"Coulomb gases and universality"
Paris, Déc. 22

Charge fluctuations in 2d Coulomb (and related) systems

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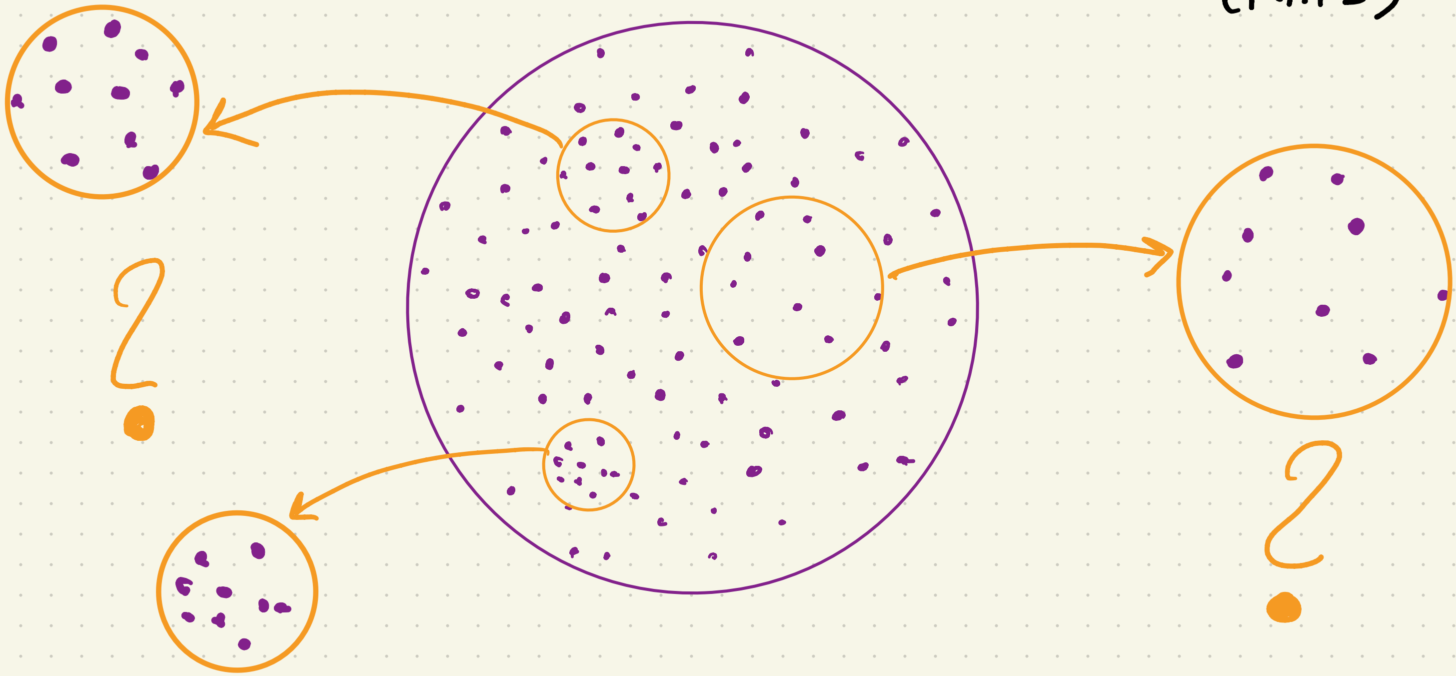


- $\mathbb{X}$ a point process on $\mathbb{R}^d$ (or the sphere, torus, a manifold...)

= random point configuration (loc. finite) $\mathbb{X} = \sum_{p \in \mathbb{X}} \delta_p$

- $\boxed{\mathrm{Points}(\mathbb{X}, \Omega)}$ (random) for each (meas.) $\Omega \subset \mathbb{R}^d$

- "Intensity" = # points / unit volume (on average)

$$\frac{\mathbb{E}[\mathrm{Points}(\mathbb{X}, \Omega)]}{|\Omega|} \text{ as } |\Omega| \to +\infty ?$$

For us, always $= 1$. $\boxed{\text{One point per unit volume in large boxes, on average}}$

Discrepancy

$$\boxed{\mathrm{Dis}(X,\Omega) = \mathrm{Points}(X,\Omega) - |\Omega|}$$

"Charge fluctuations" – Local non neutrality.

- Is $\mathbb{E}[\mathrm{Dis}(X,\Omega)] \simeq 0$? Centeredness
- How does $\mathrm{Var}(\mathrm{Dis}_\Omega) = \mathrm{Var}(\mathrm{Points}_\Omega)$ grow with $|\Omega|$? Number variance
- What is $\mathbb{P}\left(\frac{\mathrm{Dis}}{\sqrt{\mathrm{Var}(\mathrm{Dis})}} \gg 1\right)$?

Large deviations / charge fluctuations.

(cf. J. Boursier's talk for 1d Riesz)

Other questions

- $\mathrm{Cor}\left(\mathrm{Points}(\Omega), \mathrm{Points}(\Omega')\right)$ as $\mathrm{dist}(\Omega,\Omega') \to +\infty$? (cf. "Debye's screening"
- Maximal discrepancy ("equidistribution") C. Garban's talk)
- Number-rigidity (for infinite systems)

(cf. D. Dereudre's talk)

$X \cap \overline{\Omega}$

$\# X \cap \Omega = f(X \cap \overline{\Omega})$ f ?

Some systems to consider

- Poisson point process
- Lattices and their perturbations
- Zeroes of Gaussian analytic functions
- The Ginibre ensemble
- Gibbsian point processes for
  - → short-range interactions
  - → long-range interactions

↑ Coulomb!

Poisson

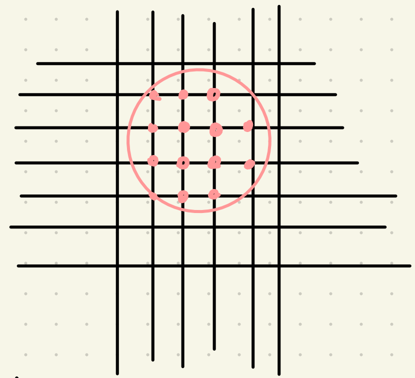
- $\mathbb{E}[\mathrm{Dis}(X,\Omega)] = 0 \quad \forall \Omega$
- $\mathrm{Var}(\mathrm{Points}(X,\Omega)) = |\Omega| \; \left(= \mathbb{E}[\mathrm{Points}(X,\Omega)]\right)$

  Scales exactly like the volume
- In a disk $D(0,R)$, the number variance $\propto R^2$
- No correlation, no rigidity

$\longleftrightarrow$ Non-interacting particles
Independent
$\left(\begin{array}{c} \text{"}\beta = 0\text{"} \\ \text{"}T = +\infty\text{"} \end{array} \right)$

Lattice(s) (e.g. $\mathbb{Z}^d$)

- Pts in Ω ? $\approx |\Omega|$ but boundary errors...



Gauss circle problem

$\boxed{\text{No randomness!}}$

- Make it stationary by choosing the origin randomly, uniformly on a fundamental domain

Stationary lattices

- $\mathbb{E}[\text{Points in } \Omega] = |\Omega|$
- $\mathrm{Var}[\text{Points in } \Omega] \simeq$ depends on the shape !

- $\boxed{\mathrm{Var}[\text{Points in a ball of radius } R] \simeq R^{d-1}}$!

! Not trivial !

Rem. This is the "slowest growth" for number variance Beck (Acta Math. '87)

- Very rigid

Infinite-range correlations

$\longleftrightarrow$ "Ground states"

"$\beta = +\infty,\ T = 0$"

Perturbed lattices

- Add iid displacements to lattice points with finite 1^{st} moment $\longrightarrow$ same number variance $\sim R^{d-1}$

  Gacs, Szás (AOP '75)

- Peres – Sly (Unpublished '14)

  Number rigidity remains in $d=1,2$ (Gaussian iid perturbat's) can be lost for $d \geq 3$ if perturbations are too big.

$\longleftrightarrow$ Physical systems near $T \approx 0$??

Zeroes of the G.E.F.

$(a_k)_{k \geqslant 0}$ iid standard (complex) Gaussian r.v.

The G.E.F.

$$f(z) := \sum_{k \geqslant 0} \frac{a_k}{\sqrt{k!}} z^k$$

converges a.s. to a $\boxed{\text{random entire function}}$

Its (random) zero set = a (random) p.p. on $\mathbb{C} \simeq \mathbb{R}^2$.

Invariant under translations/rotations (in distribution) ?

$\mathbb{E}[\text{Pts}(X, \Omega)]$

$= |\Omega|$ ✓

$\text{Var}[\text{Pts in } D(0,R)]$

$\sim R^1 = R^{2-1}$

Number-Rigid

and

Center of mass-Rigid

(Ghosh-Peres '12)

Ginibre ensemble

$(a_{ij})_{1\le i,j\le N}$ iid standard complex Gaussian r.v.

Non-Hermitian

$$A = \left[\frac{a_{ij}}{\sqrt{N}}\right]_{1\le i,j\le N}$$ Random matrix

N complex eigenvalues a.s.

$\mathrm{Sp}\, A \subset \mathbb{C} \simeq \mathbb{R}^2$ finite random p.p. $\boxed{\text{determinantal}}$

Admits a limit as $N \to +\infty$ Ginibre (JMP '65)

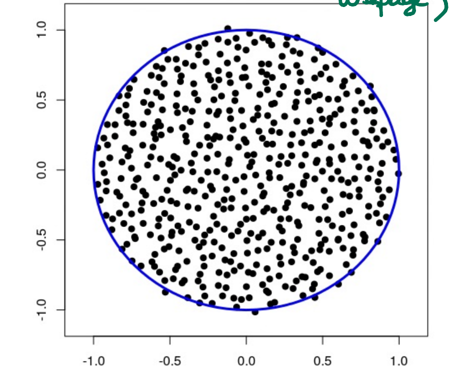
See Hough-Krishnapur-Peres-Virág 's book on determinantal p.p.
('06)

Ginibre ensemble

- Invariant under translations / rotations
- Fast decay of 2-point correlation $e^{-|x-y|^2}$
- $\mathrm{Var}[\text{Pts in } D(0,R)] \sim R^1$

  Shirai (JSP '06)
- Number - rigid ! Ghosh - Peres (Duke '12)

(A. Hardy, from Djalil's webpage)



- Fine properties are known

  Forrester's book ("Log-gases and random matrices") '10

Gibbsian point processes

Take $\Sigma_N =$ "box of size N" $\left(\begin{array}{l} \text{disk } D(0, \sqrt{N/\pi}) \\ \text{square } \left[-\frac{\sqrt{N}}{2}, \frac{\sqrt{N}}{2}\right]^2 \end{array} \right)$

$X_N =$ N-tuple of particles in Σ_N

$F_N(X_N)$ energy $\longrightarrow$ $\displaystyle \frac{dP_N^\beta(X_N)}{dX_N} = \frac{e^{-\beta F_N(X_N)}}{\int e^{-\beta F_N(X_N)}\,dX_N}$

Canonical Gibbs measure

Partition function

$\beta = 1/T$ parameter

Alternative formulat° with external confinement.

Coulomb

Replace by Riesz $\frac{1}{|x-y|^s}$?

$$\boxed{F_N(X_N) = \frac{1}{2}\iint_{x\neq y} (-\log|x-y|)\, d f_N(x)\, d f_N(y)}$$

$$f_N = \underbrace{\sum_{i=1}^{N} \delta_{x_i}}_{\text{N positive particles}} - \underbrace{\text{Lebesgue on } \Sigma_N}_{\text{uniform neutralizing background.}}$$

globally neutral system.

$\longrightarrow$ Charge fluctuations at "local" scales? $\longleftarrow$

- Hierarchical model (S. Chatterjee '18) $\longrightarrow \mathrm{Var}(D(0,R)) \sim R$ with log corrections.

Physicists' opinion(s)

- Martin – Yalcin (JSP '80) "Charge fluct. in classical Coulomb systems"

Clustering assumptions → "If the charge fluctuations are not extensive, then they are necessarily of the order of the surface" R^d R^{d-1}

Sum rules ⊕ → CLT for charge fluct.

- Lebowitz (Phys. Rev. A '83) "Charge fluctuations in Coulomb systems"

Decay of correlations in the system → Joint charge fluctuations → Gaussian with explicit covariance

- Martin (Rev. Mod. Phys '88) "Sum rules in charged fluids"

(Stillinger – Lovett 1st sum rule)

2-point correlation

$$\int_{\mathbb{R}^d} (g_2 - 1) = -1$$

equivalent

- Levesque – Weis – Lebowitz "Charge fluctuations in the 2d OCP" (JSP '00)

- Torquato – Stillinger, Torquato ('03 – '18): "Hyperuniform states of matter"

Hyperuniform systems: $\boxed{\text{number variance} \ll \text{Volume}}$

("natural" examples)

- Gabrielli – Joyce – Sylos Labini

$\boxed{\text{Superhomogeneous}}$

('02) Gabrielli – Jancovici – Joyce – Lebowitz – Pietronero – Sylos Labini

(applications to cosmology!)

The JLM prediction

Jancovici – Lebowitz – Manificat (JSP '93)

$$P\left[\text{Discrepancy in } D(0,R) \geq R^{\alpha}\right] \sim e^{-R^{\varphi(\alpha)}}$$

fast decay

$$\varphi(\alpha) = \begin{cases} 2\alpha - 1 & \alpha \in (\frac{1}{2}, 1) \\ 3\alpha - 2 & \alpha \in (1, 2) \\ 2\alpha & \alpha \geq 2 \end{cases}$$

Small
Medium } Regime
Large

For Two-dimensional Coulomb gas ("One-component plasma")

"2d OCP"

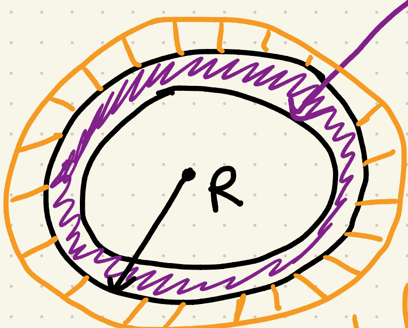
⊕ Another similar prediction for 3d OCP.

JLM (Continued)

- Clearly implies that $\boxed{\mathrm{Var}(\text{Points in } D(0,R)) = \mathcal{O}(R^1)}$

("Type I" hyperuniformity)

- Argument



Excess R^α in a thin annulus of size 1

$\simeq R^{\alpha-1}$ "excess density"

deficit on the other side

⚠ $R^{\alpha-1} \ll 1$ in the small regime

Free energy of such a "double layer" scales like ...... ?

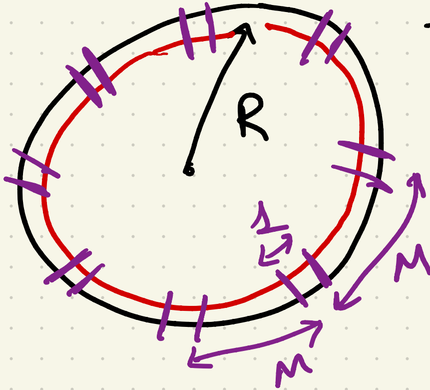
JLM (end)

- Proven for Ginibre ensemble ⟺ 2d Coulomb gas at $\beta = 2$!

Fenzl – Lambert (IMRN '20) determinantal computations

- $\boxed{\text{Proven for the zeroes of the GEF ! Nazarov – Sodin – Volberg (GAFA '07)}}$!!

- Underlying mechanism ?



Assume there are R^α too many points

(NSV argument to prove JLM for the GEF)

1. Show there must be some excess near the boundary

(For the small regime, $\alpha < 1$)

2. Cut it into pieces of size $\simeq 1$

(Group every M of them) $\rightarrow$ Take a $\boxed{\text{well-separated}}$ family carrying $\approx \frac{R^\alpha}{M}$ excess.

$\approx \frac{R}{M}$ "Boxes" of size $\simeq 1$ near the boundary

$$\sum_{i=1}^{R/M} \mathrm{Dis}(i^{th} \text{ box}) \approx \frac{R^\alpha}{M}$$

3.

a) Show the boxes are $\approx$ independent

Use that they are well-separated

b) Show that $\boxed{\mathbb{E}[\mathrm{Points}(\Omega)] - |\Omega| \approx 0}$ in each box

Use some kind of translation-invariance

Centeredness

c) Show that $\boxed{\mathrm{Var}\left[\mathrm{Points}(\Omega)\right] \approx 1}$ in each box

(use that they are of size ≈ 1)

4. Apply Bernstein's concentration inequality:

$\left(\mathrm{Dis}\left(i^{th} \text{ box}\right)\right)_{i=1,\dots,\frac{R}{M}}$ are $\approx$ independent, $\approx$ variance 1, $\approx$ centered

$$\mathbb{P}\left[\sum \mathrm{Dis}\left(i^{th} \text{ box}\right) \geq \frac{R^{\alpha}}{M}\right] \leq \exp\left(-\frac{R^{2\alpha-1}}{M}\right)$$

Tools for the 2DOCP

- Energy bounds at all scales

  full system, scale $\sqrt{N}$

  $\longrightarrow$ Total energy $F_N = O(N)$ with high proba.

  $\longrightarrow$ Energy in a domain $\Omega = O(|\Omega|)$ w.h.p.

  valid down to scales $\simeq 1$ (microscopic)

Local laws

L. '17 ; Bauerschmidt-Bourgade-Nikula-Yau '17

Armstrong-Serfaty '20

- Fluctuations and discrepancy bounds in purely energetic terms.

  $\hookrightarrow \mathrm{Var}\,[D(0,R)] \leq C R^{2}$.

Serfaty + { Sandier '10, Rougerie, Armstrong '20 }

Tools for the 2d OCP (continued)

- For φ lipschitz, purely "energetic" bounds imply

$$\mathrm{Fluct}[\varphi] := \sum_{i=1}^{N} \varphi(x_i) - \int_{\Sigma_N} \varphi(x)\, dx$$

is of order $\leq \sqrt{N}$ w.h.p.

!!

- For φ smoother (say C^4_c),

$\mathrm{Fluct}[\varphi]$ is of order 1

+ Gaussian tails.

Variance $= \|\varphi\|^2_{H^1}$

L.-Serfaty '17

BBNY '17

Serfaty '21

"Gaussian fluctuations"

...

Tools for 2dOCP (end)

- Wegner's estimates / Clustering upper bounds E. Thoma '22

$$\mathbb{E}[\mathrm{Points}(\Omega)] \leq C|\Omega| \text{ for } \underline{\underline{\text{all}}}\ \Omega$$

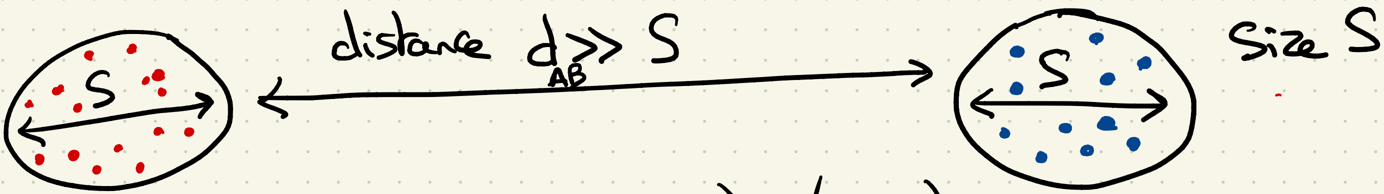
+ tail estimates for large excesses ; UB on k-point correlation functions $\forall k$.

- Missing, compared to NSV ?

  ① Almost $\perp\!\!\!\perp$

  ② Centeredness of discrepancies

Conditional approximate independence



A $\quad$ Int. Energy $\quad (A+B)\cdot(A+B) \quad$ B

$$= A\cdot A + B\cdot B + 2\,A\cdot B$$

$$A\cdot B = \iint_{A\,B} -\log|x-y| \, df_N(x)\, df_N(y)$$

$$= -\log d_{AB} \times \mathrm{Dis}(A) \times \mathrm{Dis}(B)$$

depends only on # of points in A, B

$$+\, O\left(\frac{1}{d_{AB}} \times S^2 \times S^2 \times S\right)$$

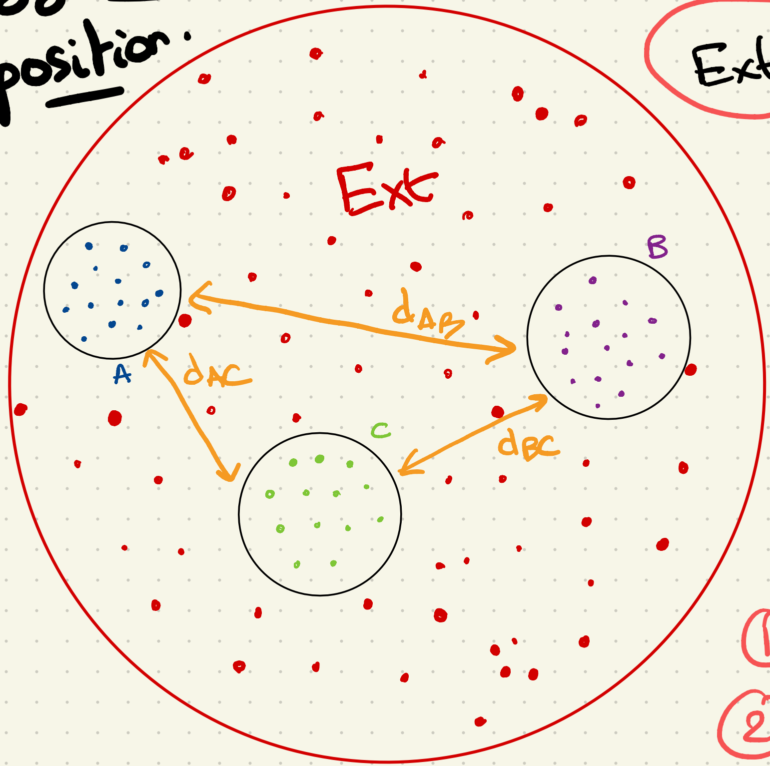
Error term!

Must be large!

# of points in A, B

size of each

Energy interaction decomposition.



Write $(A+B+C+Ext)^2$ as

$$Ext^2 + (A^2 + 2Ext \cdot A) + (B^2 + 2Ext \cdot B) + (C^2 + 2Ext \cdot C) + 2\sum Dis(i)Dis(j)\log d_{ij} + \text{Error CI}$$

Condition on :

1. Ext
2. The number of points in each box

HOPE → They are "almost independent".

Sub-systems

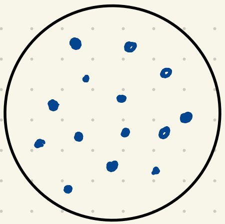
$(A^2 + 2\,\mathrm{Ext} \cdot A)$

$(B^2 + 2\,\mathrm{Ext} \cdot B)$

$(C^2 + 2\,\mathrm{Ext} \cdot C)$

Need to consider sub-systems

A



a) not globally neutral 2dOCP

b) with external potential

harmonic

given by "Ext"

Recover "everything" in that setup

(local laws, discrepancy estimates, fluctuations of smooth statistics ...)

cf. "conditional measures" Bourgade - Erdös - Yau

Bauerschmidt - Bourgade - Nikula - Yau

cf. Hierarchical model by Chatterjee.

Centeredness via translation-inv.

- If X is translation-inv. then $\mathbb{E}[\mathrm{Pts}(X,\mathcal{U})] = |\mathcal{U}|$
  so discrepancies are centered.
- In a finite system (with boundary) $\longrightarrow$ no perfect trans.-inv! Approximate?
- How does one prove translation-invariance of Gibbsian p.p.?
  Similar pb. for lattice spin systems "Absence of symmetry breaking"

  $x \in \mathbb{Z}^d \mapsto \sigma_x$ "spin" on $\mathbb{S}^1$ + interactions
  Assume interactions are invariant $\left(\text{e.g.} \sum_{x,y \in \mathbb{Z}^d} J_{xy}\, \sigma_x \cdot \sigma_y\right)$
  under rotations of all spins
  Could infinite Gibbs measures "break symmetry"?

" Mermin – Wagner (PRL'66) ": no breaking of continuous symmetries (for reasonable interactions) in dimension $d = 2$

Dobrushin – Shlosman (CMP '75)

Pfister (CMP '81)

$\boxed{\text{Spin wave argument}}$

distance to the origin

① Rotate the spins by an angle $\theta(x) = \theta_0 + \psi(x)$

such that $\begin{cases} \theta(x) = \theta_0 & \text{on } [-L, L]^2 \quad \text{large} \\ \theta(x) = 0 & \text{far away} \end{cases}$

② Show that the energy in the system satisfies

$$\left| H(X) - \left(\frac{H(X+\theta) + H(X-\theta)}{2} \right) \right| \leq C$$

bounded

uniform in L

Rotation in opposite directions

③ Magic happens!

For point processes, translation-inv. is proven similarly

① Move the points by $x + \psi(x)$

with $\left| \begin{array}{l} \psi(x) = \text{a constant vector } \psi_0 \text{ in } [-L, L]^2 \\ \psi(x) = 0 \text{ far away} \end{array} \right.$

② Show that the energy in the system satisfies

$$\left| H(X) - \left(\frac{H(X+\psi) + H(X-\psi)}{2} \right) \right| \leq C$$

bounded

uniform in L

Translation in opposite directions

③ Magic

Rem. Need to construct ψ s.t. $\boxed{x \mapsto x + \psi(x) \text{ has Jacobian } = 1}$

Fröhlich-Spencer '81 ; Georgii ; Richthammer

Magic? Try to do the same in finite volume

$$\frac{P(A+\varphi_0)}{P(A)} = \frac{\int e^{-\beta H(x)} \mathbb{1}_A(x+\varphi_0)\, dx}{\int e^{-\beta H(x)} \mathbb{1}_A\, dx}$$

Introduce your ① localized translation $\Phi(x)$

$$= \frac{\int e^{-\beta H(x)} \mathbb{1}_A(\Phi(x))\, dx}{\int e^{-\beta H(x)} \mathbb{1}_A(x)\, dx}$$

② Use that Φ has Jacobian 1

$$= \frac{\int e^{-\beta H\circ \Phi^{-1}(x)} \mathbb{1}_A(x)\, dx}{\int e^{-\beta H(x)} \mathbb{1}_A(x)\, dx}$$

③ Assume Φ has a bounded energy cost C

$$\leq e^{+\beta C} \quad \text{and} \quad \geq e^{-\beta C}$$

$\boxed{\text{If } C \text{ is } O(1) \text{ but not } o(1)\text{?}}$

• $\boxed{\text{Finite-volume}}$ "approximate" symmetry conservation proofs use spin waves with $\boxed{\text{small energy cost.}}$

cf. Friedli–Velenik's book, Chap. 9.

2. Most of the other proofs of absence of symmetry breaking use an apparently weaker property of the model than Lemma III.7.2: namely, that for any Λ, we can find $f \equiv 1$ and with $f(\alpha) = 0$ for $|\alpha|$ large, so that $|\langle d^2H/d\theta^2 \rangle| \leq c$ where c is independent of $|\Lambda|$. However, it appers that any model in which this weaker property is valid, the analog of Lemma III.7.2 holds.

B. Simon "Stat. mech. of lattice gases"

!!

Key point: Show that the energy cost of $x + \psi(x)$ is controlled by $\|\psi\|_{W^{1,2}}$

localized translation / rotation

Here: Challenges due to singularity + long-range of interaction pot.

Uses Serfaty '20 + revisiting computations. $\boxed{\text{No } \|\psi\|_{W^{2,1}} \text{ allowed!}}$

In dim 2, $W^{1,2} \not\hookrightarrow L^\infty$! (Barely)

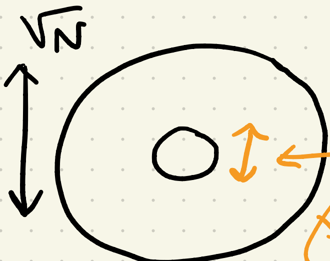
$W^{2,1}$ does!

One may find ψ which is

1) $\psi = \psi_0$ on $[-L,L]^2$
2) $\psi = 0$ far away
3) $\int |D\psi|^2 \leq \varepsilon$

(+ Jacobian condition)

However: Need to "dampen" the perturbation slowly over $L e^{1/\varepsilon}$ very large box.



Approximate translation-inv. $\forall \beta$ at scales $\ll \log N$ near the origin

cf. JM Stephan's talk about oscillations

Conclusion

Thm. (L. 23$^+$) For the 2dOCP, at all $\beta \in (0,+\infty)$

$$\boxed{\mathrm{Var}\left[\text{Points in } D(0,R)\right] \leq \frac{R^2}{(\ln R)^\delta} \quad \text{for some } \delta \in (0,1)}$$

(cf. previous estimates (Armstrong-Serfaty) $= \Theta(R^2)$)

$\longrightarrow$ it is hyperuniform. ($\oplus$ some tail estimates)
(better than Poisson)

Far from conjectured JLM law $\Theta(R)$ bound! Can do better?

Thank you

for your attention!