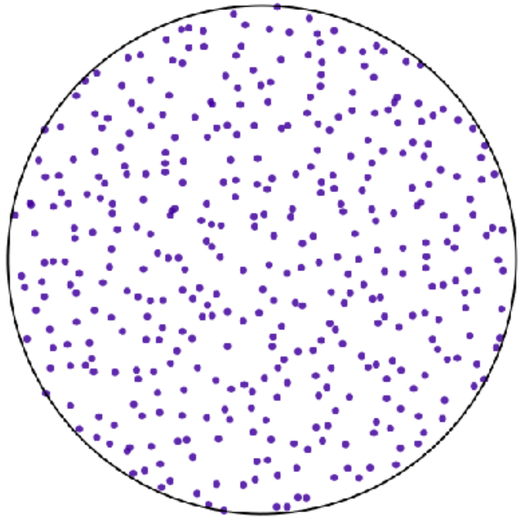
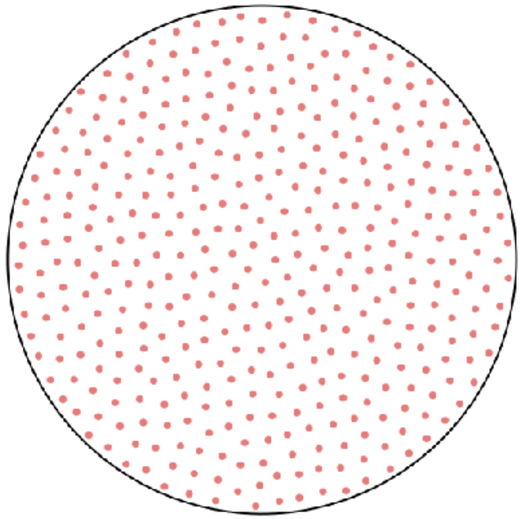
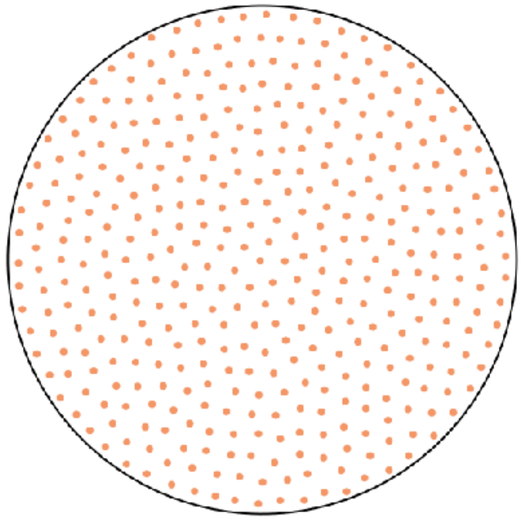
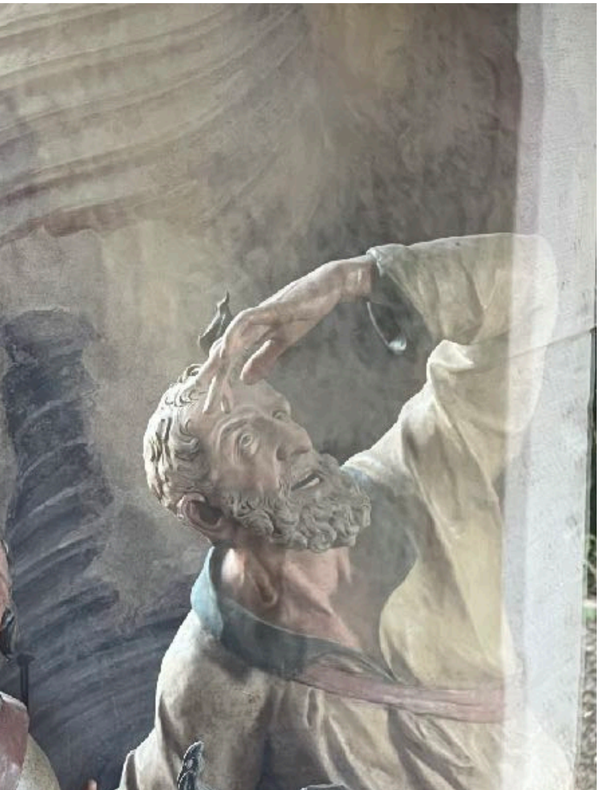






(Looking for) a phase transition in the 2DOCP

Thomas Leblé, CNRS & Université Paris-Cité

Riemann Prize Week, RISM, July 2026

Recipe for statistical physics

1. *State space* S
2. *Energy* functional $H : S \to (-\infty, +\infty]$
3. Reference measure (volume) μ on S
4. *Inverse temperature* $\beta > 0$

The state $\mathbf{X} \in S$ is *random* and distributed as:

$$\mathrm{d}\mathbb{P}(\mathbf{X}) = \frac{1}{Z} e^{-\beta H(\mathbf{X})} \mathrm{d}\mu(\mathbf{X})$$

Gibbs measure

Partition function

Boltzmann's factor

How does the typical behavior (under the Gibbs measure) change when β changes ?

Phase transition?

The state $\mathbf{X} \in S$ is a *microscopic* description: detailed, but not very "telling"

Choose an observable $\mathcal{O}(\mathbf{X})$ that says something about *macroscopic* properties
(could be a Yes/No question)

Ask for the typical behavior of $\mathcal{O}(\mathbf{X})$ under the Gibbs measure (phase)

If the behavior changes "dramatically" at a certain value of β
call this a phase transition

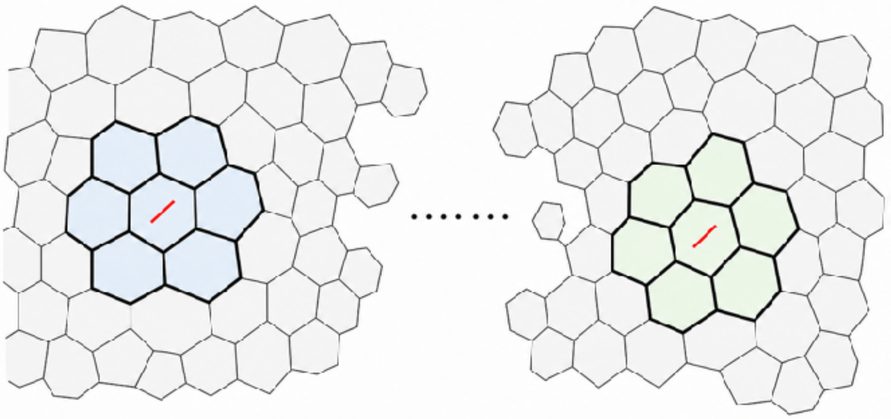
Melting transition in 2d

Melting in two spatial dimensions, as realized in thin films or at interfaces, represents one of the most fascinating phase transitions in nature, but it remains poorly understood.

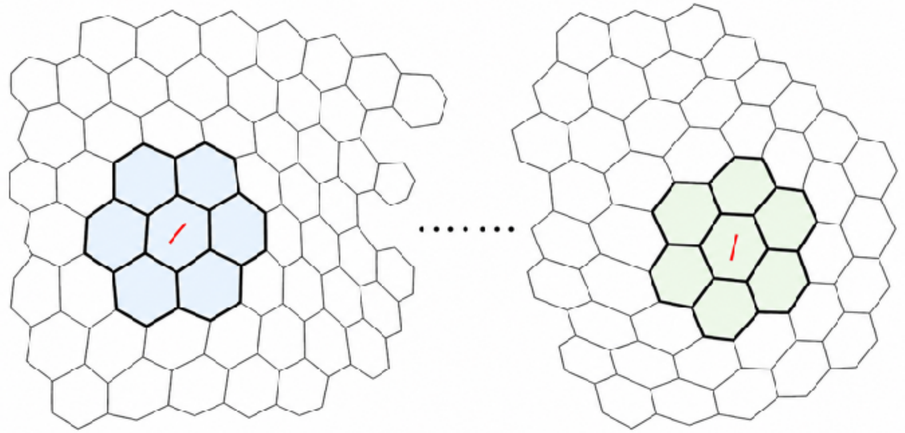
Bernard-Krauth 2011 "Two-step melting in two dimensions"

First discovered for *hard spheres* Adler-Wainwright 1962

Lots of works in the 70's, 80's - two competing scenarii… none of them correct!



One key observable
is *rotational order*



Spontaneous symmetry breaking

If the energy interaction has a symmetry (translation, rotation)
is it conserved (or broken) in the system itself?

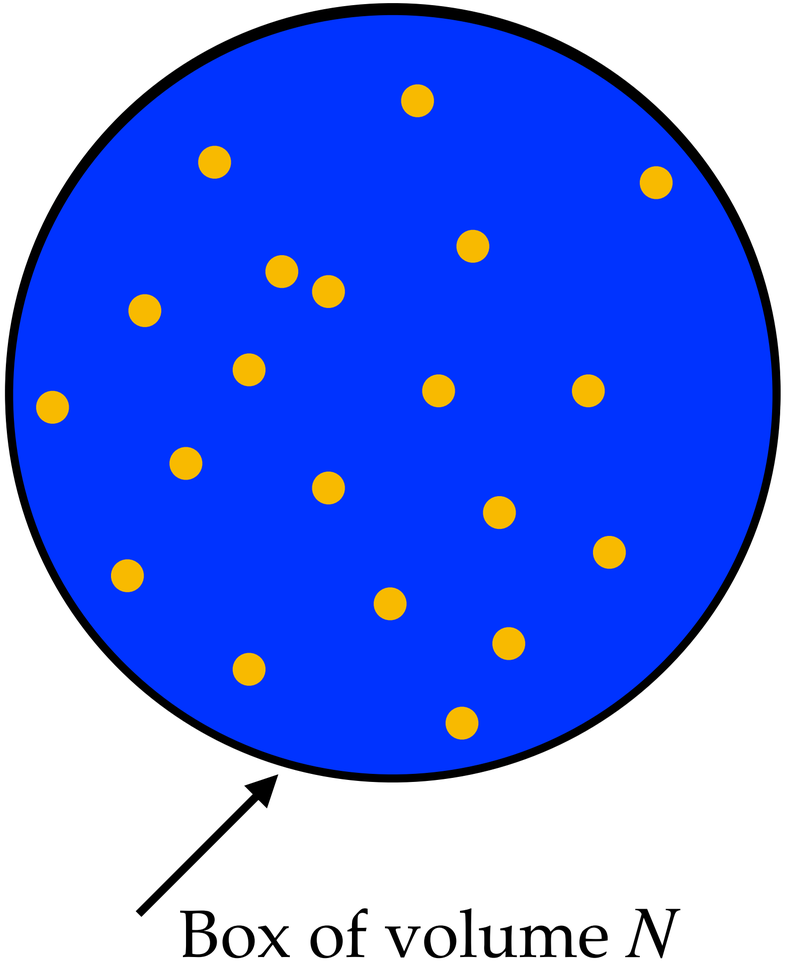
Hohenberg-Mermin-Wagner (1967)
"There is no breaking of a continuous symmetry in dimension 2"
…except when there is…

It turns out, however, that the breaking of rotation-invariance, i.e. *directional ordering*, is possible, in principle, in two-dimensional systems with connected correlations which do not fall off more rapidly than the inverse square distance (so that there is some divergent "susceptibility").

Fröhlich-Pfister 1981

Progress in **computational physics,** 2d melting remains an **open mathematical question**.

One-component plasma (OCP)



N particles $\overrightarrow{X}_N := (x_1, x_2, \ldots, x_N)$

+1 charge

Uniform "neutralizing" background

-1 density

Electrostatics = *Coulomb* potential

$= -\log r$ **in dimension 2**

Jellium, Coulomb gas, Log-gas

State space and energy

State $\overrightarrow{X}_N = (x_1, x_2, \ldots, x_N)$: a N-tuple of points in the box (particles)

Interaction energy $H_N(\overrightarrow{X}_N)$ in the state $\overrightarrow{X}_N$

$$H_N(\overrightarrow{X}_N) := \frac{1}{2} \iint_{(x,y)\in \mathrm{Box}^2,\ x\neq y} -\log|x-y| \left(\sum_{i=1}^{N} \delta_{x_i} - \mathrm{Leb}\right)(x) \left(\sum_{i=1}^{N} \delta_{x_i} - \mathrm{Leb}\right)(y)$$

No self-interaction

Positive charges

Background

2DOCP in finite volume

The Gibbs measure

$$\frac{1}{Z_{N,\beta}} e^{-\beta H_N(x_1,\dots,x_N)} dx_1 \dots dx_N$$

Lebesgue in the box

can also be written as

$$\frac{1}{Z'_{N,\beta}} e^{-\beta \sum_{i<j} -\log|x_i - x_j|} d\mu(x_1) \dots d\mu(x_N)$$

Log gas

(Truncated) Gaussian

Do physicists care?

On the ground state of the one-component classical plasma

Equilibrium properties of a two-dimensional Coulomb gas

STATISTICAL MECHANICS OF TWO-DIMENSIONAL COULOMB SYSTEMS

Cooperative Phenomena below Melting of the One-Component Two-Dimensional Plasma

On the classical two-dimensional one-component Coulomb plasma

ON "CRITICAL POINTS" IN THE TWO-DIMENSIONAL CLASSICAL JELLIUM

A Monte Carlo Study of the Classical Two-Dimensional One-Component Plasma

Remarks on the Independence of the Free Energy from Crystalline Boundary Conditions in the Two-Dimensional One-Component Plasma

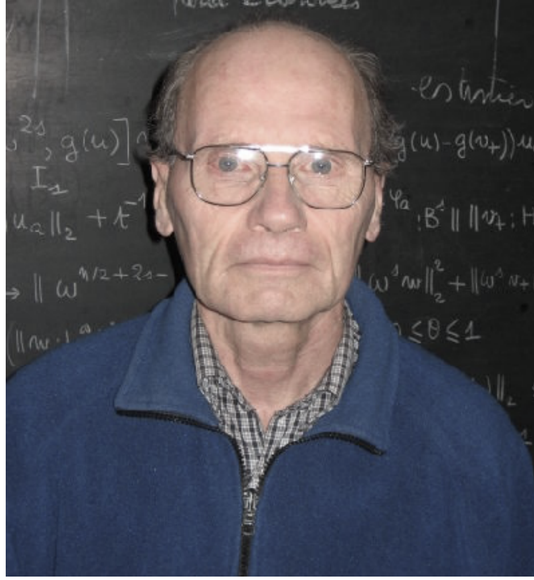
Why do they care?

- One-component plasma as toy model for matter
  small electrons (particles) versus *big protons* (background)

- Strong interest for statistical mechanics in dimension 2
  + *logarithmic interactions* appear elsewhere (vortex systems)

- Singular, long-range (not $\to 0$ fast at infinity) interactions are special
  "classical" theory developed for nice, short-range interactions.

Other motivation - RMT



JOURNAL OF MATHEMATICAL PHYSICS VOLUME 6, NUMBER 3 MARCH 1965

Statistical Ensembles of Complex, Quaternion, and Real Matrices

JEAN GINIBRE*
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received 7 July 1964)

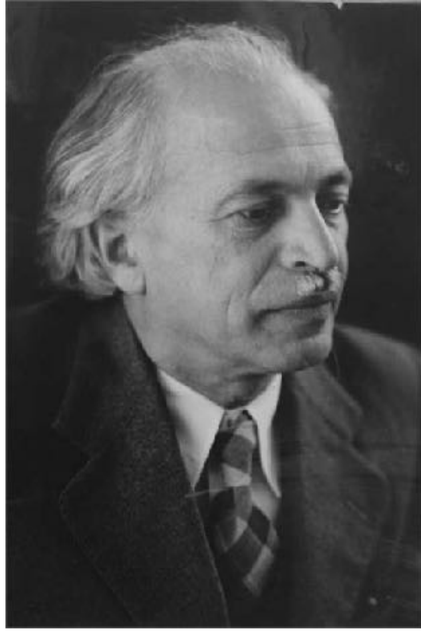
Following Dyson (*Statistical theory of the energy levels of complex systems*) Ginibre computes the distribution of eigenvalues for a $N \times N$ random matrix with **independent** complex Gaussian coefficients

$$d\mathbb{P}_{N,\beta}(x_1, \ldots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \sum_{i<j} -\log|x_i - x_j|} d\mu(x_1) \ldots d\mu(x_N)$$

$\beta = 2$

Gaussian

Other motivation - Fekete points



Fekete problem:
Place $x_1, \ldots, x_N$ points on the sphere and (try to)
maximize the product of their distances
minimize $\sum_{i \neq j} -\log|x_i - x_j|$

Problem 7: Distribution of Points on the 2-Sphere
Let $V_N(x) = \sum_{1 \leq i < j \leq N} \log \frac{1}{\|x_i - x_j\|}$, where $x = (x_1, \ldots, x_N)$, the x_i are distinct points on the 2-sphere $S^2 \subset \mathbb{R}^3$, and $\|x_i - x_j\|$ is the distance in $\mathbb{R}^3$. Denote $\min_x V_N(x)$ by V_N.

Find $(x_1, \ldots, x_N)$ *such that*

$$V_N(x) - V_N \leq c \log N, \quad c \text{ a universal constant.} \qquad (2)$$

Smale 7th problem:
Do this *approximately*
in polynomial time

Other motivation - FQHE

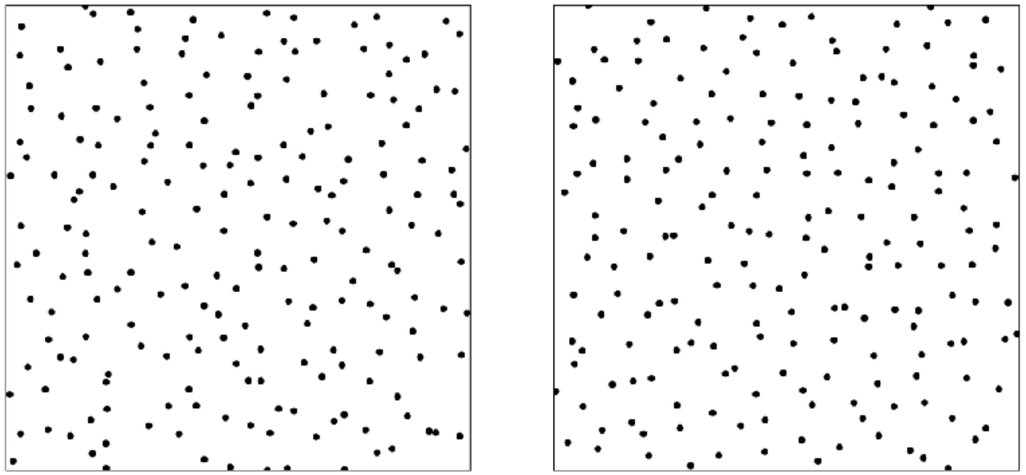
Two-dimensional quantum system of particles with interactions, in a magnetic field

Laughlin trial state:

$$|\Psi_{Lau}(x_1,\ldots,x_n)|^2 = c \prod_{1\le i<j\le n} |x_i - x_j|^{2\ell}\, d\mu(x_1)\ldots d\mu(x_n)$$

Gaussian

Other motivation - stochastic geometry



Take $(a_k)_{k\geq 0}$ independent complex Gaussian coefficients, and form:

$$f(z) := \sum_{k\geq 0} \frac{a_k}{\sqrt{k!}} z^k$$

Gaussian Entire Function (GEF)

Zeroes of f share many similarities

with a 2DOCP

Final motivation: it's hard

The logarithmic interaction is **singular** (at 0) and **long-range** (at infinity)

Power-Law $\varphi(x) = \dfrac{1}{|x|^s}$, *Riesz potential*

In dimension d :

$s > d$ short-range

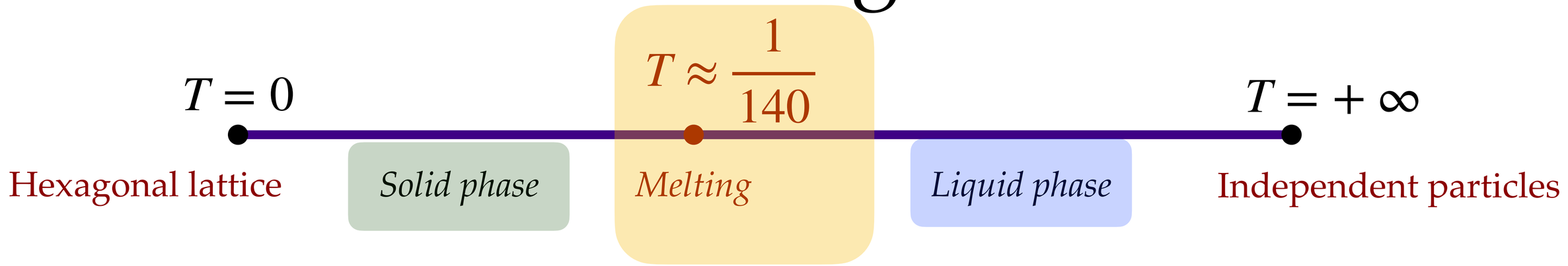
$s < d$ long-range

$s = d - 2$ Coulomb

$$s = 0 \leftrightarrow \varphi(x) = -\log|x|$$

Expect **major differences** between short- and long-range
e.g. when $s < d$, the energy ceases to be (almost-)additive w.r.t. space

Phase diagram



It is very likely that the model has a solid-fluid phase transition.

transition is located at a coupling $\Gamma = e^2/k_B T \simeq 140$.

(i) The 2D-OCP exhibits a weak first-order transition at $\Gamma = 142 \pm 3$ with a latent heat of fusion

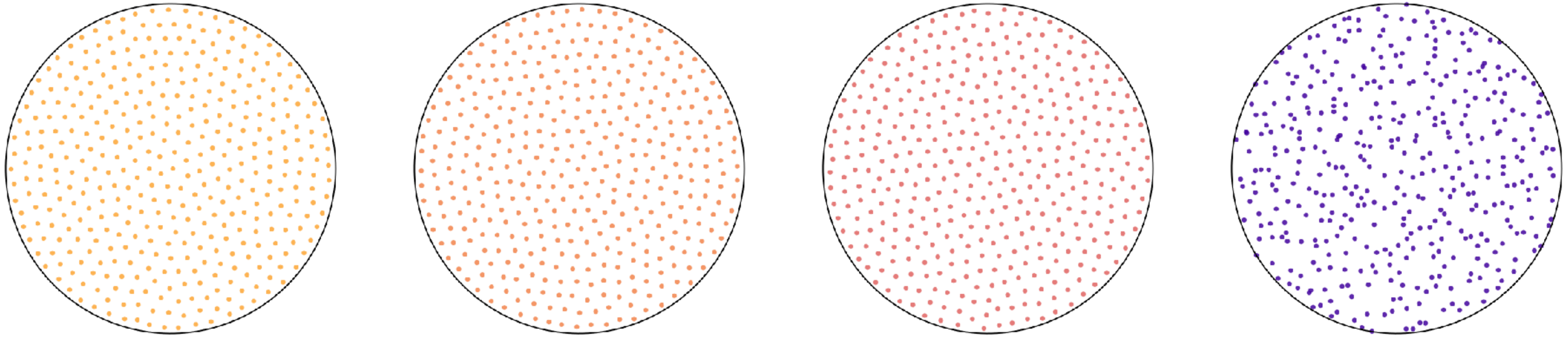
observed for $\Gamma = q^2/kT \approx 135$.

value $\gamma_C = e^2/kT \cong 137$,

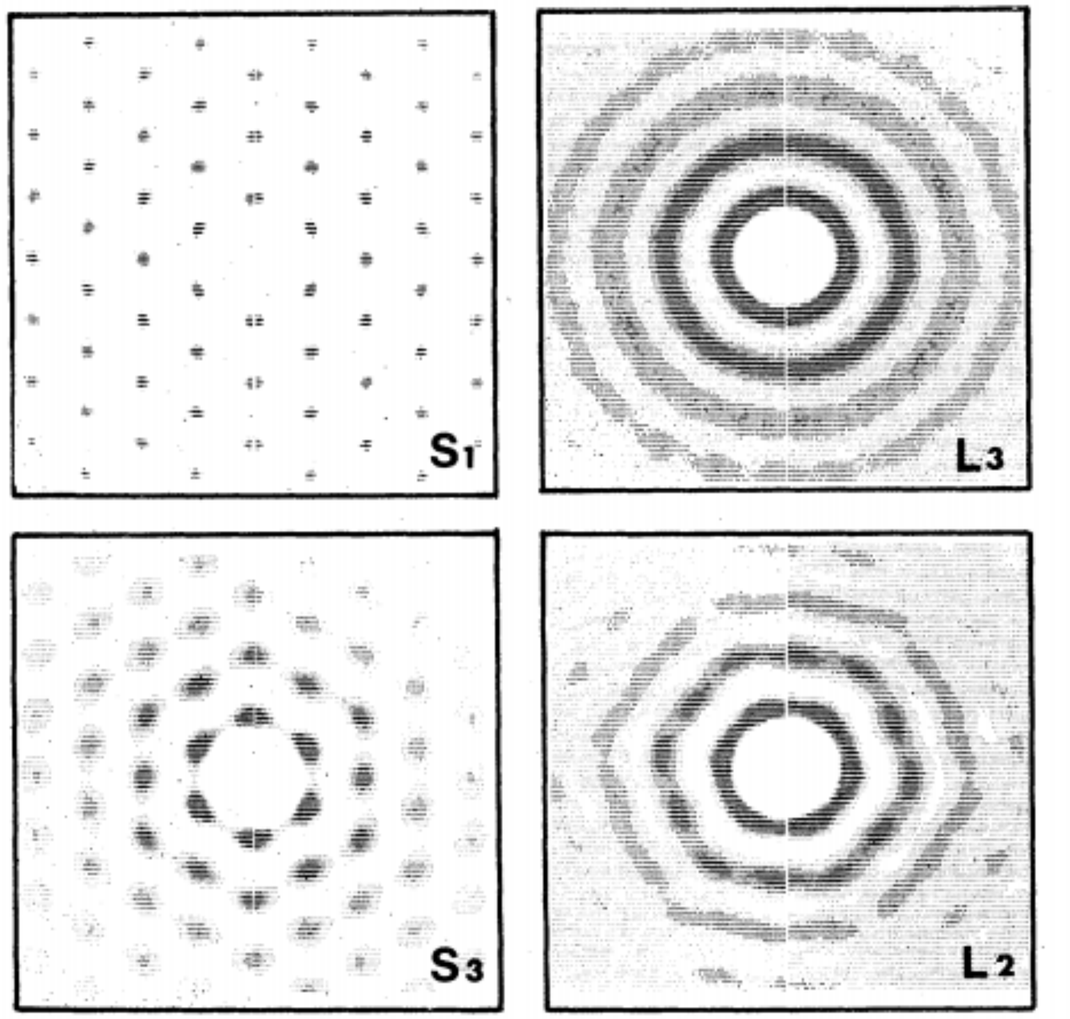
Goal: *mathematical confirmation of the physicists' predictions*

Finding an order-disorder transition for the 2DOCP

Two main open problems



1. Prove that the system forms an hexagonal lattice at zero temperature (energy minimizers)
2. Prove the existence of a melting transition around $\beta \approx 140$



Cooperative phenomena below melting of the one-component two-dimensional plasma
Choquard-Clerouin 1983

The infinite-volume issue

Phase transitions are best detected in *infinite-volume* ($N = +\infty$ particles)

It requires to make sense of infinite-volume stat. mech. objects

- Interaction energy?
- Gibbs measure?

Here the **long-range** interaction starts being annoying.

The renormalized energy (2010s)

Take an infinite point configuration $\mathbf{X}$
how to *make sense* of the logarithmic interaction between
- The infinitely many points of $\mathbf{X}$
- The infinite background given by the Lebesgue measure on $\mathbb{R}^2$

and **connect it rigorously** to its finite-volume counterpart

From the Ginzburg-Landau Model to Vortex Lattice Problems

Etienne Sandier[1,2], Sylvia Serfaty[3,4,5]

2D COULOMB GASES AND THE RENORMALIZED ENERGY

BY ETIENNE SANDIER[1] AND SYLVIA SERFATY[2]

The reasonable guess

$$\lim_{R\to\infty} \frac{1}{|\Lambda_R|} \iint_{x\neq y\in\Lambda_R\times\Lambda_R} -\log|x-y|\, d(\mathbf{X}-\mathbf{Leb})(x) d(\mathbf{X}-\mathbf{Leb})(y)$$

does not work - we are missing the "influence of infinity"

This **is** a subtle question.

…what works is the infinite-volume renormalized energy,
using the formalism of **electric fields**

Béthuel-Brézis-Hélein

Sandier-Serfaty

Rougerie-Serfaty
Petrache-Serfaty

Electric field formalism - finite volume

$$\Phi(x) = \int_{Box} -\log|x-y|\, d\left(\mathbf{X}_N - \mathbf{Leb}\right)(y)$$

Electrostatic potential

$$\mathrm{E}(x) = \int_{Box} -\nabla\log|x-y|\, d\left(\mathbf{X}_N - \mathbf{Leb}\right)(y)$$

Electric field

Compatible $-\mathrm{div}\mathrm{E} = 2\pi\left(\mathbf{X}_N - \mathbf{Leb}\right)$ in the box

$$H_N(\mathbf{X}_N) := \frac{1}{2}\iint_{(x,y)\in \mathrm{Box}^2,\ x\neq y} -\log|x-y|\left(\sum_{i=1}^{N}\delta_{x_i} - \mathrm{Leb}\right)(x)\left(\sum_{i=1}^{N}\delta_{x_i} - \mathrm{Leb}\right)(y)$$

Integration by parts $H_N(\mathbf{X}_N) \approx \frac{1}{2\pi}\int_{\mathbb{R}^2} |\mathrm{E}|^2$ to be computed in a *renormalized* fashion

Electric field formalism - infinite volume

Compatible electric field $-\mathrm{div}\mathrm{E} = 2\pi(\mathbf{X} - \mathbf{Leb})$ in the plane

Compute $\int_{\Lambda_R} |\mathrm{E}|^2$ in a renormalized fashion

Take $\lim_{R\to\infty} \frac{1}{|\Lambda_R|} \int_{\Lambda_R} |\mathrm{E}|^2$

Optimize over all choices of the field

The role of abstract fields

Work with **abstract** compatible electric fields

$-\mathrm{divE} = 2\pi(\mathbf{X} - \mathbf{Leb})$ in the plane

The freedom given by working abstract fields,
before returning to the "true one" is key

Electric fields yoga:

- Localizing
- Screening
- Gluing
- Projecting

Local fields

Work with abstract compatible
"local" electric fields in a box Λ

$$-\mathrm{divE} = 2\pi\,(\mathbf{X} - \mathbf{Leb}) \text{ in } \Lambda$$

Among them is "the best local field"
with **minimal energy**

$$\operatorname{argmin} \int_{\Lambda} |E|^2$$

Among them is the screened field
namely the one with Neumann b.c.

$$E \cdot \vec{n} = 0 \quad \text{along } \partial\Lambda$$

(its mere existence presupposes neutrality in Λ)

Screened fields - II

Why are screened fields so nice?

The importance of screened fields Screened fields are of outmost importance because they can be glued together in the following sense: if Λ^1, Λ^2 are two squares with one side in common, $\mathbf{X}^1, \mathbf{X}^2$ are two point configurations in Λ^1, Λ^2 and $\mathrm{E}^1, \mathrm{E}^2$ are two local electric fields compatible with $\mathbf{X}^1, \mathbf{X}^2$, then defining:

$$\mathbf{X}^t := \mathbf{X}^1 + \mathbf{X}^2, \quad \mathrm{E}^t := \mathrm{E}^1 1_{\Lambda_1} + \mathbf{E}^2 1_{\Lambda_2}$$

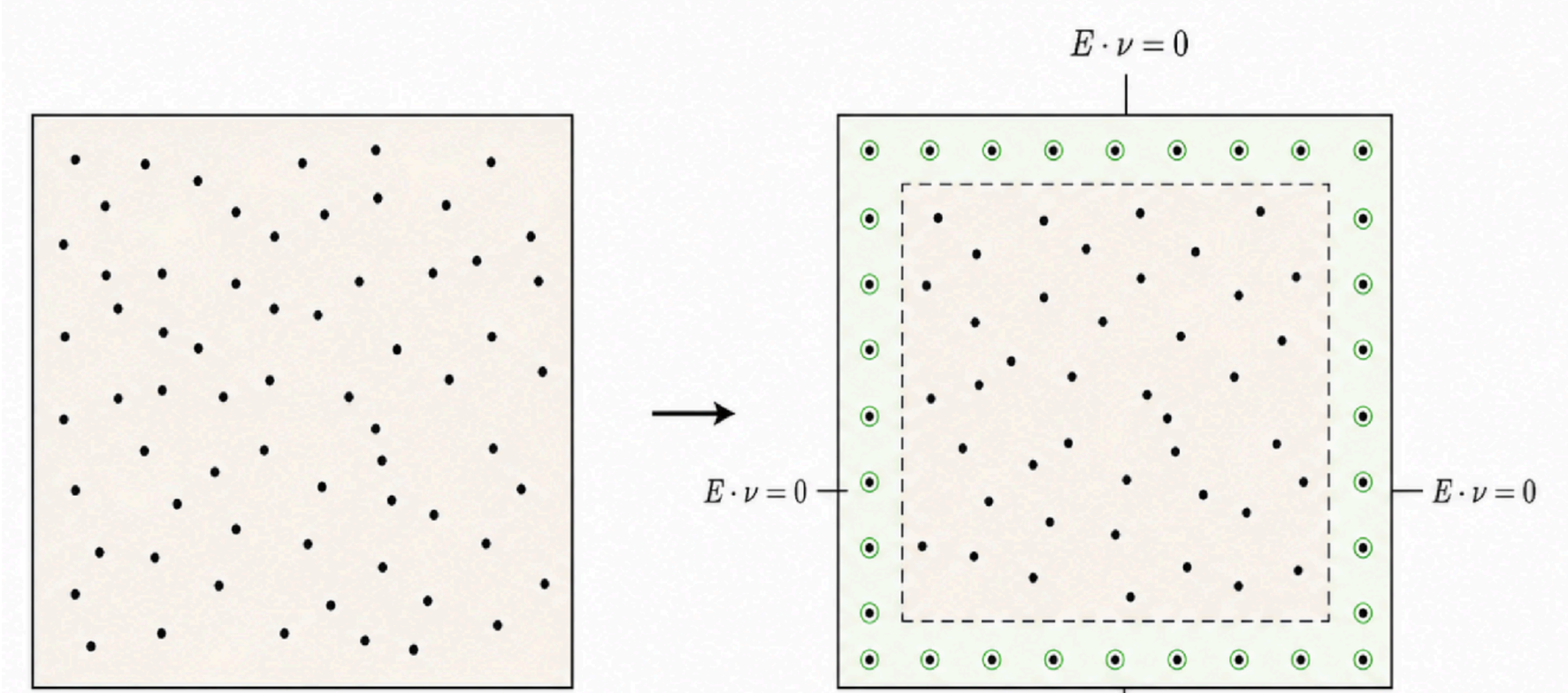
yields a "total" point configuration and a "total" electric field, but *in general,* E^t *is not compatible with* $\mathbf{X}^t$. However, if $\mathrm{E}^1, \mathrm{E}^2$ are both screened, or more generally if *their normal components agree on the common side of* Λ_1, Λ_2, then E^t is compatible with $\mathbf{X}^t$.

And their energy adds up!

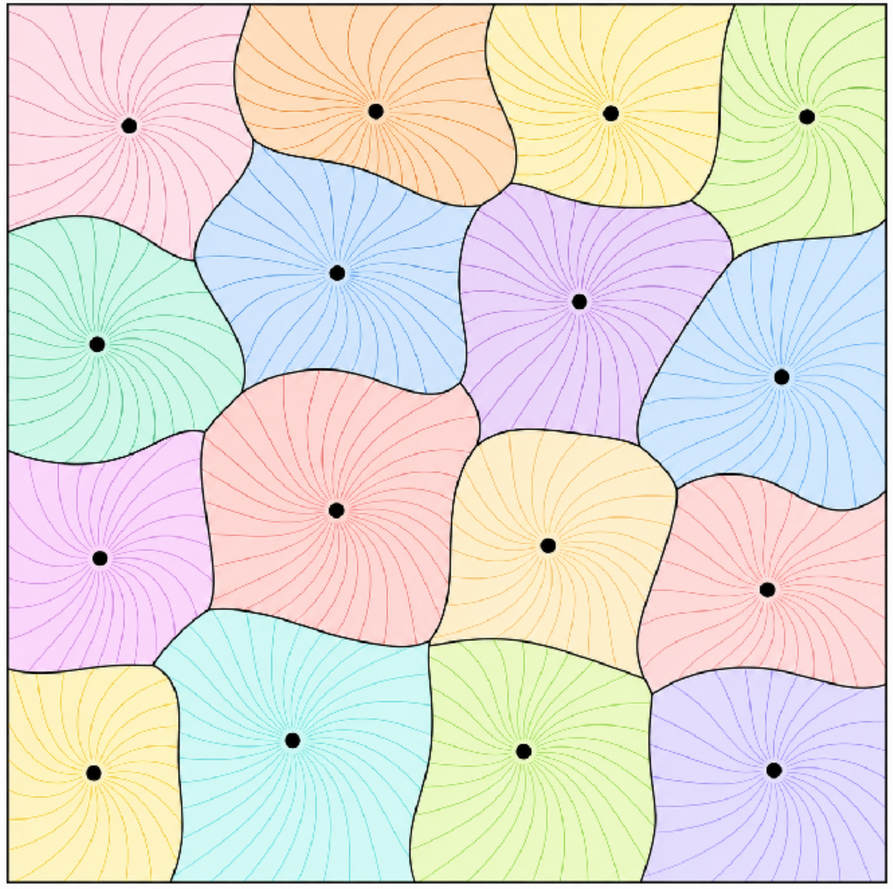
$$\int_{\Lambda_1 \cup \Lambda_2} |E^t|^2 = \int_{\Lambda_1} |E^1|^2 + \int_{\Lambda_2} |E^2|^2$$

Screening

The **screening procedure** of Sandier-Serfaty, later refined in *Rougerie-Serfaty, Petrache-Serfaty, Armstrong-Serfaty*…
is aimed at **producing screened local fields**, while
changing the underlying configuration and the energy **as little as possible**.



One tool that has not been used yet



Nazarov-Sodin-Volberg 2005 (zeroes of GAF)
Transportation by gradient flow of potential

$$\dot{X} = -\nabla\Phi(X)$$

- The set of points who "fall" on x has area 1
let's call this a Coulomb cell

- On the boundary of a Coulomb cell, one has
$\nabla\Phi \cdot \vec{n} = 0$

- The flow gives a natural decomposition
into **screened** cells. The energy is **additive**.
Problem: *cell shapes can be wild*.

Application I - Free energy

(L.-Serfaty '15)

The average microscopic behavior of a 2DOCP
looks, with very high probability
like a minimizer of the free energy functional

$$\beta \mathbb{E}_P[\mathscr{W}] + \mathscr{E}(P)$$

Average **renormalized energy**

Specific relative entropy
Measures "how independent"
the points are

Here P describes a random infinite point configuration
= the random state of the system in infinite-volume

Detecting melting?

$$\beta \mathbb{E}_P \left[\mathscr{W} \right] + \mathscr{E}(P)$$

Presumably favors order

Favors disorder

There is a competition between the two terms (order / disorder)

If for $\beta \gg 1$ one can prove:
there exists a minimizer which is not rotation-invariant
then there would be **breaking of rotational symmetry**

If for $\beta \ll 1$ one can prove:
there exists a unique minimizer, with fast decay of correlations
then there would be **a liquid phase**

Applications II

The "electric approach" through renormalized energy, screening, etc.
of Sandier-Serfaty has led to many other results / techniques for the 2DOCP:

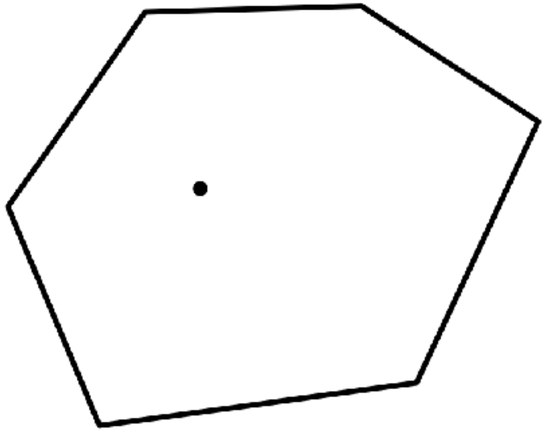
Local laws (Armstrong-Serfaty)
Expansion of partition function (Serfaty)
Fluctuations of smooth linear statistics (Serfaty)
Hyperuniformity, number-rigidity (L.)
No breaking of translational symmetry (L.)
Maximum of the electrostatic potential (Lambert-L.-Zeitouni)
Serfaty's *anisotropy trick*
(…)

But we are still far from the two goals: $\beta = +\infty$ and melting

The β-Wasserstein model - I

Wasserstein cost for one *cell* = second moment

$$\int_{\mathscr{C}} |x-t|^2 dt$$

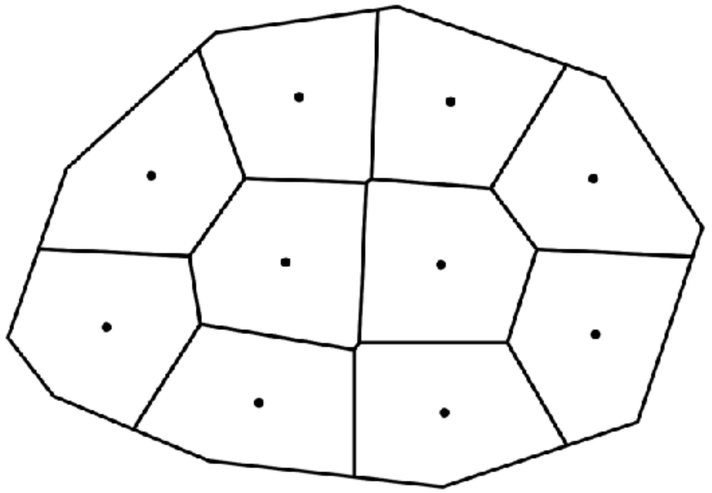


Wasserstein cost for one *finite point configuration*

$$= \sum_x \int_{\mathscr{C}(x)} |x-t|^2 dt \text{ (\textbf{optimized over choice of cells})}$$

Wasserstein cost for one *infinite point configuration*

$$= \lim_{R\to\infty} \frac{1}{|\Lambda_R|} \sum_{x\in\mathbf{X}\cap\Lambda_R} \int_{\mathscr{C}(x)} |x-t|^2 dt \text{ (\textbf{optimized again})}$$



The β-Wasserstein model - II

Consider the following *free energy* functional

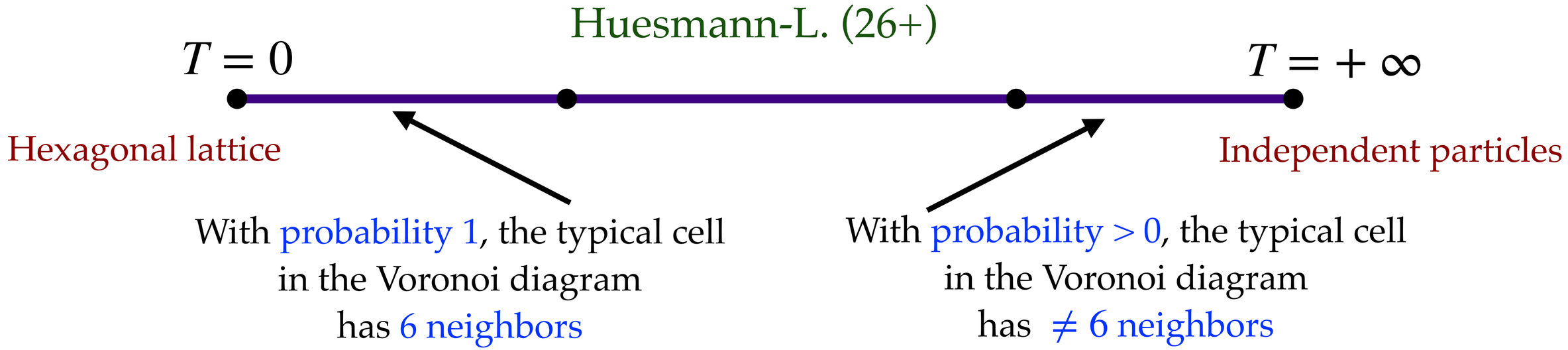
$$\beta \mathscr{W}(P) + \mathscr{E}(P)$$

Where instead of the Coulomb interaction,
we use $\mathscr{W}(P)$ = the average Wasserstein cost

Wasserstein and Coulomb energies are known to be closely related:
heuristically, the Coulomb energy is an H^{-1} distance (squared) between measures,
and so is the linearization of the 2-Wasserstein cost

Huesmann-L. 24 Coulomb energy controls the Wasserstein one,
the converse is true under some (strong) regularity assumption

The β-Wasserstein model - III



We prove the existence of two phases in the geometric arrangement of the points
At low temperature, the hexagonal structure is preserved **without defects**
At high temperature, it is destroyed

Melting transition for a 2d system… with **Wasserstein** interaction

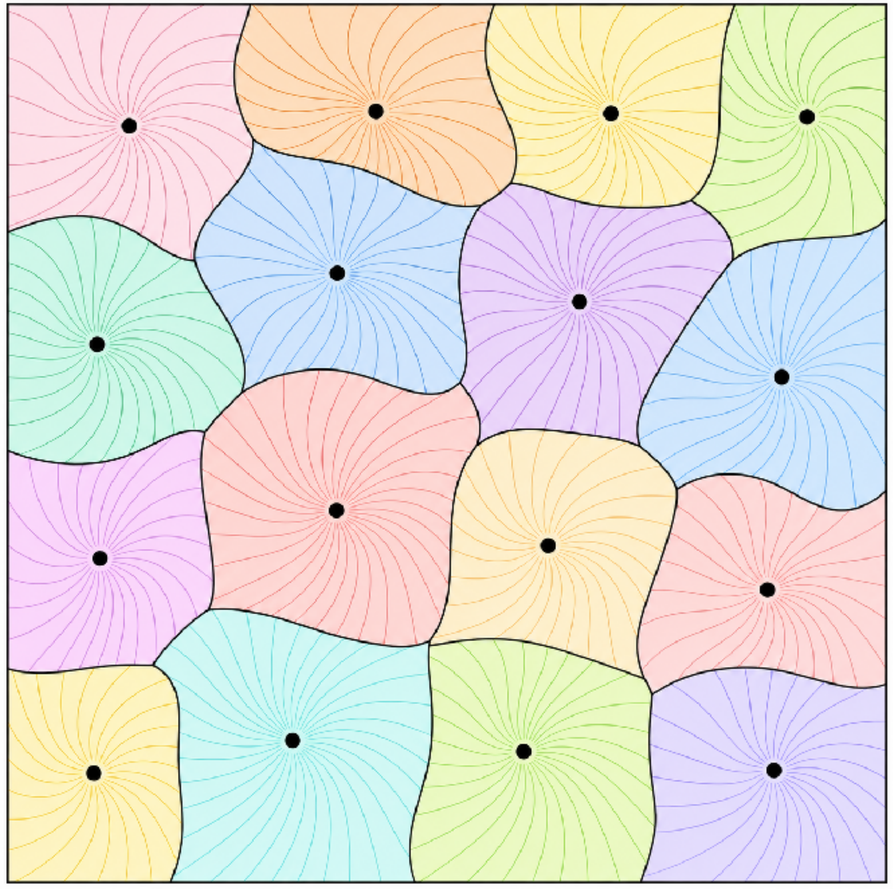
Wasserstein energy minimization

following Bourne-Peletier-Theil '13

- 2-Wasserstein cells are polygons of area 1
- Fejes Toth (1940s) Denote by $c(n)$ the second moment of a regular n-gon of area 1 about its center, then for any other n-gon of area 1, the second moment is $\geq c(n)$
- The average number of sides of a typical Wasserstein cell is 6
- The function $x \mapsto c(x)$ is convex

$$\mathscr{W}(P) = \sum_{n\geq 3} p_n \mathbb{E}_P[\textbf{Second moment} \cap \textbf{the cell is a n-gon}] \geq \sum_{n\geq 3} p_n c(n) \geq c(\sum_{n\geq 3} p_n n) = c(6)$$

Coulomb energy minimization ???



Coulomb energy

$$= \lim_{R\to\infty} \frac{1}{|\Lambda_R|} \sum_{x\in\mathbf{X}\cap\Lambda_R} \int_{\mathscr{C}(x)} |\mathrm{E}^{neu}(\mathscr{C},x)|^2$$

$$\begin{aligned} -\mathrm{div}\mathrm{E}^{neu} &= 2\pi\left(\delta_x - \mathrm{Leb}\right) && \text{in } \mathscr{C} \\ \mathrm{E}^{neu}\cdot\vec{n} &= 0 && \text{on } \partial\mathscr{C} \end{aligned}$$

Cells are not polygons
but have 6 “sides” on average

Can **you** minimize this?

Thank you for your attention!