1 Motivation and goal of the course

1.1 Random matrices: from statistics to nuclear physics

At the end of the nineteenth century, Hurwitz [Hur97] constructed the invariant measure on the orthogonal and unitary groups. Thirty years later, Wishart [Wis28] computed the law of the empirical covariance matrix of a Gaussian sample. In both cases, random matrices appeared as tools.

They became objects of study in their own right with Wigner. In quantum mechanics, the energy levels of a system are the eigenvalues of its Hamiltonian, a self-adjoint matrix. For heavy nuclei, writing down this matrix from first principles is hopeless. Wigner's idea was to replace it by a *random* self-adjoint matrix, in which, roughly speaking, all interactions are equally likely, and to hope that the statistics of the levels would be *universal*.

Definition 1.1 (Wigner matrices)**.** A (real) *Wigner matrix* of size N is a random symmetric matrix $H_N = (h_{ij})_{1\le i,j\le N}$ whose entries $(h_{ij})_{i\le j}$ are independent and centered, with $\mathbb{E}[h_{ij}^2] = \frac{1}{N}$ for $i < j$ and $\mathbb{E}[h_{ii}^2] = \frac{\sigma^2}{N}$. In the complex case, H_N is Hermitian, the entries $(h_{ij})_{i<j}$ are complex with $\mathbb{E}|h_{ij}|^2 = \frac{1}{N}$, and the diagonal is real.

Theorem (Wigner's semicircle law [Wig55, Wig58])**.** If the entries have uniformly bounded moments, the empirical distribution $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ of the eigenvalues of H_N converges weakly, almost surely, to the semicircle law $\frac{1}{2\pi}\sqrt{(4-x^2)_+}\,\mathtt{d}x$.

The limit does not depend on the law of the entries: this is a first instance of universality. Wigner also guessed a formula for the distribution of the spacing between consecutive levels (the "Wigner surmise").

Why Gaussian entries: invariance

The basis in which the Hamiltonian is written has no physical meaning, so *the invariances of the physical system under changes of basis should be invariances of the law of the random matrix.*

- For a system invariant under time reversal (with integer spin), the Hamiltonian can be taken real symmetric, and the changes of basis are orthogonal. The law of H_N should then be invariant under $H \mapsto OHO^{\mathsf{T}}$, $O \in O(N)$.
- Without time-reversal symmetry, H_N is Hermitian and the symmetry group is $U(N)$.
- With time-reversal symmetry and half-integer spin, H_N is quaternionic self-dual and the group is $Sp(N)$.

Among Wigner matrices, the requirement "independent entries *and* invariant law" singles out the Gaussian ones (see [Meh04]). For instance, in the real case, the law with density proportional to $e^{-\frac{N}{4}\mathrm{Tr}(H^2)}$, where $\mathrm{Tr}(H^2) = \sum_i h_{ii}^2 + 2\sum_{i<j} h_{ij}^2$, has independent Gaussian entries of variances $\frac{1}{N}$ off the diagonal and $\frac{2}{N}$ on it, and it is clearly invariant under conjugation by $O(N)$. Around 1960, Porter and Rosenzweig [PR60] and Mehta and Gaudin [MG60, Gau61] studied these Gaussian ensembles. Since the law is invariant, the eigenvectors are Haar distributed and independent of the eigenvalues, and the Weyl integration formula gives the joint law of the eigenvalues explicitly (it is the density below with $\beta = 1, 2, 4$). This gave the Gaussian Orthogonal, Unitary and Symplectic Ensembles (GOE, GUE, GSE). For the GUE, the eigenvalues form a *determinantal* point process, and, after zooming in the bulk at the scale of the mean spacing, the limit is the determinantal process with the *sine kernel* $\frac{\sin\pi(x-y)}{\pi(x-y)}$. Universality of these local statistics, for Wigner matrices with general independent entries, was proven fifty years later [EY17].

1.2 Dyson: the log-gas and the threefold way

In a famous series of papers [Dys62b], Dyson laid the foundations of the field. He made three contributions that matter here.

- *The threefold way.* He classified the relevant ensembles according to the symmetries of the physical system [Dys62c]: real, complex and quaternionic matrices. In all three cases, the joint density of the eigenvalues is
$$\frac{1}{Z_{N,\beta}} \prod_{i<j} |x_i - x_j|^\beta \, e^{-\frac{\beta N}{4} \sum_i x_i^2}, \qquad \beta = 1, 2, 4.$$

- *The Coulomb gas.* This density is $e^{-\beta \mathsf{H}_N}$, with $\mathsf{H}_N = \frac{1}{2} \sum_{i \neq j} - \log |x_i - x_j| + N \sum_i \frac{x_i^2}{4}$: the Gibbs measure, at inverse temperature β, of N unit charges on the line, interacting through the *two-dimensional* Coulomb potential $-\log|x-y|$ and confined by a quadratic potential. This is the *log-gas*. The three symmetry classes become three temperatures.

- *Dynamics.* He introduced the Brownian motion of the eigenvalues [Dys62a], the origin of the dynamical approach to random matrices [EY17].

1.3 β-ensembles

From the point of view of statistical physics, nothing singles out $\beta = 1, 2, 4$: the Gibbs measure makes sense for every $\beta > 0$. These *β-ensembles* remained without a matrix model until Dumitriu and Edelman [DE02] found, for every $\beta > 0$, a random *tridiagonal* matrix with independent entries whose eigenvalues have the law above. Since then, they have been studied in their own right.

The Gaussian β-ensemble. For $N \geq 2$ and $\beta > 0$, the GβE is the probability measure on $\mathbb{R}^N$
$$\mathtt{d}\mathbb{P}_{N,\beta}(x_1, \dots, x_N) := \frac{1}{\mathsf{Z}_{N,\beta}} \prod_{i<j} |x_i - x_j|^\beta \, e^{-\frac{\beta N}{4} \sum_i x_i^2} \, \mathtt{d}x_1 \dots \mathtt{d}x_N, \tag{1.1}$$
in the normalization where the off-diagonal entries of the matrices have variance $\frac{1}{N}$. We will *not* use any matrix model: (1.1) is our starting point. It is a *canonical Gibbs measure*, $\mathtt{d}\mathbb{P}_{N,\beta} \propto e^{-\beta \mathsf{H}_N} \mathtt{d}\mathsf{X}_N$, with the energy
$$\mathsf{H}_N(\mathsf{X}_N) := \overbrace{\frac{1}{2} \sum_{i \neq j} - \log |x_i - x_j|}^{\text{repulsion}} + \overbrace{\sum_i N \frac{x_i^2}{4}}^{\text{confinement}}$$
of N unit charges, repelling each other through the two-dimensional Coulomb interaction $-\log|x-y|$ and confined by an external potential. The logarithm is singular at short distance (but integrable), and long-range: neither $\log|x|$ nor its derivative is integrable at infinity. Both features will matter.

Macroscopic behavior. In terms of the empirical measure $\mu_N := \frac{1}{N} \sum_i \delta_{x_i}$, $\mathsf{H}_N = N^2 \left(\frac{1}{2} \iint_{x \neq y} - \log|x-y| \mathtt{d}\mu_N \mathtt{d}\mu_N + \int \right.$ is of order N^2, while the entropy of N particles is of order N. So μ_N concentrates on the minimizer of the energy functional
$$\mathcal{I}(\mu) := \frac{1}{2} \iint - \log|x-y| \mathtt{d}\mu(x) \mathtt{d}\mu(y) + \int \frac{x^2}{4} \mathtt{d}\mu(x),$$
which is the semicircle law $\mathtt{d}\mu_{\mathrm{sc}} = \rho_{\mathrm{sc}}(x)\mathtt{d}x$, $\rho_{\mathrm{sc}}(x) := \frac{1}{2\pi} \sqrt{(4-x^2)_+}$. Ben Arous and Guionnet [BAG97] proved the large deviation principle, informally $\mathbb{P}_{N,\beta}(\mu_N \approx \mu) \approx e^{-\beta N^2 (\mathcal{I}(\mu) - \mathcal{I}(\mu_{\mathrm{sc}}))}$, and the fluctuations of smooth linear statistics are of order one, and Gaussian [Joh98]:

the macroscopic behavior does not depend on β.

At the microscopic scale, by contrast, the behavior depends on β, but not on the confining potential: local statistics are universal [BEY14].

1.4 The Sine_β process

Let $\mathsf{x} \in (-2, 2)$ be a point in the bulk. Near x, the typical distance between particles is $(N\rho_{\mathrm{sc}}(\mathsf{x}))^{-1}$. We zoom in and consider the random point configuration

$$\mathbf{X}_{N,\mathsf{x}} := \sum_{i=1}^{N} \delta_{N\rho_{\mathrm{sc}}(\mathsf{x})(x_i - \mathsf{x})},$$

which has, on average, one point per unit length near the origin.

Theorem (Valkó–Virág [VV09])**.** As $N \to \infty$, the law of $\mathbf{X}_{N,\mathsf{x}}$ converges to a point process Sine_β which does not depend on x.

The tridiagonal models opened the way to the microscopic scale, through random operators and stochastic differential equations. In the bulk, Sine_β is described by a family of coupled diffusions, the "Brownian carousel" [VV09]; Killip and Stoiciu [KS09] obtained the limit of the circular β-ensemble, and the two limits coincide [Nak14, VV20]; later, Sine_β was described as the spectrum of a random Dirac operator [VV17]. At the edge, the limit is Airy_β, the spectrum of a random Schrödinger operator [RRV11].

For $\beta = 2$, Sine_β is the determinantal process with the sine kernel. For general β, many properties are now known:

- the number of points in a large interval has a logarithmic variance and satisfies a CLT [KVV12];
- the probability of n points in a fixed interval decays like $e^{-\mathtt{c}_\beta n^2 \log n}$ [HV17], as for a Coulomb gas (compare with (2.3));
- Sine_β is number-rigid: the configuration outside an interval determines the number of points inside [CN18, DHLM21];
- it converges to the Poisson process as $\beta \to 0$ [AD14] and to the lattice as $\beta \to \infty$.

1.5 The goal of the course

All these results rely on the matrix models. None of them, *including the existence of* Sine_β, is proven from the Gibbsian structure alone. The question of the course is:

Can we prove the existence of Sine_β*, and characterize it, using only the fact that the* β*-ensemble is the Gibbs measure of a log-gas?*

The answer is yes (Theorem 1.2 below).

1.6 The statistical physics approach, and the main result

A line of research takes Dyson's observation seriously and studies β-ensembles as *Gibbs measures*: at the macroscopic scale, this goes back to Boutet de Monvel, Pastur and Shcherbina [BdMPS95]; at the microscopic scale, to Sandier and Serfaty [SS12, SS15]. They introduced a *renormalized energy* for infinite configurations of charges, defined through the electric field they generate, and started a programme mostly carried out by Serfaty and her co-authors [Ser24]: next-order expansions of the free energy [RS16, PS17], a large deviation principle for the microscopic configuration seen from a typical point [LS17], whose rate function, $\beta\times$ energy + entropy, suggests that the microscopic limit minimizes a free energy, CLTs by transport [BLS18], and local laws by energy methods [Leb17, BBNY17, AS21, Pei22] (for β-ensembles, optimal rigidity had been obtained earlier by loop equations [BEY14]). On the side of infinite-volume Gibbs measures, Sine_β was shown to satisfy canonical DLR equations [DHLM21], and to be the unique minimizer of the free energy [EHL21, HM25].

We carry this programme to its end, existence included. The statement needs three notions, made precise in Sections 6 and 8.

- A point process is *good* if its counts have all moments of the natural order and its discrepancies $\mathbf{X}(I) - |I|$ are logarithmic in $|I|$. $\mathcal{G}$ denotes the stationary good processes.

- The *free energy* $\mathcal{F}_\beta^{\mathrm{int}}$ of a stationary process is β times its energy per unit length, plus its entropy per unit length relative to the Poisson process: the same balance as in a finite Gibbs measure.
- The *canonical DLR equations* express that, conditionally on the exterior of any bounded interval and on the number of points inside, the configuration inside is a finite Gibbs measure for the log-gas.

Theorem 1.2 (Existence and characterization of Sine_β)**.** Let $\beta > 0$ and $\mathsf{x} \in (-2, 2)$. As $N \to \infty$, the law of $\mathbf{X}_{N,\mathsf{x}}$ converges to a point process $\mathrm{Sine}_\beta \in \mathcal{G}$, which does not depend on x. Moreover:

1. Sine_β is the unique minimizer of $\mathcal{F}_\beta^{\mathrm{int}}$ on $\mathcal{G}$ (among processes whose energy per unit length is well defined as a limit);
2. Sine_β is the unique stationary good solution of the canonical DLR equations.

The proof follows the chain:

$$\begin{aligned} \mathrm{G}\beta\mathrm{E} &\xrightarrow{\text{Sections 4–5}} && \text{local laws, uniformly in } N \\ \text{limit point P of } \mathbf{X}_{N,\mathsf{x}} &\xrightarrow{\text{Section 8}} && \mathrm{P} \text{ is good and } \mathrm{P} \in \mathrm{DLR}_\beta \\ &\xrightarrow{\text{Section 9}} && \mathrm{P} \text{ is stationary, so } \mathrm{P} \in \mathcal{G} \\ &\xrightarrow{\text{Sections 7, 10}} && \mathcal{F}_\beta^{\mathrm{int}}(\mathrm{P}) = \min_{\mathcal{G}} \mathcal{F}_\beta^{\mathrm{int}} \\ &\xrightarrow{\text{Section 11}} && \text{the minimizer is unique, so all limit points coincide.} \end{aligned}$$

In words: the local laws give limit points that remember the Gibbs structure (DLR); the DLR equations force translation invariance and minimality of the free energy; and the free energy has a unique minimizer. Unlike most of the literature above, *we never use electric fields*: everything happens on the line, with one object, the discrepancy function, whose Gagliardo seminorm is the energy (Section 2).

References

For the history of random matrices, see [DF17] and the introductions of [Meh04, For10]; for the mathematical theory, [AGZ10, For10], and [EY17] for universality. For log-gases and Coulomb gases as statistical mechanics systems, see the lecture notes [Ser24] and the survey [Lew22]. The existence of Sine_β is [VV09, KS09]; the statistical physics characterizations that these notes build on are [LS17, DHLM21, EHL21].

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