Hyperuniformity vs. finite 2-Wasserstein distance to a lattice

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Goal: comparing two properties of *stationary point processes*:

- **Hyperuniformity (small number variance in large disks)**
- **Finite transportation distance to a lattice**

With D. Dereudre, D. Flimmel, and M. Huesmann (2024):

- *(Non)-hyperuniformity of perturbed lattices*
- *The link between hyperuniformity, Coulomb energy, and Wasserstein distance to Lebesgue for two-dimensional point processes*.

Number variance

- $\mathbf{X}$ a point configuration (loc. finite collection of points in $\mathbb{R}^{\mathrm{d}}$)
- $\mathcal{D}_r$ the disk of radius $r > 0$, $|\mathcal{D}_r|$ its Lebesgue measure.
- $\mathrm{Points}(\mathbf{X}, \mathcal{D}_r)$ the number of points of $\mathbf{X}$ inside $\mathcal{D}_r$

If $\dot{\mathbf{X}}$ is a random point configuration, write:

$$\sigma_{\dot{\mathbf{X}}}(r) := \frac{1}{|\mathcal{D}_r|} \mathrm{Var}\left[\mathrm{Points}(\dot{\mathbf{X}}, \mathcal{D}_r)\right].$$

"Rescaled number variance" for the point process $\dot{\mathbf{X}}$.
Interest in the asymptotic behavior of $\sigma_{\dot{\mathbf{X}}}(r)$ as $r \to \infty$.

Hyperuniformity

A point process / random point configuration $\dot{\mathbf{X}}$ is said to be **hyperuniform** when:

$$\lim_{r\to\infty} \sigma_{\dot{\mathbf{X}}}(r) = 0.$$

1. For "iid points" (Poisson point process), $\sigma(r) = 1$.
2. For points on a "stationary lattice", $\sigma(r) = O(r^{-1})$.
3. This cannot be beaten. Beck, Acta Mathematica 1987 ("Class-I hyperuniform")

Gabrielli, Joyce, Sylos Labini, 2002 super-homogeneous
Torquato, Stillinger, 2003 hyperuniform

Interest?

"Order within disorder".

- Lattices represent "usual" order: correlations persists at infinity.
- The Ginibre ensemble from RMT (eigenvalues of a $N \times N$ matrix with i.i.d. complex Gaussian entries, then $N \to \infty$, Ginibre '64) is class-I hyperuniform.
- The two-point correlation function of the Ginibre ensemble decays as $e^{-|x-y|^2}$!!

No "order" in the usual sense of long-range correlations. And yet "suppression of charge fluctuations" Martin-Yalcin, Lebowitz 80's one-component plasmas / Coulomb gases.
Torquato 2018: survey of examples of various nature.

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Claim (Gabrielli, 2004)

In dimension 2, dependent perturbations of a lattice + finite second moment $\implies$ hyperuniform.

Perturbations of a lattice?

- $(p_x)_{x\in\mathbb{Z}^\mathrm{d}}$ "perturbation field" with $\mathbb{Z}^\mathrm{d}$-invariant distribution
- Identically distributed + more invariance conditions.
  Independent versus dependent case. Size of perturbations.
- u uniform in $[0,1]^d$ (independent of perturbations)
- Perturbed stationary lattice:

$$\dot{\mathbf{X}} := \sum_{x\in\mathbb{Z}^\mathrm{d}} \delta_{x+p_x+u}$$

Finite Wasserstein distance

Possible definition: we say that $\dot{\mathbf{X}}$ has finite s-Wasserstein distance to the stationary lattice when $\dot{\mathbf{X}}$ can be written as a perturbed lattice:

$$\dot{\mathbf{X}} := \sum_{x \in \mathbb{Z}^d} \delta_{x+p_x+u}$$

for some $\mathbb{Z}^d$-**invariant** perturbation field s.t. $\mathbb{E}[|p_0|^s] < +\infty$.

Wasserstein / Optimal transport interpretation?

First result: from Wasserstein to Hyperuniform

The physics literature (Gabrielli, 2004, Kim-Torquato 2018) claims that: in dimension 2, perturbations with finite second moment cannot break hyperuniformity.

Theorem (Dereudre-Flimmel-Huesmann-L. '24)

- *In dimension 2, finite 2-Wasserstein distance to a stationary lattice implies hyperuniformity.*
  No information on the speed of $\sigma(r) \to 0$ in general!
- *In dimension ≥ 3, there are arbitrarily small perturbations of $\mathbb{Z}^d$ s.t. $\sigma(r) \to +\infty$.*
- *In any dimension, there are hyperuniform processes such that $\sigma(r) \to 0$ arbitrarily slowly.*

Converse results?

(2024) Lachièze-Rey, Yogeshwaran, and Butez, Dallaporta, Garcia-Zelada: results of the type "hyperuniform + conditions imply finite 2-Wasserstein distance".

Theorem (Huesmann-L.)

- *Hyperuniformity does not imply finite 2-Wasserstein distance to a lattice in general.*
- *However, if we impose that $\sigma(r) \to 0$ in a reasonable way*

$$\sigma(r) \leq Cr^{-\epsilon}, \quad \sigma(r) \leq C \ln(r)^{-1-\epsilon}, \quad \sum_{n \geq 1} \sigma(2^n) < +\infty \; (\star\text{-}\mathbf{HU})$$

then $\star$-**HU** $\implies$ *finite 2-Wasserstein distance to a lattice.*

Summary

In dimension 2:

- Two implications

$$\star\textbf{-HU} \implies \text{finite 2-Wasserstein to a lattice} \implies \textbf{HU}.$$

- "Gap" consists of processes such that $\sigma(r) \to 0$ very slowly.
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Typology of HU processes (Torquato):

1. Class-I (lattices + i.i.d perturbations, Ginibre...) $\sigma(r) \sim Cr^{-1}$
2. Class-II $\sigma(r) \sim Cr^{-1} \ln r$
3. Class-III $\sigma(r) \sim Cr^{-\epsilon}$, for $\epsilon > 0$.
4. ....

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3. "reduced covariance measure" has an explicit expression:

$$\alpha_{red}^{(2)}(B) = \mathbb{E}\left[\sum_{x \in \mathbb{Z}^d} \mathbf{1}\left\{x + p_x - p_0 \in B\right\}\right] \tag{1}$$

Fourier side: "structure factor" S

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6. Difficult part: justifying that **only** small frequencies matter.

$\star$-HU implies finite Wasserstein distance

- True "Wasserstein" interpretation: need a theory of optimal transportation for point processes. see Martin's talk, Thursday 2pm "Optimal Transport and the Role of Entropy". Infinite-volume theory, constructed as "per unit volume" limits for invariant objects.

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- In finite-volume, classical inequalities relating:
  1. 2-Wasserstein distance between $\mathbf{m}_0$ and Lebesgue measure in a square.
  2. The negative Sobolev H^{-1} norm of $\mathbf{m}_0$ – Lebesgue in that square.

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  Ledoux "On optimal matching of Gaussian samples" 2019:

$$\mathrm{W}_2(\mathbf{m}_0, \text{Lebesgue}) \leq C \|\mathbf{m}_0 - \text{Lebesgue}\|_{H^{-1}}$$

  (Any dimension!)

$\star$-HU implies finite Wasserstein distance - II

Controlling $H^{-1} \implies$ controlling 2-Wasserstein distance.
H^{-1} norm of $\mathbf{m}_0 -$ Lebesgue is related to *Coulomb energy*.

$$\iint g(x-y)d\left(\mathbf{m}_0 - \text{Lebesgue}\right)(x)d\left(\mathbf{m}_0 - \text{Lebesgue}\right)(y),$$

where g is the Coulomb kernel.

Theorem (HL)

1. *In any dimension, finite Coulomb energy of a point process implies finite 2-Wasserstein distance to the Lebesgue measure.*
2. *The converse is (barely false) but becomes true under uniform density condition.*
3. *In dimension 2, $\star$-HU is equivalent (!) to finite Coulomb.*

The last step relies on a recent result of Sodin-Yakir-Wenmann.

Counter-example: hyperuniform but not finite distance

Take $\Lambda_N := [-\sqrt{N}, \sqrt{N}]^2$. Throw $|\Lambda_N|$ i.i.d. points in Λ_N.

- Average Wasserstein distance between the points and Lebesgue (or a portion of lattice...) is $CN^2 \log N$ Ajtai, Komlos, Tusnady 1984.
- Within Λ_N, the rescaled number variance is of order 1...

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Tile $\mathbb{R}^2$ with copies of Λ_N, throw $|\Lambda_N|$ i.i.d. points in each copy.

- The **transportation cost per unit volume** grows like $\log N$
- The rescaled number variance becomes small at large scales.

Choose N randomly in such a way that:

$$\sum_{n \geq 1} \mathbb{P}(N = n) \log n = +\infty.$$

Thank you for your attention!