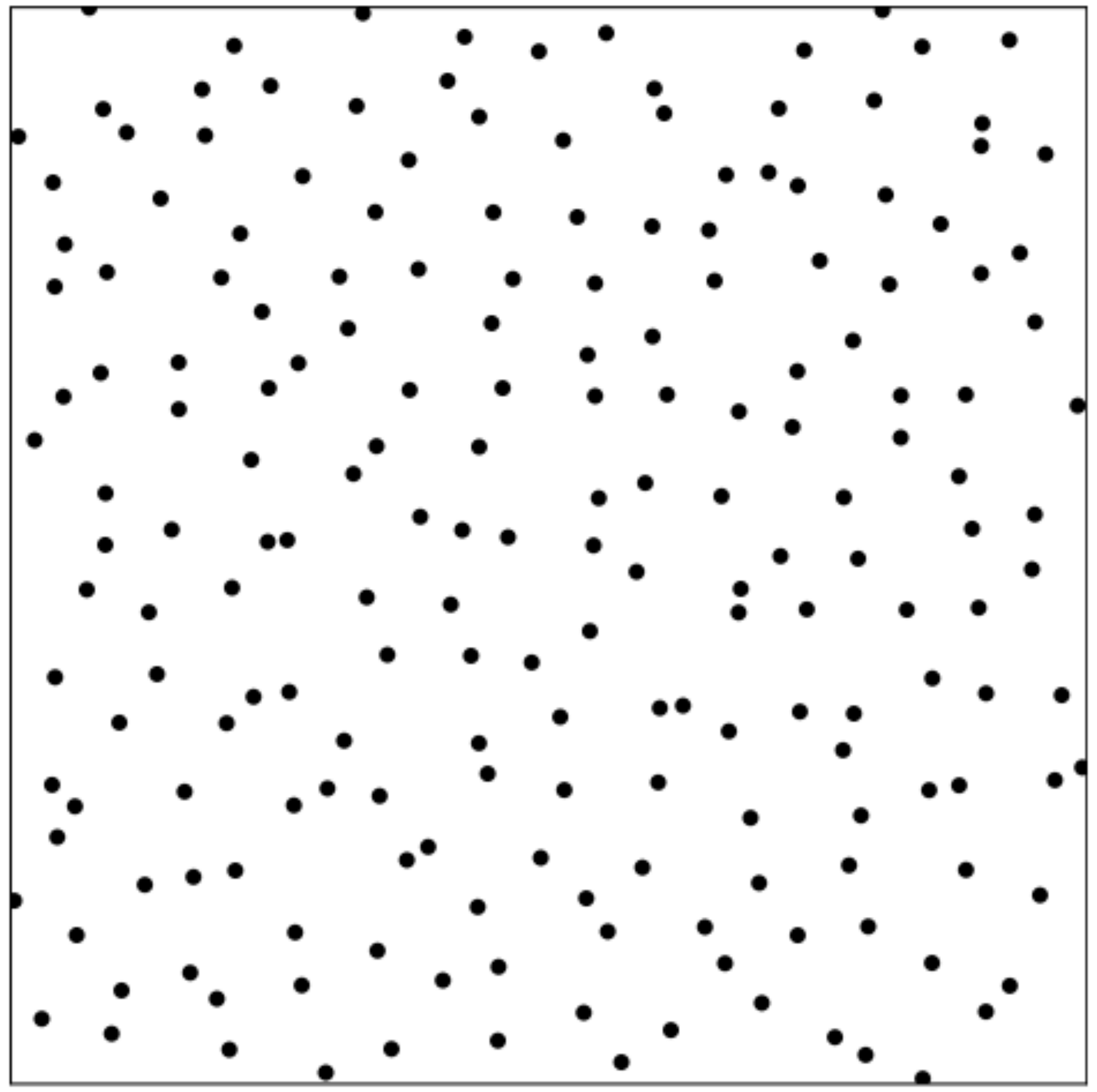
Gaussian Kac Polynomial
Zeroes of P_N share many similarities with a 1d log-gas



Independent points

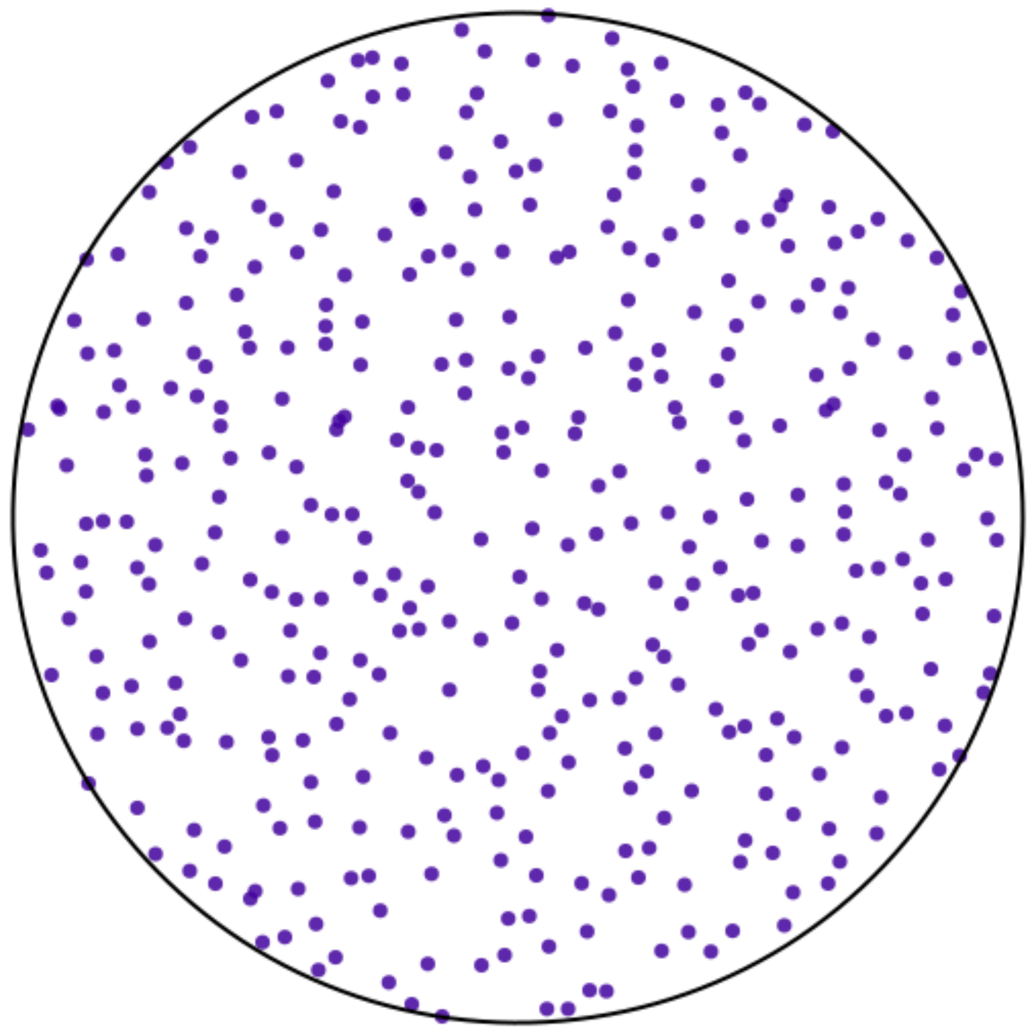


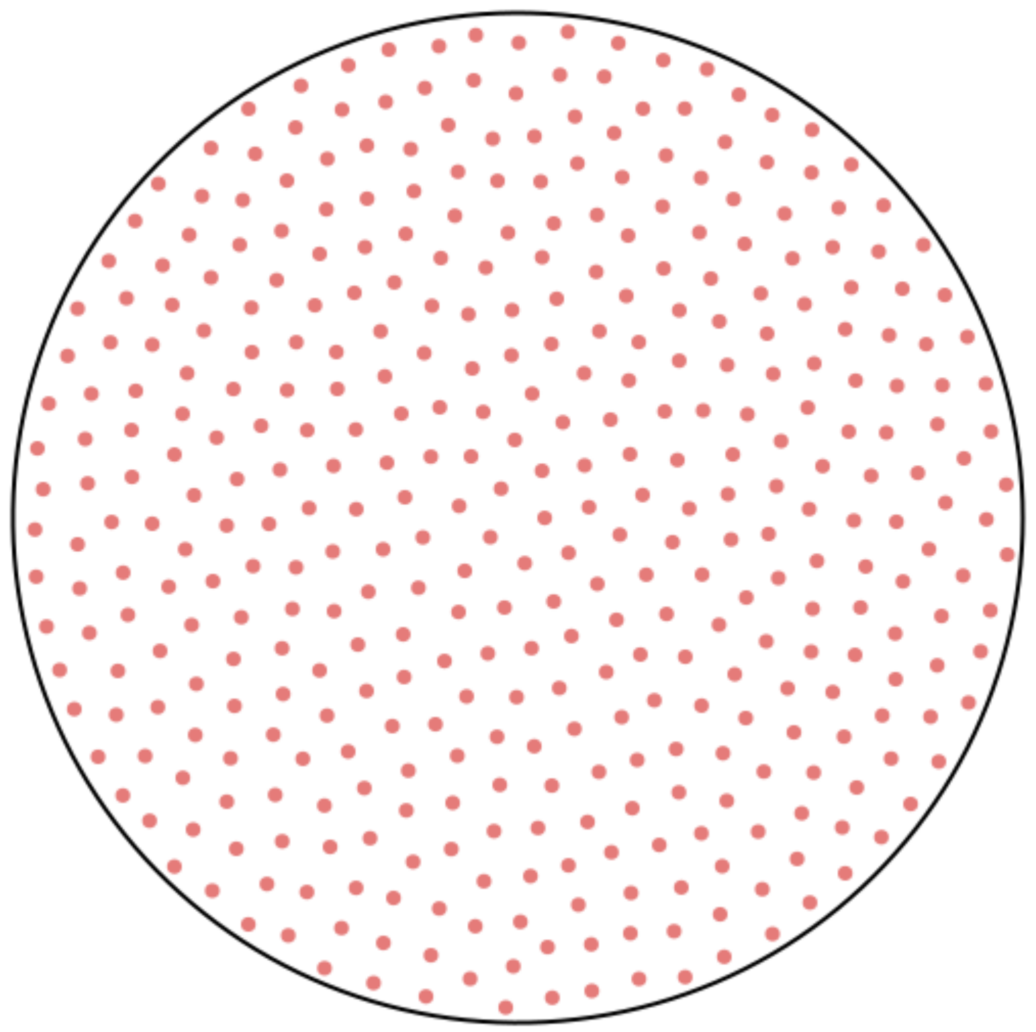
Ginibre ensemble (2DOCP)

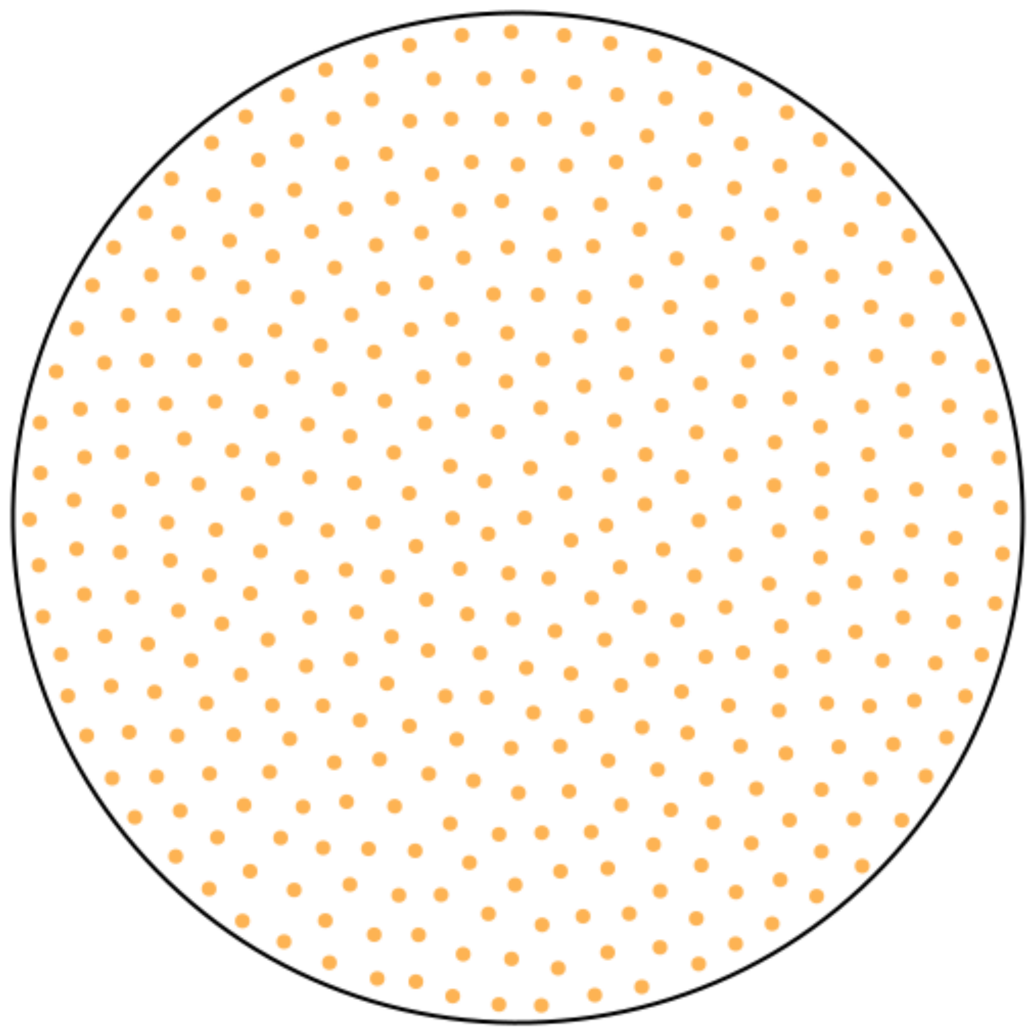


Zeroes of GEF

β small







β large

A 2d log-gas

Infinite-volume limit

- We would like to let $N \to \infty$, obtain **infinite** random point configuration
- Existence of a limit / limit points is hard in general! *Uniqueness matters*.
- Known:
  For β-ensembles (Gaussian and Circular): **Sine-β point process** (Valkó-Virág '09)
  For 2d log-gas at $\beta = 2$: **Ginibre point process** ('64), general β is open
  For some long-range 1d Riesz gases: **Riesz-β point process** (Boursier '21)

Instead of studying the Gibbs measure $\mathbb{P}_{N,\beta}$ as $N \to \infty$, study the
infinite-volume variational problem of minimizing free energy?

Free energy minimization

Dimension d

Config = point configurations in $\mathbb{R}^d$

$\mathscr{P}(\text{Config})$ = probability measures on Config

Given the interaction potential φ, define (??) an *energy density* $H_\varphi : \text{Config} \to \mathbb{R}$

Minimize $\beta \, \mathbb{E}_{\mathrm{P}}[H_\varphi] + \mathscr{E}(\mathrm{P})$ over $\mathscr{P}(\text{Config})$

Average energy density under P

Entropy ***density*** **of** P

In classical cases, this describes the infinite-volume limit "equilibrium" state(s)

Random point configurations

This leads us to study *random point configurations* in some generality:

- Because one can hope to solve the free energy minimization problem
- To derive good notions of order and disorder
- Because they might be *good candidates for deterministic questions*!
- Because it's fun? *Stochastic geometry*.

Some classes and operations

- Determinantal point processes (DPP): correlation functions expressed through a *determinant* $\rho_n(x_1, \ldots, x_n) = \det(K(x_i, x_j))$ *Explicit computations!* Examples: 1d log-gas at $\beta = 2$, 2d log-gas at $\beta = 2$
- Perturbed lattices: move lattice points at random. *Large class*!
- **Gibbs** p.p.: have a local "density" of the form $e^{-\beta H}$ for some *energy* H
- Operations: superposition/thinning, mixture. Displacement interpolation?

Order / Disorder for random point configurations

Order through correlations

1-point correlation $\rho_1(x)$ = "probability" of having a particle at x
2-point $\rho_2(x, y)$ = "probability" of having a particle at x **and** a particle at y

Decay of correlations ?

$\rho_2(x, y) - \rho_1(x)\rho_1(y) \to 0$ as $|x - y| \to \infty$

Exponential (« fast ») decay = "disorder"
Polynomial (« slow ») decay = "quasi-order"
No decay = "order"

Repulsion ?

$\rho_2(x, y) \ll \rho_1(x)\rho_1(y)$ for $|x - y| \ll 1$

Determinantal point processes are repulsive

But: correlations are hard

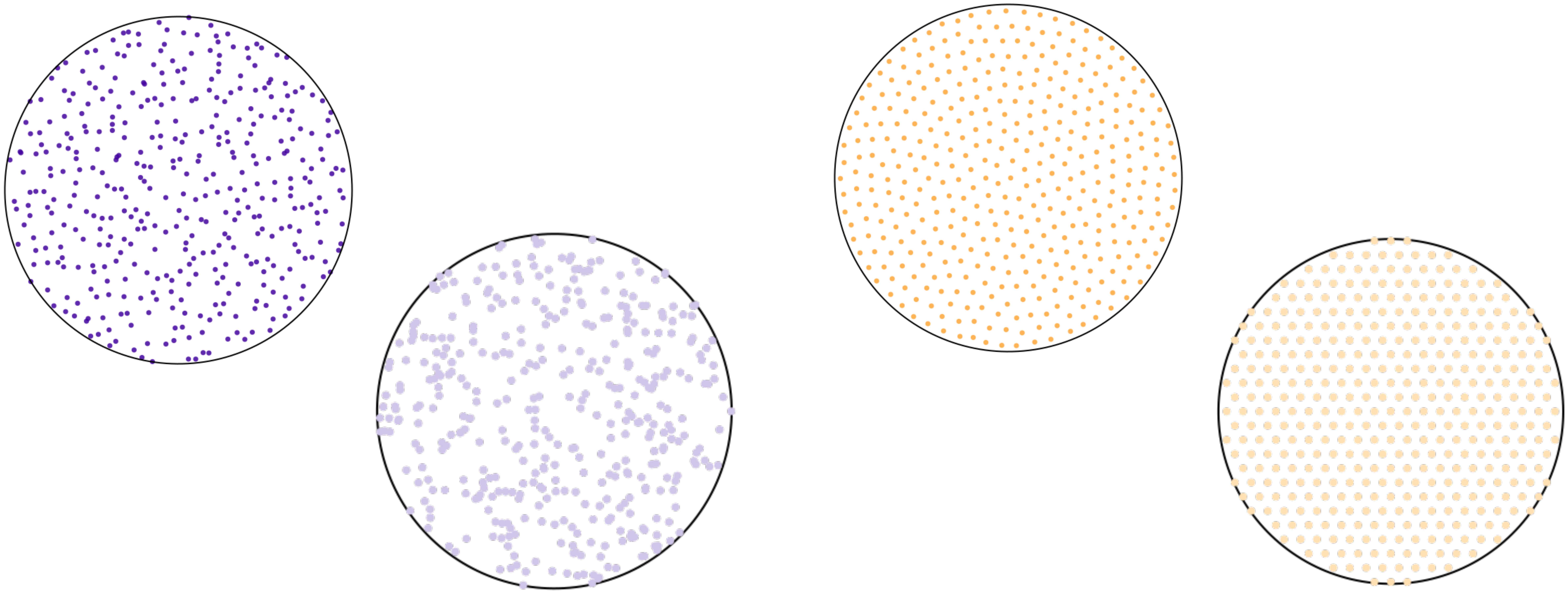
- Correlation functions - or simply their decay - are *very hard to compute*
- **Recent progress for the 1d log-gas**
- For the 2d log-gas, only known case for 2-point correlation is $\beta = 2$:

  decay is $e^{-|x-y|^2}$ for $|x-y| \gg 1$ (very fast!)

  vanishes like $|x-y|^2$ for $|x-y| \ll 1$ (*repulsion*)

Moreover, some regularity of those systems is **not** captured
by the usual “decay of correlations” criterion
One sometimes speaks of *order within disorder*

Other faces of order



1. Size of fluctuations 2. Rigidity with respect to the exterior 3. Transportation cost

Size of *fluctuations*

Take a big ball of radius $R \gg 1$ and count **how many points fall into it**.
What is the ***variance*** of this quantity (measures "*charge fluctuations*") ?
For **independent** points: grows like R^d.
For a **lattice** with independent perturbations, or uniform shift: like R^{d-1}.

Physics literature: **Anything with variance** $o(R^d)$ **is interesting**.
Hyperuniform or *super-homogenous* systems. Torquato.

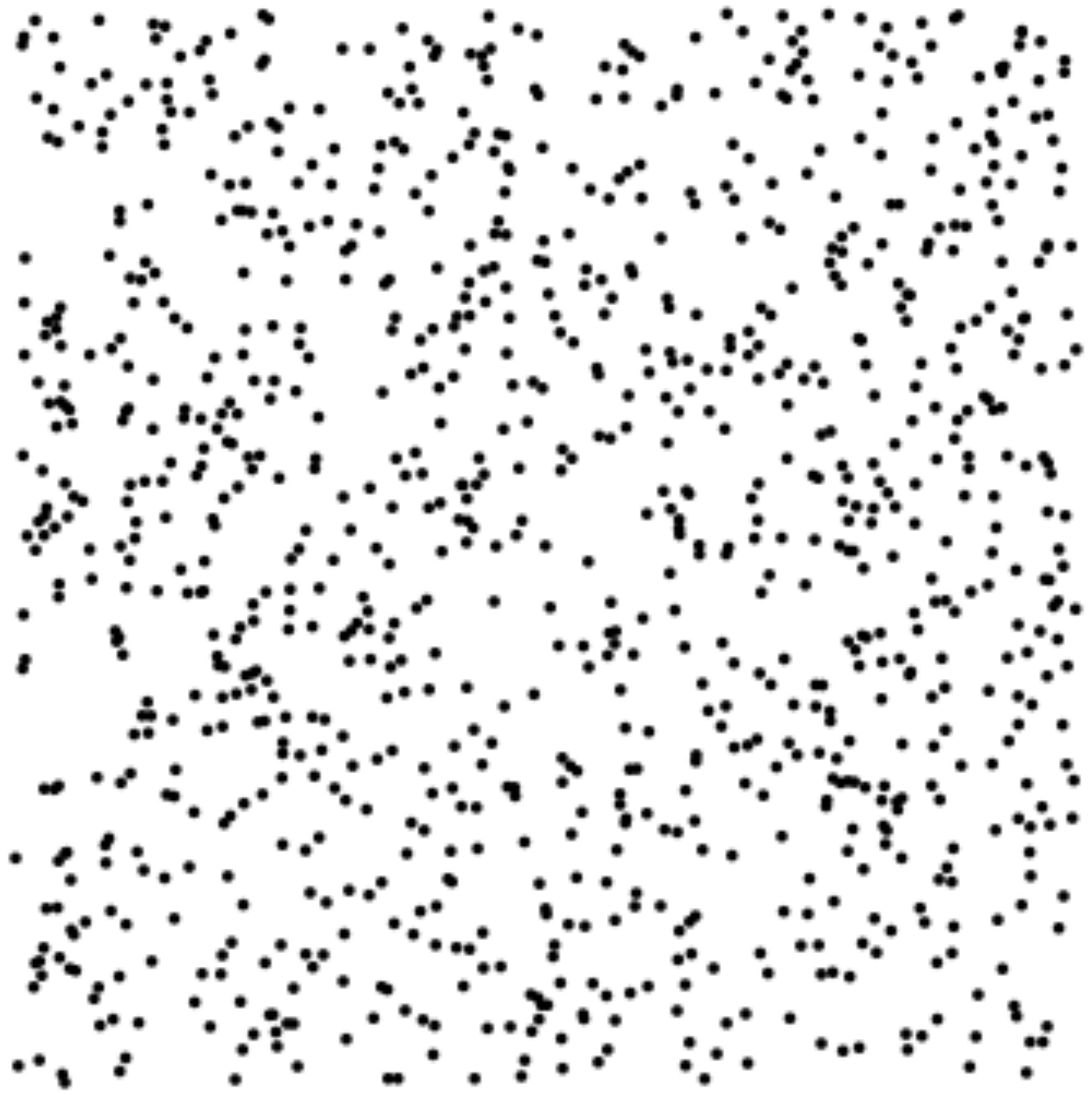
Instead of counting points, what about

$$\text{Variance}\left[\sum_{i=1}^{N} f(x_i)\right] \text{ for a nice function } f : \mathbb{R}^d \to \mathbb{R}?$$

Fluctuations of linear statistics

Rigidity properties

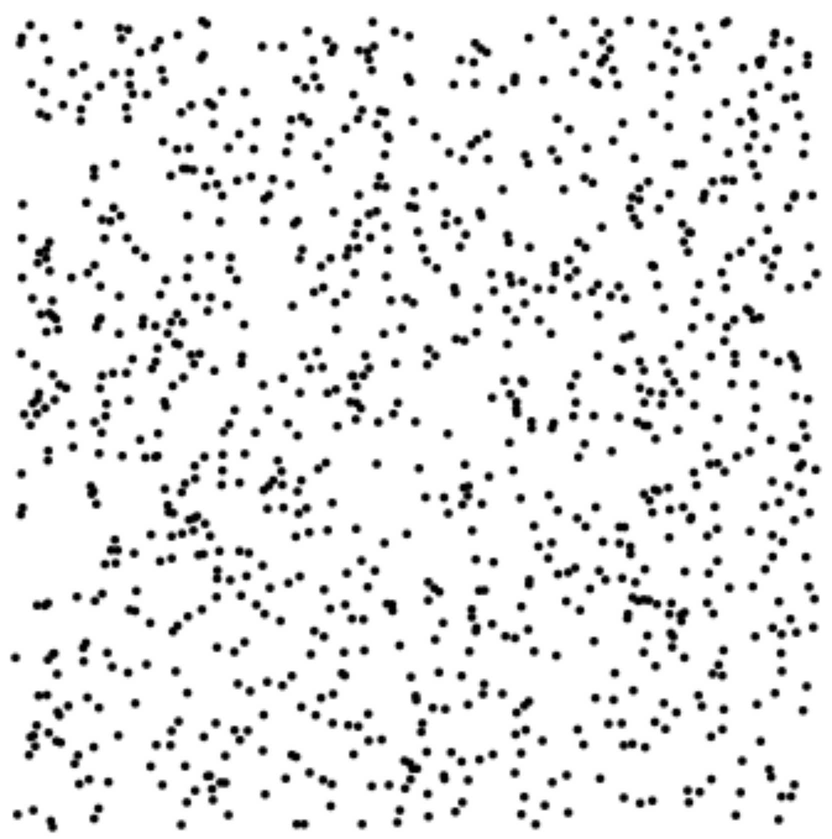
Ghosh-Peres '12



What can we say about the *inside* region from looking *outside*?

- (Almost surely) guess the **number** of points? *Number-rigid.*
- (Almost surely) guess the **center of mass**? *Barycenter-rigid*.
- (Almost surely) guess the full **configuration**? *Fully-rigid.*

Who is rigid?



- A Poisson point process (independent points)
  No rigidity
- A lattice (perfectly ordered)
  Fully-rigid

What about lattice + i.i.d. perturbations? Surprisingly subtle question **Peres-Sly '14**

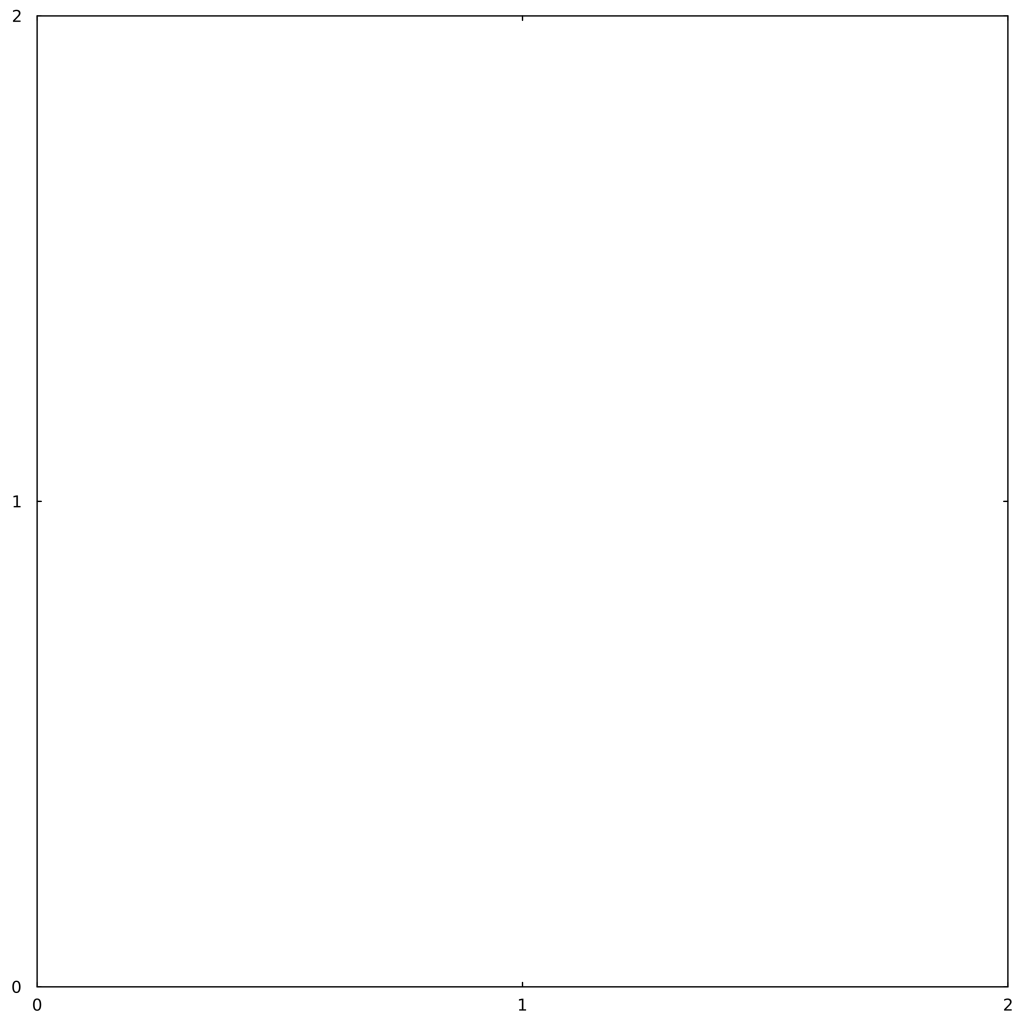
1d Log-gas **is** **Chhaibi-Najnudel '15**
Number-rigid $\forall \beta > 0$

Zeroes of GEF are
number-rigid **and** **Ghosh-Peres '12**
barycenter-rigid

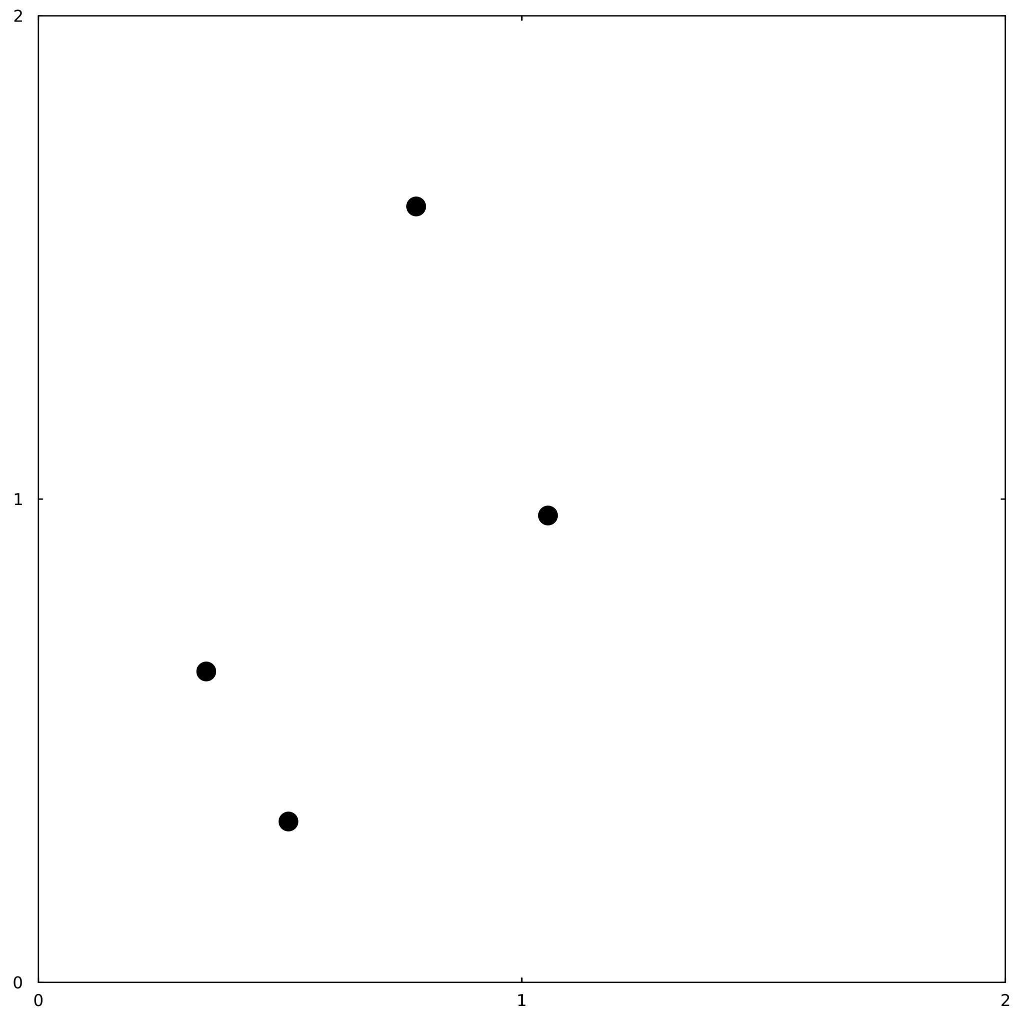
Some 1d Riesz gases
are not number-rigid
Dereudre-Vasseur '18
Some **are**
Dereudre-Digneaux '25

2d Log-gas **is**
Number-rigid $\forall \beta > 0$
L. '24

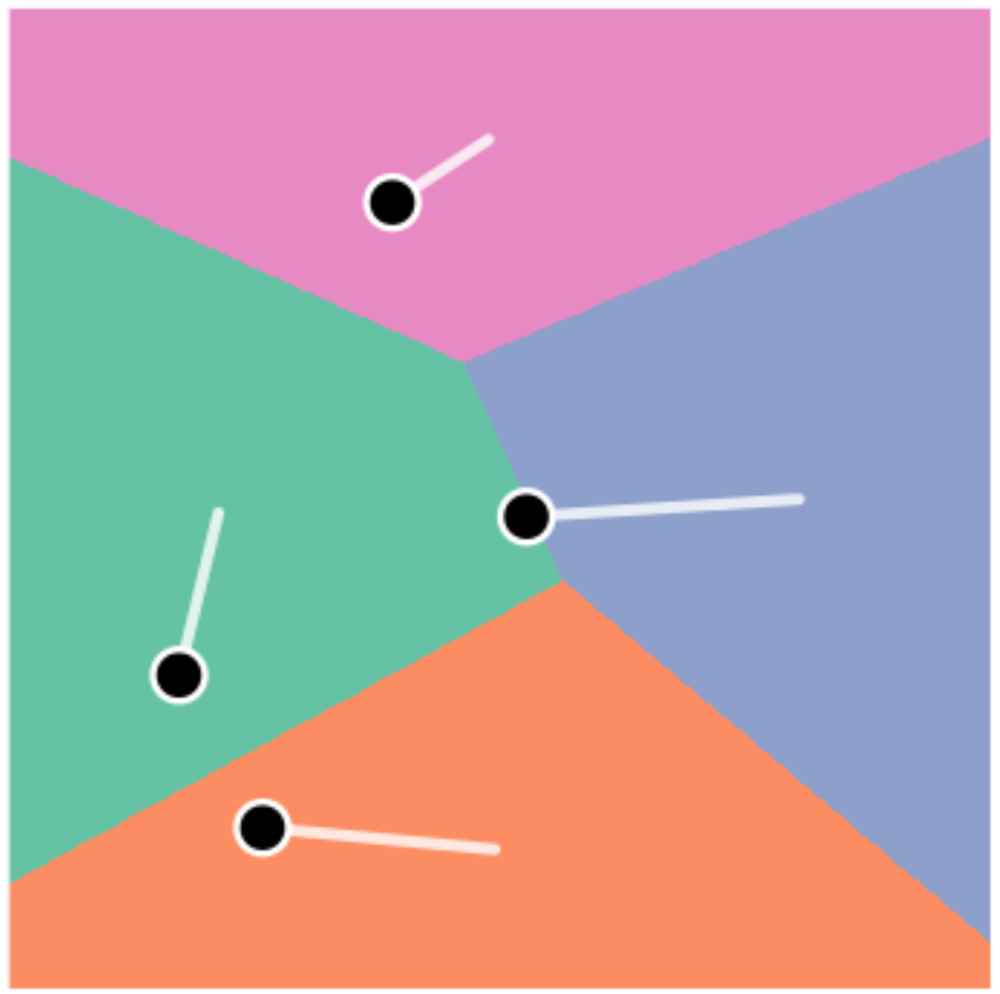
Transportation cost



Take a $L \times L$ square



Place L^2 points



Associate to each point x a **cell** C_x of area 1

Quadratic transportation cost for the x-cell: $\int_{C_x} |x - t|^2 \mathrm{d}t$

Transportation cost $\sum_x \int_{C_x} |x - t|^2 \mathrm{d}t$

What is the *minimal* cost?

Wasserstein "order" in $d = 1{,}2$

If the points are on a grid (*order*) the cost (per point) is **bounded**

Ajtai-Komlos-Tusnady '84

If points are drawn i.i.d. at random (*disorder*) average cost **diverges as** $L \to +\infty$ in $d = 1{,}2$.

In dimension 3 and above, this does not capture a meaningful difference between order and disorder

Summary of order / disorder

- "Classical" notion of order through correlation functions
- Some alternative notions: hyperuniformity, small fluctuations of linear statistics, rigidity *à la* Ghosh-Peres, Wasserstein distance to Lebesgue
- **Model-dependent** answers but few (if any) β-dependent answers yet.

Final word: the $\beta = +\infty$ case

Minimizing "$\beta \times$ Energy − logarithmic volume (Entropy)"

For $\beta = +\infty$, this boils down an *energy minimization problem*

Deterministic, but *notoriously difficult…*

Hard to study $\beta \gg 1$ when you don't know $\beta = +\infty$…

Free energy minimization is an "extension" of
this problem in a random setting.

Thank you for your attention